DIRAC EQUATION WITH HIDDEN EXTRA SPINS: A GENERALIZATION OF KÄHLER EQUATION*

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It is observed that the Dirac anticommutation relations admit some peculiar reducible representations implying the existence of additional spins 1/2 commuting with the usual spin 1/2 and being decoupled from the magnetic field in the Dirac equation. In the case of the simplest representation of this class the Dirac equation becomes equivalent to the Kähler equation. So, such a class of reducible representations gives a generalization of Kähler equation, realizing the Dirac square-root procedure in the case of arbitrary total spin. A possible relation of the generalized Kähler equation to the problem of fermion generations is sketched.

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1. Introduction

In this paper we would like to point out that the Dirac anticommutation relations admit some peculiar reducible representations implying the existence of additional spin-1/2 operators commuting with the usual spin-1/2 operator and being decoupled from the magnetic field in the Dirac equation. So, these extra spins are hidden in the Dirac equation, unless the mass is dependent on them.

Let us start with the Dirac equation in an external electromagnetic field,

$$[\Gamma \cdot (p-eA)-m]\psi = 0, \tag{1}$$

where $p_\mu = i\partial/\partial x^\mu$, $[m, \Gamma^\mu] = 0$ and

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}. \tag{2}$$

Our observation is that for the Dirac anticommutation relations (2) there exists the following sequence of representations:

$$\Gamma^\mu = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i^\mu, \tag{3}$$

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where the matrices γ_i^μ, $i = 1, 2, \ldots, N$, span the sequence of Clifford algebras defined by the anticommutation relations

$$\{\gamma_i^\mu, \gamma_j^\nu\}_- = 2\delta_{ij} g^{\mu\nu}. \tag{4}$$

It can be seen that for $N \geqslant 2$ the representations (3) are reducible (cf. Eqs. (13) and (16)).

Using the usual 4×4 Dirac matrices γ^μ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, the matrices γ_i^μ and $\gamma_i^5 = i\gamma_i^0\gamma_i^1\gamma_i^2\gamma_i^3$ (where $\{\gamma_i^\mu, \gamma_i^5\} = 0$) can be represented for $N = 1$ as

$$\gamma_1^\mu = \gamma^\mu, \quad \gamma_1^5 = \gamma^5, \tag{5}$$

for $N = 2$ as

$$\begin{aligned} \gamma_1^\mu &= \gamma^\mu \otimes \mathbf{1}, & \gamma_1^5 &= \gamma^5 \otimes \mathbf{1}, \\ \gamma_2^\mu &= \gamma^5 \otimes i\gamma^5\gamma^\mu, & \gamma_2^5 &= \mathbf{1} \otimes \gamma^5, \end{aligned} \tag{6}$$

for $N = 3$ as

$$\begin{aligned} \gamma_1^\mu &= \gamma^\mu \otimes \mathbf{1} \otimes \mathbf{1}, & \gamma_1^5 &= \gamma^5 \otimes \mathbf{1} \otimes \mathbf{1}, \\ \gamma_2^\mu &= \gamma^5 \otimes i\gamma^5\gamma^\mu \otimes \mathbf{1}, & \gamma_2^5 &= \mathbf{1} \otimes \gamma^5 \otimes \mathbf{1}, \\ \gamma_3^\mu &= \gamma^5 \otimes \gamma^5 \otimes \gamma^\mu, & \gamma_3^5 &= \mathbf{1} \otimes \mathbf{1} \otimes \gamma^5, \end{aligned} \tag{7}$$

for $N = 4$ as

$$\begin{aligned} \gamma_1^\mu &= \gamma^\mu \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}, & \gamma_1^5 &= \gamma^5 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}, \\ \gamma_2^\mu &= \gamma^5 \otimes i\gamma^5\gamma^\mu \otimes \mathbf{1} \otimes \mathbf{1}, & \gamma_2^5 &= \mathbf{1} \otimes \gamma^5 \otimes \mathbf{1} \otimes \mathbf{1}, \\ \gamma_3^\mu &= \gamma^5 \otimes \gamma^5 \otimes \gamma^\mu \otimes \mathbf{1}, & \gamma_3^5 &= \mathbf{1} \otimes \mathbf{1} \otimes \gamma^5 \otimes \mathbf{1}, \\ \gamma_4^\mu &= \gamma^5 \otimes \gamma^5 \otimes \gamma^5 \otimes i\gamma^5\gamma^\mu, & \gamma_4^5 &= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \gamma^5 \end{aligned} \tag{8}$$

and so on. Here, $\mathbf{1}$ is the 4×4 unit matrix. Note that for $i \neq j$ $[\gamma_i^\mu, \gamma_j^5] = 0$ and $[\gamma_i^5, \gamma_j^5] = 0$ though $\{\gamma_i^\mu, \gamma_j^\nu\} = 0$. In the van der Waerden representation of γ^μ where γ^5 is diagonal, all matrices γ_i^5 get a diagonal structure.

In the representation (3) the Lorentz transformations of $\psi(x)$ in the Dirac equation (1) are generated by

$$J^{\mu\nu} = L^{\mu\nu} + \sum_{i=1}^{N} \tfrac{1}{2}\sigma_i^{\mu\nu}, \tag{9}$$

where

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \quad \sigma_i^{\mu\nu} = \frac{i}{2}[\gamma_i^\mu, \gamma_i^\nu] \tag{10}$$

Note that for $i \neq j$ $[\gamma_i^\mu\ \sigma_j^{\varrho\sigma}] = 0$ and $[\sigma_i^{\mu\nu}, \sigma_j^{\varrho\sigma}] = 0$ though $\{\gamma_i^\mu, \gamma_j^\nu\} = 0$.

Denoting $\sigma_i^k = \frac{1}{2}\varepsilon^{klm}\sigma_i^{lm}$ and $\alpha_i^k = i\sigma_i^{k0} = \gamma_i^0\gamma_i^k = \gamma_i^5\sigma_i^k$, $k = 1, 2, 3$, one gets

$$\begin{gathered} [\sigma_i^k, \sigma_j^l] = 2i\delta_{ij}\varepsilon^{klm}\sigma_j^m, \quad \{\sigma_i^k, \sigma_i^l\} = 2\delta^{kl}, \\ [\sigma_i^k, \gamma_j^5] = 0 = [\sigma_i^k, \gamma_j^0] \end{gathered} \tag{11}$$

and

$$\{\alpha_i^k, \alpha_i^l\} = 2\delta^{kl}, \quad \{\alpha_i^k, \gamma_i^0\} = 0. \tag{12}$$

2. *Generalized Kähler equation*

Now, let us define for any $N = 1, 2, 3, \ldots$ the new matrices

$$\Gamma_1^\mu = \Gamma^\mu, \quad \Gamma_2^\mu, \ldots, \Gamma_N^\mu \tag{13}$$

by means of the Euler linear combinations of the previous matrices $\gamma_1^\mu, \gamma_2^\mu, \ldots, \gamma_N^\mu$. For instance, for $N = 3$ we get

$$\begin{aligned} \Gamma_1^\mu &= \frac{1}{\sqrt{3}}(\gamma_1^\mu+\gamma_2^\mu+\gamma_3^\mu), \\ \Gamma_2^\mu &= \frac{1}{\sqrt{2}}(\gamma_1^\mu-\gamma_2^\mu), \\ \Gamma_3^\mu &= \frac{1}{\sqrt{6}}(\gamma_1^\mu+\gamma_2^\mu-2\gamma_3^\mu). \end{aligned} \tag{14}$$

Then, the matrices (13) satisfy the anticommutation relations of the type (4):

$$\{\Gamma_i^\mu, \Gamma_j^\nu\} = 2\delta_{ij}g^{\mu\nu}, \tag{15}$$

So, after a proper change of the representations (5)–(8) (and so on) for γ_i^μ and γ_i^5, the matrices Γ_i^μ and $\Gamma_i^5 = i\Gamma_i^0\Gamma_i^1\Gamma_i^2\Gamma_i^3$ (where $\{\Gamma_i^\mu, \Gamma_i^5\} = 0$) can be represented by means of γ^μ, γ^5 and $\mathbf{1}$ through the equations of the type (5)–(8) (and so on). Notice that for $i \neq j$ $[\Gamma_i^\mu, \Gamma_j^5] = 0$ and $[\Gamma_i^5, \Gamma_j^5] = 0$ though $\{\Gamma_i^\mu, \Gamma_j^\nu\} = 0$.

In particular, we have in this new representation

$$\Gamma_1^\mu = \gamma^\mu \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{N-1 \text{ times}} \tag{16}$$

so that Eq. (1) takes the form

$$[\gamma\cdot(p-eA)-m]\psi = 0, \tag{17}$$

where N-1 bispinor indices of $\psi(x)$ (except for the first one) are free, unless m is an operator acting on them. It is interesting to note that for $N = 2$ Eq. (17) is equivalent [1] to the Kähler equation [2, 3]. Thus, Eq. (1) together with Eqs. (3) and (4) gives us a generalization of the Kähler equation for an arbitrary N. It realizes the Dirac square-root procedure in the case of arbitrary total spin.

It can be shown that in Eq. (9) the equality

$$\sum_{i=1}^{N} \tfrac{1}{2}\sigma_i^{\mu\nu} = \sum_{i=1}^{N} \tfrac{1}{2}\Sigma_i^{\mu\nu} \tag{18}$$

holds, where

$$\Sigma_i^{\mu\nu} = \frac{i}{2}\,[\Gamma_i^\mu, \Gamma_i^\nu]. \tag{19}$$

Thus, the Lorentz-group generators (9) can be rewritten as

$$J^{\mu\nu} = L^{\mu\nu} + \sum_{i=1}^{N} \tfrac{1}{2}\,\Sigma_i^{\mu\nu}. \tag{20}$$

Notice that for $i \neq j$ $[\Gamma_i^\mu, \Sigma_j^{\varrho\sigma}] = 0$ and $[\Sigma_i^{\mu\nu}, \Sigma_j^{\varrho\sigma}] = 0$ though $\{\Gamma_i^\mu, \Gamma_j^\nu\} = 0$. Denoting $\Sigma_i^k = \frac{1}{2}\varepsilon^{klm}\Sigma_i^{lm}$ and $A_i^k = i\Sigma_i^{ko} = \Gamma_i^0\Gamma_i^k = \Gamma_i^5\Sigma_i^k$, $k = 1, 2, 3$, one obtains

$$[\Sigma_i^k, \Sigma_j^l] = 2i\delta_{ij}\varepsilon^{klm}\Sigma_j^m,$$

$$\{\Sigma_i^k, \Sigma_i^l\} = 2\delta^{kl},$$

$$[\Sigma_i^k, \Gamma_j^5] = 0 = [\Sigma_i^k, \Gamma_j^0] \tag{21}$$

and

$$\{A_i^k, A_i^l\} = 2\delta^{kl}, \quad \{A_i^k, \Gamma_i^0\} = 0. \tag{22}$$

Since in the Dirac equation (1) the electromagnetic field is coupled only to the matrices $\Gamma^\mu = \Gamma_1^\mu$, a particle satisfying such an equation can display in the magnetic field only the usual spin 1/2 described by the operator $\frac{1}{2}\vec{\Sigma} = \frac{1}{2}\vec{\Sigma}_1$. It is true for any $N = 1, 2, 3, \ldots$. The additional spins 1/2 described by the operators $\frac{1}{2}\vec{\Sigma}_i$, $i = 2, 3, \ldots, N$, are decoupled from the magnetic field. So they remain hidden in the Dirac equation (1) unless the mass m is an operator built up of the matrices Γ_i^μ, $i = 2, 3, \ldots, N$. But then, m should be constructed in such a way that $[m, \Gamma_1^\mu] = 0$ and $[m, J^{\mu\nu}] = 0$ and also $[m, P] = 0$, where $P = \eta_N\Gamma_1^0\Gamma_2^0 \ldots \Gamma_N^0 P_{\rm orbit}$ (with $\eta_N = \varepsilon_N i^{\frac{1}{2}N(N-1)}$ and $\varepsilon_N^2 = 1$) is the total parity of $\psi(x)$; $P^\dagger = P = P^{-1}$. Thus, in general, m may depend on Γ_i^μ via $\Gamma_i^5\Gamma_j^5$ and

$$i\Gamma_i^\mu\Gamma_{j\mu} = i\Gamma_i^0\Gamma_j^0(1-\vec{A}_i\cdot\vec{A}_j) = i\Gamma_i^0\Gamma_j^0(1-\Gamma_i^5\Gamma_j^5\vec{\Sigma}_i\cdot\vec{\Sigma}_j),$$

$$\tfrac{1}{2}\,\Sigma_i^{\mu\nu}\Sigma_{j\mu\nu} = \vec{\Sigma}_i\cdot\vec{\Sigma}_j+\vec{A}_i\cdot\vec{A}_j = (1+\Gamma_i^5\Gamma_j^5)\vec{\Sigma}_i\cdot\vec{\Sigma}_j, \tag{23}$$

where $i, j = 2, 3, \ldots, N$ and $i \neq j$.

3. *Which is the physical Lorentz group?*

The assumption of $[m, J^{\mu\nu}] = 0$ telling us that m is a scalar under the Lorentz group generated by $J^{\mu\nu}$, is very natural and consistent with our construction of Γ_i^μ, $i = 1, 2, \ldots, N$. However, there is a priori another option where the physical Lorentz group corresponding to the theory of relativity is generated not by $J^{\mu\nu}$ but rather by

$$J^{\mu\nu}_{\rm visible} = L^{\mu\nu} + \tfrac{1}{2}\,\Sigma^{\mu\nu}, \tag{24}$$

with $\Sigma^{\mu\nu} = \Sigma_1^{\mu\nu}$. Then m, being invariant under this restricted Lorentz group, is not obliged by the theory of relativity to be invariant under the residual Lorentz group generated by

$$J^{\mu\nu}_{\text{hidden}} = J^{\mu\nu} - J^{\mu\nu}_{\text{visible}} = \sum_{i=2}^{N} \tfrac{1}{2}\Sigma_i^{\mu\nu}. \tag{25}$$

In this option, the additional spins 1/2 described by the operators $\frac{1}{2}\vec{\Sigma}_i$, $i = 2, 3, \ldots, N$, are not connected in any way with the physical space-time and so they should correspond to some other internal degrees of freedom [4]. So, if all matrices γ_i^μ, $i = 1, 2, \ldots, N$, are connected with the physical space-time (as it is consistent with our construction of the matrices Γ_i^μ, $i = 1, 2, \ldots, N$), this second option is logically excluded.

From Eq. (1) and its Hermitian conjugate we readily deduce that

$$\frac{\partial}{\partial x^\mu}(\psi^\dagger \Gamma_1^0 \Gamma_1^\mu \psi) = 0 \tag{26}$$

as well as

$$\frac{\partial}{\partial x^\mu}(\psi^\dagger \Gamma_1^0 \Gamma_2^0 \ldots \Gamma_N^0 \Gamma_1^\mu \psi)$$

$$= \frac{1}{i}(-1)^N \psi^\dagger [m, \Gamma_1^0 \Gamma_2^0 \ldots \Gamma_N^0]\psi + \frac{1}{i}[1+(-1)^N]\psi^\dagger \Gamma_1^0 \Gamma_2^0 \ldots \Gamma_N^0 m\psi. \tag{27}$$

Note that the assumption of $[m, P] = 0$ implies that $[m, \Gamma_1^0\Gamma_2^0 \ldots \Gamma_N^0] = 0$ (if m is $\vec{x}$-independent). Thus, for N odd Eq. (27) gives

$$\frac{\partial}{\partial x^\mu}(\psi^\dagger \Gamma_1^0 \eta_{N-1} \Gamma_2^0 \ldots \Gamma_N^0 \Gamma_1^\mu \psi) = 0, \tag{28}$$

where we introduced the previously defined phase factor η_{N-1} in order to have to do with a Hermitian operator. Note that $P_{\text{hidden}} = \eta_{N-1}\Gamma_2^0 \ldots \Gamma_N^0$ is the residual parity, while $P_{\text{visible}} = \eta_1 \Gamma_1^0 P_{\text{orbit}}$ gives us the restricted parity. Here, $P = P_{\text{visible}} P_{\text{hidden}}$ if $\varepsilon_N = \sqrt{(-1)^{N-1}}\varepsilon_1 \varepsilon_{N-1}$ where $\eta_1 = \varepsilon_1$, e.g. $\varepsilon_1 = 1$. $\varepsilon_{N-1} = 1$ and $\varepsilon_N = \sqrt{(-1)^{N-1}}$. In the representation of the type (5)–(8) (and so on) for Γ_i^μ, $i = 1, 2, \ldots, N$, we obtain in the case of N odd

$$P_{\text{hidden}} = \varepsilon_{N-1} \mathbf{1} \otimes \underbrace{\gamma^0 \otimes \ldots \otimes \gamma^0}_{N-1 \text{ times}}, \tag{29}$$

while

$$P_{\text{visible}} = \varepsilon_1 \gamma^0 \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{N-1 \text{ times}} P_{\text{orbit}}. \tag{30}$$

Thus, for N odd

$$P = \varepsilon_N \underbrace{\gamma^0 \otimes \gamma^0 \otimes \ldots \otimes \gamma^0}_{N \text{ times}} P_{\text{orbit}}. \tag{31}$$

We can see that the current appearing in Eq. (26) is always conserved, but it is not covariant under the Lorentz group generated by $J^{\mu\nu}$ though it is covariant under the restricted Lorentz group generated by $J^{\mu\nu}_{\text{visible}}$. In contrast, the current in Eq. (28) is covariant under the Lorentz group generated by $J^{\mu\nu}$ and for N odd it is conserved.

Thus, under the assumption that the physical Lorentz group corresponding to the theory of relativity is generated by $J^{\mu\nu}$, the probability interpretation of $\psi(x)$ is not permitted for N even (in particular, it is not permitted for $N = 2$ i.e., when Eq. (17) is equivalent to the Kähler equation). For N odd such an interpretation is allowed if

$$\psi^\dagger \eta_{N-1} \Gamma_2^0 \ldots \Gamma_N^0 \psi > 0, \tag{32}$$

where the spatial integral of the left-hand side is constant in time due to Eq. (28). Then for N odd the Dirac equation (1) may be interpreted as a wave equation describing the probability amplitude. Of course, for N even or odd the total spin of $\psi(x)$ is integer or half-integer, respectively (though in both cases the magnetically visible spin is 1/2, the rest of spin being magnetically hidden). However, the probability interpretation of $\psi(x)$ coincides with the half-integer total spin.

In order to guarantee the inequality (32) to be satisfied, we may impose the condition

$$\eta_{N-1} \Gamma_2^0 \ldots \Gamma_N^0 \psi = \psi \tag{33}$$

or $P_{\text{hidden}} \psi = \psi$, which for N odd is consistent with the wave equation (1) since then due to Eq. (28), P_{hidden} is a constant of motion. Of course, for N odd P_{hidden} commutes with the corresponding Hamiltonian

$$H = \Gamma_1^0 \vec{\Gamma}_1 \cdot (\vec{p} - e\vec{A}) + \Gamma_1^0 m + eA^0, \tag{34}$$

where m may depend on $\Gamma_i^5 \Gamma_j^5$, $i\Gamma_i^\mu \Gamma_{j\mu}$ and $\frac{1}{2} \Sigma_i^{\mu\nu} \Sigma_{j\mu\nu}$, $i, j = 2, 3, \ldots, N$.

Concluding, under the assumption that the physical Lorentz group corresponding to the theory of relativity is generated by $J^{\mu\nu}$, the Dirac equation (1) together with Eqs. (3) and (4) may be interpreted for $N = 1, 3, 5, \ldots$ as a sequence of wave equations, if only the sequence of conditions (33) is satisfied. For any $N = 1, 3, 5, \ldots$ the wave function $\psi(x)$ has visible spin 1/2 described by $\frac{1}{2} \vec{\Sigma}_1$ and hidden spin $0, 1, \ldots, \frac{1}{2}(N-1)$ described by $\sum_{i=2}^{N} \frac{1}{2} \vec{\Sigma}_i$. Thus, in particular, for $N = 1$ one gets one Dirac particle with hidden spin 0, while for $N = 3$ one obtains two Dirac particles with hidden spin 0 and two Dirac particles with hidden spin 1 since in each of these cases there are two eigenvalues of the operator $\Gamma_2^5 \Gamma_3^5$ commuting with $P_{\text{hidden}} = \eta_2 \Gamma_2^0 \Gamma_3^0$. They are equal to 1 and -1, the eigenvalue of P_{hidden} being kept equal to 1 by the condition (33).

4. Possible interpretation in terms of fermion generations

The question of the possible physical interpretation of our generalized Kähler equation given by formulae (1), (3) and (4) is entirely open. However, it would be attractive indeed to try to interpret the Dirac particles following from this equation as the experimentally observed generation replicas of basic leptons (ν_e or e^-) or quarks (u or d). For instance, in the case of the electron e^-, one may try to make the following tentative identification:

	N	visible spin	hidden spin	$\Gamma_2^5\Gamma_3^5$	$\vec{\Sigma}_2\cdot\vec{\Sigma}_3$
e^-	1	1/2	0	0	0
μ^-	3	1/2	0	-1	-3
τ^-	3	1/2	0	1	-3
$(?)_\mu^-$	3	1/2	1	-1	1
$(?)_\tau^-$	3	1/2	1	1	1

and so on (but with more and more quantum numbers involved) for $N = 5, 7, \ldots$, unless a principle terminating the sequence $N = 1, 3, 5, \ldots$ exists.

Note that in the case when hidden-spin triplets ($(?)_\mu^-$ and $(?)_\tau^-$) exist in the electron family there should also exist the corresponding hidden-spin triplets ($(?)_\mu^0$ and $(?)_\tau^0$) in the neutrino family. Then, assuming that the hidden spin is conserved separately, the decay rate for $W^- \to (?)_{\mu,\tau}^- + \overline{(?)_{\mu,\tau}^0}$ (if energetically allowed) should get no extra multiplicity factor and this is also true for $(?)_{\mu,\tau}^- \to W^- + (?)_{\mu,\tau}^0 \to \mu^- + \bar{\nu}_\mu + (?)_{\mu,\tau}^0$. This fact together with the actual experimental data on τ^- [5] would not exclude an alternative identification of τ^- with a hidden-spin triplet.

After the second quantization is performed, the generalized Kähler fermions can be formally included into the scheme of the standard model in place of the usual sequential fermions. In this case $m = 0$ and fermion masses arise from Higgs mechanism and radiative corrections.

For instance, in the case of leptons there are a weak isospin doublet

$$\psi_L = \begin{pmatrix} \psi_L^{(\nu)} \\ \psi_L^{(e)} \end{pmatrix} \tag{35}$$

and a singlet $\psi_R^{(e)}$, where

$$\psi^{(\nu)} = \begin{pmatrix} \psi_{N=1}^{(\nu)} \\ \psi_{N=3}^{(\nu)} \\ \cdots \end{pmatrix}, \qquad \psi^{(e)} = \begin{pmatrix} \psi_{N=1}^{(e)} \\ \psi_{N=3}^{(e)} \\ \cdots \end{pmatrix} \tag{36}$$

denote the quantized (multicomponent) fields of the neutrino and electron families and $\psi_{L,R} = \frac{1}{2}(1 \pm \Gamma_1^5)\psi$ are the chiral components. Then, for the sake of a phenomenological discussion, one may consider the following general lepton-higgs coupling:

$$\begin{aligned} &\psi_{LN=1}^\dagger \Gamma_1^0 h\phi\psi_{RN=1} + \text{h.c.} \\ &+ \psi_{LN=3}^\dagger \Gamma_1^0 (h_S\phi_S + h_P\phi_P\Gamma_2^5\Gamma_3^5 + h_V\phi_V i\Gamma_2^\mu\Gamma_{3\mu} \\ &+ h_A\phi_A i\Gamma_2^5\Gamma_3^5\Gamma_2^\mu\Gamma_{3\mu} + h_T\phi_T \tfrac{1}{2}\Sigma_2^{\mu\nu}\Sigma_{3\mu\nu})\psi_{RN=3} + \text{h.c.} + \ldots, \end{aligned} \tag{37}$$

where

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \phi_S = \begin{pmatrix} \phi_S^+ \\ \phi_S^0 \end{pmatrix}, \text{ etc.,} \tag{38}$$

are six higgs weak-isospin doubles, while h, h_S, etc., stand for coupling constants. In Eq. (37) the condition (33) was applied.

The coupling (37) is invariant under the total Lorentz group (if Eq. (33) is taken into account) and conserves the total parity. It may happen that in the experiment a special and simplified case of this general phenomenological coupling is realized.

Referring to our tentative identification of leptons and making use of Eqs. (23) we can rewrite Eq. (37) in the form

$$\begin{aligned} &(\overline{\nu}_e, \overline{e^-})_L h\phi e_R^- + \text{h.c.} \\ &+(\overline{\nu}_\mu, \overline{\mu^-})_L(h_S\phi_S - h_P\phi_P - 2h_V\phi_V + 2h_A\phi_A)\mu_R^- + \text{h.c.} \\ &+(\overline{\nu}_\mu, \overline{\tau^-})_L(h_S\phi_S + h_P\phi_P + 4h_V\phi_V + 4h_A\phi_A - 6h_T\phi_T)\tau_R^- + \text{h.c.} \\ &+(\overline{(?)^0_\mu}, \overline{(?)^-_\mu})_L(h_S\phi_S - h_P\phi_P + 2h_V\phi_V - 2h_A\phi_A)\,(?)^-_{\mu R} + \text{h.c.} \\ &+(\overline{(?)^0_\tau}, \overline{(?)^-_\tau})_L(h_S\phi_S + h_P\phi_P + 2h_T\phi_T)\,(?)^-_{\tau R} + \text{h.c.} + \dots . \end{aligned} \tag{39}$$

Here, for $N = 3$ the eigenvalue 1 of $P_{\text{hidden}} = i\Gamma_2^0\Gamma_3^0$ was used in accordance with the condition (33). In the tree approximation we obtain from Eq. (39) the following masses:

$$\begin{aligned} m_e &= hv, \\ m_\mu &= h_Sv_S - h_Pv_P - 2h_Vv_V + 2h_Av_A, \\ m_\tau &= h_Sv_S + h_Pv_P + 4h_Vv_V + 4h_Av_A - 6h_Tv_T, \\ m_{(?)_\mu^-} &= h_Sv_S - h_Pv_P + 2h_Vv_V - 2h_Av_A, \\ m_{(?)_\tau^-} &= h_Sv_S + h_Pv_P + 2h_Tv_T, \end{aligned} \tag{40}$$

where

$$v = \langle\phi^0\rangle, \quad v_S = \langle\phi_S^0\rangle, \text{ etc.,} \tag{41}$$

are vacuum expectation values. Of course, all masses in the neutrino family are zero.

For example, if $h_P = 0$ and $h_A = 0$, Eqs. (40) reduce to the form

$$\begin{aligned} m_e &= hv, \\ m_\mu &= h_Sv_S - 2h_Vv_V, \\ m_\tau &= h_Sv_S - (2-\varepsilon)h_Vv_V, \\ m_{(?)_\mu^-} &= h_Sv_S + 2h_Vv_V, \\ m_{(?)_\tau^-} &= h_Sv_S + \tfrac{1}{3}(6-\varepsilon)h_Vv_V, \end{aligned} \tag{42}$$

where ε is defined as

$$(6-\varepsilon)h_V v_V = 6h_T v_T. \tag{43}$$

Formulae (42) show that

$$\begin{aligned} m_\tau &= m_\mu + \varepsilon h_V v_V, \\ m_{(?)_\mu^-} &= \tfrac{1}{3}\, m_\mu + \tfrac{2}{3}\, m_\tau + \tfrac{2}{3}\,(6-\varepsilon)h_V v_V, \\ m_{(?)_\tau^-} &= \tfrac{2}{3}\, m_\mu + \tfrac{1}{3}\, m_\tau + \tfrac{2}{3}\,(6-\varepsilon)h_V v_V \end{aligned} \tag{44}$$

and predict the sum rules

$$\begin{aligned} m_\mu + 3m_{(?)_\mu^-} &= m_\tau + 3m_{(?)_\tau^-}, \\ (4-\varepsilon)m_\mu + \varepsilon m_{(?)_\mu^-} &= 4m_\tau. \end{aligned} \tag{45}$$

They also determine $h_S v_S$, $h_V v_V$ and ε in terms of three mass combinations

$$\begin{aligned} h_S v_S &= \tfrac{1}{2}\,(m_{(?)_\mu^-} + m_\mu) > 0, \\ h_V v_V &= \tfrac{1}{4}\,(m_{(?)_\mu^-} - m_\mu) > 0, \\ \varepsilon h_V v_V &= m_\tau - m_\mu > 0. \end{aligned} \tag{46}$$

The second sum rule (45) leads to

$$m_{(?)_\mu^-} = \frac{4}{\varepsilon}\,(m_\tau - m_\mu) + m_\mu \tag{47}$$

and then the first one implies

$$m_{(?)_\tau^-} = \frac{4}{\varepsilon}\,(m_\tau - m_\mu) - \tfrac{1}{3}\,(m_\tau - 4m_\mu). \tag{48}$$

Here, $\varepsilon > 0$ is expected to be small in order to give large $m_{(?)_\mu^-}$ and $m_{(?)_\tau^-}$ (then $h_S v_S > 0$ and $h_V v_V > 0$ are large).

At present we cannot say more about the Higgs mechanism for our generalized Kähler leptons since v, v_S, etc., as well as h, h_S. etc., are a priori arbitrary. The experimentum crucis for generalized Kähler leptons as discussed in this paper would be, of course, the existence of leptonic hidden-spin triplets.

Summarizing, if the generalized Kähler equation provided a correct description of the problem of fermion generations, then the Dirac square-root procedure would turn out equally adequate for the structure of electron family as it has been proved to be for the electron itself.

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[2] E. Kähler, *Rendiconti di Matematica* **21**, 425 (1962). It is worthwhile to mention that this equation was discovered much earlier by Ivanenko and Landau (*Z. Phys.* **48**, 341 (1928)), soon after the discovery of Dirac equation (P.A.M. Dirac, *Proc. R. Soc.* **117**, 610 (1928)). I am indebted to Andrzej Trautman for his calling my attention to this seldom noticed fact.
[3] W. Graf, *Ann. Inst. H. Poincaré* **A29**, 85 (1978).
[4] For $N = 2$ these other degrees of freedom appearing in Eq. (17) were interpreted in Ref. [1] as being responsible for four fermion generations. Cf. also W. Królikowski, *Acta Phys. Pol.* **B14**, 533 (1983); **B17**, 813 (1986).
[5] Cf. e.g. M. L. Perl, The tau and beyond: future research on heavy leptons, SLAC-PUB-4819, January 1989.