KINKS UNDER CONSTANT FORCES

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The behaviour of two-dimensional kinks under constant forces is fully analyzed in this paper. The most general solution is found and shown to be equivalent to the two-dimensional "bounce" in asymmetric potentials. The conditions under which the solution hereby presented behaves as an oscillating elliptic kink are also discussed.

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After the large development that soliton physics has experienced in the last decade, several authors have now turned their attention to the problem of perturbed solitons. Since many soliton solutions (either of topological character on not) have already been discovered for scalar, spinor or gauge fields (or combination of these), one of the main questions now is to look at the problem of solitonic solutions perturbed in various ways. For topological solitons this is an interesting question since we do not know "a priori" whether the topological invariants that ensure stability for the soliton are strong enough to mantain solitonic behaviour with the same stable properties, when external forces are included in the physical system.

One of the first problems solved along this line has been the behaviour of kinks under velocity-dependent forces or "friction". The kink-friction system has been studied by several authors (see Refs [1-2]) and finally exactly solved by the author and collaborator ([3]). Also, a sort of friction arising from internal degrees of freedom for scalar fields has been studied recently and the energies and invariant charges have also been calculated exactly [4].

In this paper we shall turn our attention to a system formed by a kink-like soliton under perturbation caused by a constant force F. This problem is in a way essentially equivalent to the problem of the "bounce" already discussed recently by the author but with some differences. The specific problem of the constant force has also been partially discussed in the recent literature ([5, 6 and 7]), buth the full analysis and discussion has not yet been presented. It is the aim of this paper to develop the total solution of this problem as well as fully analyse the correspondent physical consequences. Our attention is mainly focused on the conditions under which the solution is of oscillatory nature. This is done by using the general form of the solution in terms of elliptic functions and the limiting and particular cases are studied separately. We also find in the course of our work, several other interesting properties in regard to the critical nature of the external force applied and also some constraints for the case of the oscillating solution which have not yet been discussed before.

In Section 1 we analyse the critical behaviour of the non-linear scalar potential including a constant force F. Section 2 is entirely devoted to the exact solution of the correspondent non-linear ordinary differential equation. The Section 3 is a full discussion of the oscillatory nature of the solution for various ranges of external forces and energies. We close with a Section of Conclusions.

1. The physical system

We start with the field equation for the kink when a constant force is included in the dynamical evolution equation.

$$\frac{d^2Q}{dt^2} - c^2\frac{d^2Q}{dx^2} - K_2Q + K_4Q^3 - F = 0, \tag{1}$$

where K_2 and K_4 are coupling constants and F is the constant force applied to the system. As always ([3–4]) we use dimensionless variables defining:

$$\left(\frac{K_4}{K_2}\right)^{1/2} Q(x,t) = g(x,t); \quad z = \left(\frac{K_2}{c^2}\right)^{1/2} \frac{x - vt}{\left[1 - \left(\frac{v}{c}\right)^2\right]^{1/2}}; \quad a = \left(\frac{K_4}{K_2^3}\right)^{1/2} F \tag{2}$$

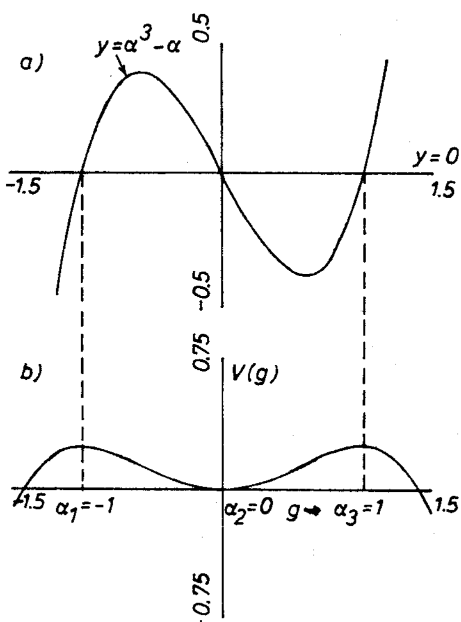

Fig. 1. a) Graphical solution of the equation $\alpha^3-\alpha = a$ for $a = 0$. b) The potential $V(g)$ for this case

and, indeed, $v < c$. Using (2), the field equation (1) transforms to:

$$\frac{d^2g}{dz^2} - g^3 + g + a = 0. \tag{3}$$

There exists an obvious first integral of (3), namely the energy, given by:

$$\varepsilon = \tfrac{1}{2}\left(\frac{dg}{dz}\right)^2 - \tfrac{1}{4}\, g^4 + \tfrac{1}{2}\, g^2 + ag. \tag{4}$$

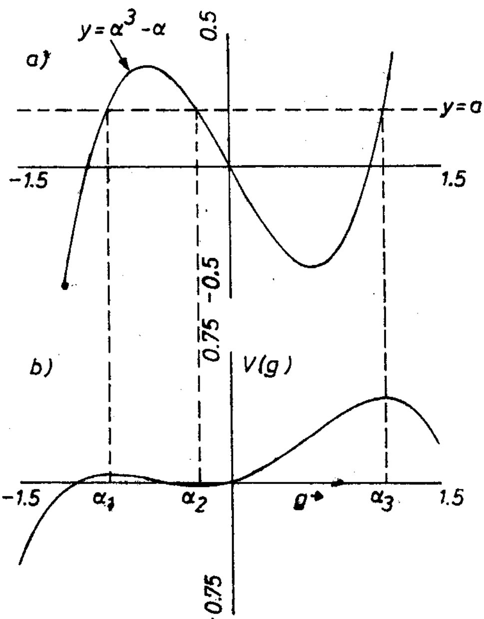

Fig. 2. a) Graphical solution of the equation $\alpha^3 - \alpha = a$ for $0 < a < [\frac{4}{27}]^{1/2}$. b) The potential $V(g)$ for this case

Equation (3) can easily be interpreted as the one representing the motion of a particle moving in the potential:

$$V(g) = -\tfrac{1}{4}\, g^4 + \tfrac{1}{2}\, g^2 + ag. \tag{5}$$

We can restrict ourselves trivially to the case $a > 0$ (the $a < 0$ is just the same as the above for $g \to -g$).

We first note that there is a critical value of a for which the potential drastically changes its form. The maxima and minima of $V(g)$ correspond to those constant values α of g satisfying the algebraic equation

$$a = \alpha(\alpha^2 - 1). \tag{6}$$

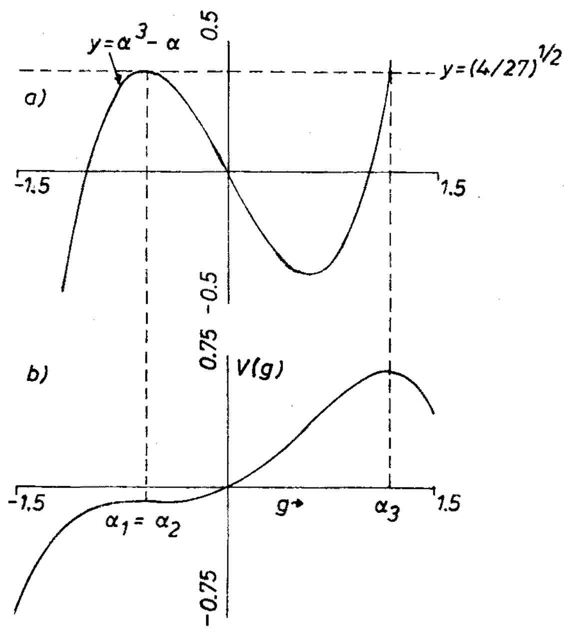

Fig. 3. a) Graphical solution of the equation $\alpha^3-\alpha = a$ for $a = [\frac{4}{27}]^{1/2}$. b) The potential $V(g)$ for this case

But (6) has three real solutions ($\alpha_1 \leqslant \alpha_2 \leqslant \alpha_3$) if $a < [\frac{4}{27}]^{1/2} = a_0$ and only one real solution if $a > a_0$. As it is shown in Figs 1, 2 and 3, the potential has two maxima and one minimum if $a < a_0$ and only one maximum if $a \geqslant a_0$. Therefore the range in which oscillatory solutions can be found is $0 \leqslant a < a_0$. The corresponding range of energies is $V(\alpha_2) \leqslant \varepsilon \leqslant V(\alpha_1)$ for $V(\alpha) = \frac{1}{4}\alpha^2(3\alpha^2-2)$. We shall restrict ourselves to these ranges of a and $V(\alpha)$ henceforth. The roots $(\alpha_1, \alpha_2, \alpha_3)$ are restricted within this range to the intervals

$$-1 \leqslant \alpha_1 \leqslant -(3)^{-1/2} \tag{7a}$$

$$-(3)^{-1/2} \leqslant \alpha_2 \leqslant 0 \tag{7b}$$

$$1 \leqslant \alpha_3 \leqslant 2(3)^{-1/2} \tag{7c}$$

as can easily be deduced from the Figs 1 to 3.

2. *The solution*

The general solution of equation (3) can be written in terms of elliptic functions (see Ref. [8]) as:

$$g(z) = A+B\,\frac{\mathrm{sn}\,(D(z-z_0)\,|m^2)}{1+C\,\mathrm{sn}\,(D(z-z_0)\,|m^2)}, \tag{8}$$

where z_0 is an arbitrary integration constant and A, B and D can be expressed in terms of

C and m as:

$$A = \frac{C}{\lambda}[1+m^2-2C^2], \tag{9a}$$

$$B = \frac{2}{\lambda}[1-C^2][m^2-C^2], \tag{9b}$$

$$D^2 = \frac{2}{\lambda^2}[1-C^2][m^2-C^2], \tag{9c}$$

$$\lambda = +[2m^2(1+m^2)+[(1+m^2)^2-12m^2]C^2+2(1+m^2)C^4]^{1/2}. \tag{9d}$$

Finally, C and m can be expressed in terms of a and ε through the expressions

$$a = \frac{2C}{\lambda^3}[1-m^2]^2[C^4-m^2], \tag{10a}$$

$$\varepsilon = \frac{1}{4\lambda^4}[(1-m^2)^2C^2-4m^2(1-C^2)^2][(1-m^2)^2C^2-4(m^2-C^2)^2]. \tag{10b}$$

Thus, the arbitrary integration constants appearing in the solution (8) are only ε and z_0. Since we are interested in the oscillatory case, the allowed range for C must be

$$m^{1/2} \leqslant C \leqslant 1. \tag{11}$$

Notice $C = m^{1/2}$ yields $a = 0$ and $C = 1$ yields $a = a_0$ in (10a). For this range $D^2 < 0$ and $\mathrm{sn}(D(z-z_0)\,|m^2)$ oscillates between the values 1 and m^{-1}. Therefore, as can easily be

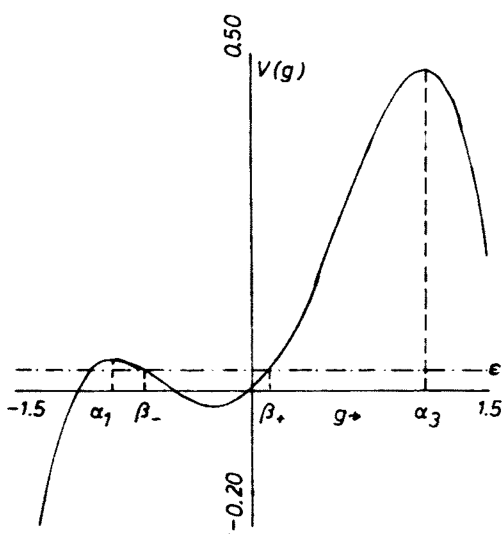

Fig. 4. The potential $V(g)$ for $0 \leqslant a \leqslant [\frac{4}{27}]^{1/2}$ where the turning points β_+ and β_-, corresponding to the oscillatory solution are displayed

seen from Fig. 4, the solution $g(z)$ oscillates between the values β_+ and β_- given by

$$\beta_+ = A + \frac{B}{1+C} = \frac{2m^2+(1-m^2)C+2C^2}{\lambda}, \tag{12a}$$

$$\beta_- = A + \frac{B}{m+C} = \frac{2m-(1-m^2)C-2mC^2}{\lambda}. \tag{12b}$$

A particular case of interest is the limiting case in which $V(\alpha_1) = \varepsilon$. This represents the behaviour for $m = 0$. In this case we can express through (10a) C as a function of α_1 as:

$$\alpha_1 = -[1+2C^2]^{-1/2} \tag{13}$$

with the help of (13) we can write down the solution just in terms of α_1. This solution is

$$g(z) = \frac{1-2\alpha_1^2}{\alpha_1} - \left[\frac{1-\alpha_1^2}{2}\right]^{1/2} \left[\frac{3\alpha_1^2-1}{\alpha_1^2}\right] \frac{\cosh[(3\alpha_1^2-1)^{1/2}(z-z_0)]}{1-\frac{1}{\alpha_1}\left[\frac{1-\alpha_1^2}{2}\right]^{1/2}\cosh[(3\alpha_1^2-1)^{1/2}(z-z_0)]} \tag{14}$$

and the energy corresponding to this solution is $\varepsilon = V(\alpha_1) = \frac{1}{4}\alpha_1^2(3\alpha_1^2-2)$.

3. *The oscillating kink*

We address now ourselves the question as to whether the conditions for an oscillating kink have to be somehow restricted. For an oscillating kink whithout external force applied to it, the field equation is

$$\frac{d^2 g_1}{dz^2} - g_1^3 + g_1 = 0 \tag{15}$$

whose oscillatory solution is well known

$$g_1(z) = \left[\frac{2m_1^2}{1+m_1^2}\right]^{1/2} \operatorname{sn}([1+m_1^2]^{-1/2}(z-z_1)\,|m_1^2) \tag{16}$$

and energy given by

$$\varepsilon_1 = \frac{1}{2}\left[\frac{dg_1}{dz}\right]^2 - \frac{1}{4}g_1^4 + \frac{1}{2}g_1^2 = \frac{m_1^2}{(1+m_1^2)^2} \tag{17}$$

so that $0 \leqslant \varepsilon_1 \leqslant \frac{1}{4}$.

If $z = z_0$ is the space-time point in which the force is applied, then (15), (16) and (17) hold for $z < z_0$. Therefore, for $z = z_0$ the field configuration is in a state $g_0 = g(z_0)$ with energy ε_1.

Now, the force is applied from $z > z_0$. The dimensionless "force" has been called

a throughout the paper. Thus, the field equation is now:

$$\frac{d^2 g_2}{dz^2} - g_2^3 + g_2 + a = 0 \tag{18}$$

whose solution has already been described in Section 2 (8, 9 and 10) and the energy is given by

$$\varepsilon_2 = \tfrac{1}{2}\left(\frac{dg_2}{dz}\right)^2 - \tfrac{1}{4}\, g_2^4 + \tfrac{1}{2}\, g_2^2 + a g_2 = \tfrac{1}{2}\left(\frac{dg_2}{dz}\right)^2 + V(g_2) \tag{19}$$

such that $V(\alpha_2) \leqslant \varepsilon_2 \leqslant V(\alpha_1)$, where $\alpha_1 \leqslant \alpha_2$ and (α_1, α_2) are the two smallest roots of the algebraic equation $\alpha^3 - \alpha = a$.

Let us now analyze the conditions under which if g_1 is an oscillating solution, g_2 could also be an oscillating solution. Firstly, we note that the dimensionless force must verify

$$0 \leqslant a \leqslant [\tfrac{4}{27}]^{1/2}$$

because otherwise the effective potential (Fig. 4) does not exhibit oscillating ranges. Additionally, we can ask also the question of whether the system is in a state, where $g(z_0)$ and $g'(z_0)$ are arbitrary. In other words. Let a kink-like system be with energy ε_1 taking the value $g_0 \equiv g_1(z_0)$ at the space-time point z_0 in which the force is applied. Now we ask about the conditions that must be fulfilled by g_0 and ε_1 in order the system would continue to be oscillating after the force is applied. For continuity and smoothness of the solution we know that

$$g_0 \equiv g_1(z_0) \equiv g_2(z_0)$$

$$\left(\frac{dg_1}{dz}\right)_{z=z_0} = \left(\frac{dg_2}{dz}\right)_{z=z_0}.$$

Using these conditions in (17) and (19) we find easily the relationship between ε_1 and ε_2, given by

$$\varepsilon_2 = \varepsilon_1 + a g_0 \tag{20}$$

or, for the potential $V(g_2)$ in (19)

$$V(\alpha_2) - \varepsilon_1 \leqslant a g_0 \leqslant V(\alpha_1) - \varepsilon_1. \tag{21}$$

On the other hand, it is trivial to note that (Fig. 4) $\alpha_1 \leqslant g_0$. Using this inequality in (21) we find

$$\varepsilon_1 \leqslant V(\alpha_1) - a\alpha_1 \tag{22}$$

or, equivalently, using $0 \leqslant \varepsilon_1 \leqslant \frac{1}{4}$, we obtain

$$0 \leqslant \varepsilon_1 \leqslant \tfrac{1}{4}\, \alpha_1^2 (2 - \alpha_1^2). \tag{23}$$

Notice that this range is smaller as the applied force increases. Furthermore, as soon as ε_1 has been fixed within the range (23) we easily obtain the range of validity for g_0 through (21). This is given by:

$$\frac{1}{a}(V(\alpha_2)-\varepsilon_1) \leqslant g_0 \leqslant \frac{1}{a}(V(\alpha_1)-\varepsilon_1). \tag{24}$$

To summarize. For a kink-like system, with energy ε_1, to which at the point z_0 an external constant force is applied, would be an oscillating system, the following conditions must hold:

$$0 \leqslant a \leqslant |\tfrac{4}{27}|^{1/2} \Rightarrow -1 \leqslant \alpha_1 \leqslant -[3]^{-1/2}, \tag{25a}$$

$$0 \leqslant \varepsilon_1 \leqslant \tfrac{1}{4}\alpha_1^2(2-\alpha_1^2), \tag{25b}$$

$$\frac{1}{a}(V(\alpha_2)-\varepsilon_1) \leqslant g_0 \leqslant \frac{1}{a}(V(\alpha_1)-\varepsilon_1). \tag{25c}$$

As was first pointed out (see Refs. [5–7]), anotheri nteresting question would be whether a non-linear kink-like non-oscillating solution would become an oscillating one under the action of a constant force. We shall show now that this possibility cannot be attained under the conditions analyzed in this paper. To see this, we first note that a non-oscillating kink before the action of the force, should be such that $\varepsilon_1 > \frac{1}{4}$ or $g_0 < -1$. But if $\varepsilon_1 > \frac{1}{4}$, then (25b) would fail to hold since $\frac{1}{4}\alpha_1^2(2-\alpha_1^2) \leqslant \frac{1}{4}$. Furthermore if $g_0 < -1$ then since $\alpha_1 \leqslant g_0$ would yield $\alpha_1 < -1$, against the inequality (25a). Therefore, the kink-like system here analyzed cannot oscillate under the action of the applied force unless it were already oscillating before the force were applied. Even in this case the conditions (25a, b, c) must hold.

4. *Conclusions*

We have presented in this paper a full account of the solution of the kink-constant force system. Three different features can be emphasized: a) The critical nature of the force. Outside some ranges there is no kink neither topological oscillations. This fact is in sharp constrast with results in linear field theory in which no such constraint exists. b) The conditions for oscillating kinks are much stronger than it was expected and a delicate balance of parameters such as energies and initial conditions is needed in order to have solutions with the required properties. c) Also, the system is closely related to that of the bounce which we expect to be classicaly unstable. Therefore this is a case in which external perturbations (constant force) induce instabilities on the unperturbed soliton. On the contrary, systems as kink-friction studied previously ([3]) are stable and the topological charge is conserved. In the case under consideration in this paper, the physical features are closer to those of non-topological solitons on Q-balls, which do not exhibit conserved topological currents ([4]). Some other results concerning different (non-constant) forces that might exhibit topological properties are now under study and will be the subject of a forthcoming paper.

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