ON SOME CLASS OF SYSTEMS WITH SECOND CLASS CONSTRAINTS

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A class of dynamical systems with second class constraints which might be viewed as systems with first class constraints supplemented with gauge constraints (being a half of the set of original ones) is selected. Its quantization by path integral method, both, in a unitary and relativistic gauges is performed.

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It is well known that first class constraints when supplemented with gauge constraints become of the second class, and relevant system can be quantized by path integral method with the use of Fradkin-Senjanovic measure [1–9]. In canonical quantization difficulties due to factor ordering in Dirac brackets arise, and thus reversed trend started [10–12] in which one converts second class constraints into the first class. However, in general, this requires introducing additional degrees of freedom. Quantization follows then according to the BRST method [13, 14].

It seems that one should not omit rather special case when a system having second class constraints can be viewed as a system with first class constrains supplemented with gauge constraints being a half of the original ones. In this case one can quantize it, in a unitary gauge, without expanding its phase space. We would like to spell out the conditions under which this can happen indeed.

Let us consider a system with n bosonic degrees of freedom, for simplicity,

$$\Gamma = \{q^k, p_k, k = 1, \ldots, n\}. \tag{1}$$

Let a system has $2m < 2n$ of second class constraints

$$C_a(q, p) = 0, \quad \det \|\{C_a, C_b\}\| \big|_{C=0} \neq 0, \quad a, b = 1, \ldots, 2m. \tag{2}$$

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Since the matrix $C \equiv \|\{C_a, C_b\}\|$ is antisymmetric and nonsingular it can be brought [8], via a nonsingular transformation L, to the form

$$LCL^T = J = \begin{Vmatrix} 0 & , Q \\ -Q, & 0 \end{Vmatrix}, \quad Q = \begin{Vmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{Vmatrix}_m^1. \tag{3}$$

Upon introducing an equivalent set of constraints

$$\tilde{C}_a \equiv L_{ab} C_b \tag{4}$$

one finds

$$\tilde{C}|_{C=0} = \|\{\tilde{C}_a, \tilde{C}_b\}\| \,|_{C=0} = \|J_{ab}\|, \; a, b, = 1, \ldots, 2m. \tag{5}$$

Therefore, taking into account the structure of matrix J one gets for $C_\alpha \equiv \tilde{C}_\alpha$ and $G^\alpha \equiv G_{2m-\alpha+1}$, $\alpha = 1, \ldots, m$ the relations

$$\{C_\alpha, C_\beta\}\,|_{C=0} = 0, \{G^\alpha, G^\beta\}\,|_{C=0} = 0, \{C_\alpha, G^\beta\}\,|_{C=0} = \delta_{\alpha\beta},$$

$$\alpha, \beta = 1, \ldots, m. \tag{6}$$

In order to declare the constraints C_α, $\alpha = 1, \ldots, m$ the first class ones, and G^α, $\alpha = 1, \ldots, m$ as their gauge partners, one should verify first the consistency conditions

$$\dot{C}_a = \{C_a, H\} + \mu^p\{C_a, C_p\} = 0, C_p - \text{primary constraints} \quad a = 1, \ldots, 2m \tag{7}$$

from which the coefficients μ^p are determined. One finds for the equivalent constraints

$$\dot{C}_\alpha = \{C_\alpha, H\} + \mu^p\{C_\alpha, C_p\} = \{C_\alpha, H\} + v_{2m-\alpha+1} = \text{l.c.}(C_a),$$

$$\dot{G}^\alpha = \{G^\alpha, H\} + \mu^p\{G^\alpha, C_p\} = \{G^\alpha, H\} - v_\alpha = \text{l.c.}\,(C_a), \quad \alpha = 1, \ldots, m, \tag{8}$$

where $v_\alpha \equiv \mu^p L^{-1}{}_{p\alpha}$, and similarly for $v_{2m-\alpha+1}$, and l.c. (C_a) stands for a "linear combination of C_a, $a = 1, \ldots, 2m$". One sees that in order to ensure the conditions

$$\{H, C_\alpha\}|_{C=0} = 0, \alpha = 1, \ldots, m \tag{9}$$

one must require the vanishing of all the coefficients

$$v_{2m-\alpha+1}|_{C=0} = \mu^p L^{-1}_{p2m-\alpha+1|C=0} = 0, \quad \alpha = 1, \ldots, m. \tag{10}$$

The conditions (8) then will determine all the remaining coefficients v_α, $\alpha = 1, \ldots, m$ as it is for a system with first class constraints (C_α) supplemented with the unitary gauge constraints (G^α). To get precisely this situation one assumes further that

$$\{C_\alpha, C_\beta\} = C_\gamma U^\gamma_{\alpha\beta}, \quad \{H, C_\alpha\} = C_\beta V^\beta_\alpha, \quad \alpha, \beta, \gamma = 1, \ldots, m, \tag{11}$$

which are stronger requirements than (5) and (8). Quantization of the system can readily be performed without introducing any new variables. One gets for the generating functional

$$Z = \int \prod_t dqdp \prod_{\alpha=1}^{m} \delta(C_\alpha)\delta(G^\alpha) \exp\{i \int dt(p\dot{q}-H)\}$$
$$= \int \prod_t dqdpS(C_a) \exp\{i \int dt(p\dot{q}-H)\}. \tag{12}$$

This comes about since the Fradkin-Senjanovic measure

$$S(C_a) \equiv |\det \|\{C_a, C_b\}\||^{1/2} \prod_{a=1}^{2m} \delta(C_a) \tag{13}$$

is invariant, both, under the renumeration of constraints, and under the replacement by equivalent constraints [16]

$$S(C_a) = S(\tilde{C}_a) = S(C_{p(a)}),\ p \in S_{2m}. \tag{14}$$

In relativistic gauges $G^\alpha = -\dot{\kappa}^\alpha + \chi^\alpha(q, p, \kappa, \pi)$, $\alpha = 1, \ldots, m$, where κ^α and π_α are Lagrange multipliers for the constraints C_α, χ^α respectively, one has to expand phase space adding $4m$ ghost variables $(Q^a) = (\Phi^\alpha, P^\alpha)$, $(P_a) = (\bar{P}_\alpha, \bar{\Phi}_\alpha)$, $a = 1, \ldots, 2m$, to the action. Fermionic degrees of freedom require to consider Γ as a Grassmann algebra, and to replace Poisson brackets and determinants with their super-generalizations [17–20].

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