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# EXACT SOLUTION OF THE EQUATIONS OF NULL GEODESIC DEVIATION IN SCHWARZSCHILD SPACE-TIME

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The method of solving equations of geodesic deviation is presented. We apply this method for the case of null geodesic lines in the Schwarzschild space-time. The exact solution of the equation of geodesic deviation is given.

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## 1. Introduction

The relative motion of test particles is one of the most important sources of information about the gravitational field and space-time geometry. This motion is described by the equations of geodesic deviation. There are two methods of solving equations of geodesic deviation: one introduced by Bażański [1, 2, 3] which is based on equations of the Hamilton-Jacobi type and the other, used e.g. by Fuchs [4], which consists in application of the first integrals corresponding to a given Killing vector field.

In this paper, using first integrals method modified for the case of null geodesic lines, we obtain the exact solution of the equations of geodesic deviation in the Schwarzschild space-time. We consider the geodesic deviation between null geodesic lines, which physically means that we investigate the relative motion of photons. In Sect. 2 we give several indispensable informations about the equations of geodesic deviation. In Sect. 3 we discuss the method of solving these equations. Then in Sect. 4 we present the exact solution of the equations of null geodesic deviation. Moreover we consider the special case when one of the freely falling photons moves along a central geodesic line. It should be stressed that the solution found here cannot be obtained from the Fuchs solution [4] by simply taking the mass of the particle to be zero, but requires all the computations to be performed from the very beginning with the assumption that the world lines of particles are null geodesics.

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## 2. *The geodesic deviation vector*

Let $M$ be a Riemannian manifold endowed with a coordinate system $\{x^\alpha\}$ valid in a region $U \subset M$ and also endowed with a one-parameter family of geodesic lines. In such a coordinate system each geodesic $\Gamma_\omega$, parametrized by a parameter $\tau \in [a, b] = I \subset \mathbf{R}$ and numbered by $\omega \in [c, d] = J \subset \mathbf{R}$ is then described by a set of functions $\eta^\alpha(\tau, \omega): I \times J \to \mathbf{R}$ where $\eta^\alpha(\tau, \omega) = x^\alpha \circ \Gamma_\omega(\tau)$. There are two vectors connected with each pair $(\tau, \omega)$: a tangent vector and a geodesic deviation vector

$$u^\alpha(\tau, \omega) = \frac{\partial \eta^\alpha(\tau, \omega)}{\partial \tau}. \tag{2.1a}$$

$$\varrho^\alpha(\tau, \omega) = \frac{\partial \eta^\alpha(\tau, \omega)}{\partial \tau}. \tag{2.1b}$$

These vectors satisfy the following equation

$$\frac{D\varrho^\alpha}{d\tau} = u^\alpha{}_{;\beta}\varrho^\beta. \tag{2.2}$$

Thus, if we define now a congruence of geodesic lines we can obtain from the definition (2.1b) the geodesic deviation vector at each point of the space-time. It follows from (2.1) and (2.2) that the vector $\varrho^\alpha$ satisfies the equations of geodesic deviation

$$\frac{D^2\varrho^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta\varrho^\gamma u^\delta = 0. \tag{2.3}$$

It should be clearly underlined that equations (2.2) and (2.3) describe two geometrically different situations. Even to write down equation (2.2) one must know the vector field $u^\alpha$ in the whole region of space-time, while to formulate (2.3) it is sufficient to know $u^\alpha$ and $R^\alpha{}_{\beta\gamma\delta}$ along a geodesic line. Thus, solution of the equation (2.3) gives a geodesic deviation vector field only along the "basic" geodesic line and not in the whole region of space-time. The equation (2.3) follows directly from (2.1) and (2.2) whereas the proof of a sort of opposite statement is not trivial and has some geometric consequences [5].

## 3. *The first integral*

Let us consider a space-time with a group of isometries and a corresponding Killing vector field $\xi^\alpha$ which satisfies the Killing equation

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0. \tag{3.1}$$

One can easily prove that any Killing vector $\xi_\alpha$ satisfies the equations of geodesic deviation [6]

$$\frac{D^2\xi_\alpha}{d\tau^2} + R^\lambda{}_{\gamma\beta\alpha}u^\beta u^\gamma \xi_\lambda = 0. \tag{3.2}$$

Multiplying equation (2.3) by $\xi_\alpha$ and equation (3.1) by $\varrho^\alpha$ and subtracting one from the other, after straightforward calculations, one obtains the first integral of the equations of geodesic deviation

$$\frac{D\varrho^\alpha}{d\tau}\overset{(a)}{\xi}_\alpha - \varrho^\alpha \frac{D\overset{(a)}{\xi}_\alpha}{d\tau} = \Sigma_a, \tag{3.3}$$

where the index "$a$" numbers the Killing vectors. Let $u^\alpha$ be a vector tangent to a geodesic line $\Gamma_\omega$. Along this line the quantity

$$C_a(\omega) = \overset{(a)}{\xi}_\alpha u^\alpha \tag{3.4}$$

is conserved (see e.g. [7]). For a static spherically symmetric space-time the equations (3.4) are conservation laws of the energy and angular momentum of a particle moving along the geodesic $\Gamma_\omega$. By covariant differentiation in the direction of $\varrho^\alpha$, using (3.1) we obtain from (3.4) the equation (3.3) in which

$$\Sigma_a = \frac{DC_a(\omega)}{d\omega}. \tag{3.5}$$

From (3.5) it follows that the constants $\Sigma_\alpha$ appearing in (3.3) describe respectively a difference of energies and of angular momenta of two test particles, one taken along the basic geodesic line and the other along neighbouring geodesic, pointed at by vector $\varrho^\alpha$. To solve (2.3) we will need one additional equation. It is easy to show that the derivative of $u^\alpha \varrho_\alpha$ in the direction of $u^\alpha$ is equal to zero. Thus

$$u^\alpha \varrho_\alpha = C \tag{3.6}$$

where $C$ is constant along $\Gamma_\omega$. And now, using (3.6) and (3.3) we are able to solve the equations of null geodesic deviation.

## 4. *Exact solution of the equations of geodesic deviation*

The metric of the Schwarzschild space-time is of the form

$$ds^2 = A(r)dt^2 - A(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.1}$$

where

$$A(r) = 1 - \frac{r_g}{r}. \tag{4.2}$$

Without any loss of the generality of our consideration, we can choose as a basic geodesic line a geodesic lying in the plane $\theta = \frac{\pi}{2}$. In this case the tangent vector $u^\alpha$ is

$$u^\alpha = \left[\frac{\omega}{A(r)}, -[A(r)B(r)]^{1/2}, 0, \frac{J}{r^2}\right], \tag{4.3}$$

where

$$B(r) = \frac{\omega^2}{A(r)} - \frac{J^2}{r^2}. \tag{4.4}$$

The constants $\omega$, $J$ are respectively the frequency and the angular momentum of the photon. There are four Killing vectors in the Schwarzschild space-time:

$$\overset{(0)\alpha}{\xi} = (1, 0, 0, 0); \quad \overset{(1)\alpha}{\xi} = (0, 0, \sin\phi, \operatorname{ctg}\theta\cos\phi);$$

$$\overset{(3)\alpha}{\xi} = (0, 0, 0, 1); \quad \overset{(2)\alpha}{\xi} = (0, 0, -\cos\phi, \operatorname{ctg}\theta\sin\phi); \tag{4.5}$$

Inserting $\overset{(0)\alpha}{\xi}$, $\overset{(2)\alpha}{\xi}$, $\overset{(3)\alpha}{\xi}$, into (3.3) and computing the covariant derivatives we obtain three equations for four components of the vector $\varrho^\alpha$

$$\frac{d\varrho^t}{d\tau} = \frac{\Sigma_0}{A(r)}, \tag{4.6a}$$

$$\frac{d\varrho^\theta}{d\tau} + \varrho^\theta \frac{J}{r^2}\operatorname{tg}\phi = \frac{\Sigma_2}{r^2\cos\phi}, \tag{4.6b}$$

$$\frac{d\varrho^\phi}{d\tau} + \varrho^r \frac{2J}{r^3} = -\frac{\Sigma_3}{r^2}. \tag{4.6c}$$

The fourth equation follows from (3.6)

$$\omega\varrho^t + \left[\frac{B(r)}{A(r)}\right]^{\frac{1}{2}} \varrho^r - J\varrho^\phi = C, \tag{4.6d}$$

where $C$ is a constant. From (4.6d) and (4.6c) we get

$$\frac{d\varrho^\phi}{d\tau} + \frac{2J^2}{r^3}\left[\frac{A(r)}{B(r)}\right]^{\frac{1}{2}} \varrho^\phi = -\frac{\Sigma_3}{r^2} - \frac{2J}{r^3}\left[\frac{A(r)}{B(r)}\right]^{\frac{1}{2}} [C - \omega\varrho^t(\tau)]. \tag{4.7}$$

Solving (4.6a), (4.6b) and (4.7) with the initial condition $\varrho^\alpha(\tau_0) = (\varrho^t_c, \varrho^r_0, \varrho^\theta_0, \varrho^\phi_0)$ we obtain (using (4.6d)) the exact solution of the equations of null geodesic deviation.

$$\varrho^t(\tau) = \varrho^t_c + K_0(\tau, \tau_0) \tag{4.8a}$$

$$\varrho^\theta(\tau) = \exp[-K_2(\tau, \tau_0)]\left(\varrho^\theta_0 + \int_{\tau_0}^{\tau} \frac{\Sigma_2}{r^2\cos\phi} \exp[K_2(\tau, \tau_0)]d\tau\right), \tag{4.8b}$$

$$\varrho^\phi(\tau) = \exp[-K_3(\tau, \tau_0)]\left(\varrho^\phi_0 - \int_{\tau_0}^{\tau}\left[\frac{\Sigma_3}{r^2} + \frac{2J}{r^3}\left(\frac{A(r)}{B(r)}\right)^{\frac{1}{2}}\right.\right.$$

$$\times [C - \omega \varrho_0^t - \omega K_0(\tau, \tau_0)] \Bigg] \exp [K_3(\tau, \tau_0)] d\tau \Bigg), \tag{4.8c}$$

$$\varrho^r(\tau) = \left[\frac{A(r)}{B(r)}\right]^{\frac{1}{2}} [J\varrho^\phi(\tau) + C - \omega \varrho^t(\tau)], \tag{4.8d}$$

where

$$K_0(\tau, \tau_0) = \int\limits_{\tau_0}^{\tau} \frac{\Sigma_0}{A(r)} \, d\tau, \tag{4.9a}$$

$$K_2(\tau, \tau_0) = \int\limits_{\tau_0}^{\tau} \frac{J \operatorname{tg} \phi}{r^2} \, d\tau, \tag{4.9b}$$

$$K_3(\tau, \tau_0) = \int\limits_{\tau_0}^{\tau} \frac{2J^2}{r^3} \left[\frac{A(r)}{B(r)}\right]^{\frac{1}{2}} d\tau. \tag{4.9c}$$

We need some relations between the Schwarzschild coordinates $(t, r, \theta, \phi)$ and the afine parameter $\tau$ to calculate the integrals appearing in (4.8). One can obtain these relations from the integrals of the motion for a photon in the Schwarzschild space-time (see e.g. [8])

$$\frac{dt}{d\tau} = \frac{\omega}{A(r)}, \tag{4.10a}$$

$$\frac{d\phi}{d\tau} = \frac{J}{r^2}, \tag{4.10b}$$

$$\left(\frac{dr}{d\tau}\right)^2 = A(r)^2 \left(\frac{dt}{d\tau}\right)^2 - A(r) \left(\frac{d\phi}{d\tau}\right)^2. \tag{4.10c}$$

For the case of a central geodesic line $\left(\text{for which } \frac{d\phi}{d\tau} = 0\right)$, we choose $\phi = 0$, $J = 0$, $\frac{dr}{d\tau} = -\omega$. The integrals $K_0$, $K_2$, $K_3$, can then be evaluated explicitly. In this case the solution (4.8) takes the form

$$\varrho^t(r) = \varrho_c^t - \frac{\Sigma_0}{\omega} \left[r_g \ln \frac{r - r_g}{r_0 - r_g} + r - r_g\right], \tag{4.11a}$$

$$\varrho^r(r) = \frac{A(r)}{\omega} \left[\frac{\omega}{A(r_0)} \varrho_0^r + \Sigma_0 \left(r_g \ln \frac{r - r_g}{r_0 - r_g} + r - r_g\right)\right], \tag{4.11b}$$

$$\varrho^{\theta}(r) = \varrho_0^{\theta} - \frac{\Sigma_2}{\omega}\left(\frac{1}{r} - \frac{1}{r_0}\right), \tag{4.11c}$$

$$\varrho^{\phi}(r) = \varrho_0^{\phi} + \frac{\Sigma_3}{\omega}\left(\frac{1}{r} - \frac{1}{r_0}\right). \tag{4.11d}$$

The $\varrho^t(r)$ component of geodesic deviation vector has a singularity for $r \to r_g$. It corresponds to the fact that the time of free fall on the horizon is infinite in the Schwarzschild coordinates. Remaining components of $\varrho^\alpha$ and the invariant $\varrho^\alpha \varrho_\alpha$ are not singular for $r \to r_g$. The general solution given by (4.8) (as well as the special one given by (4.11)) allows one, by a suitable choice of initial conditions and constants appearing in (4.8) and (4.11), to describe various physical situations connected with relative motion of freely falling photons.

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