RENORMALIZATION OF THE SCALAR QUANTUM FIELD THEORY IN THE OPTIMIZED EXPANSION*

BY ANNA OKOPIŃSKA**

Institute of Physics, Warsaw University, Białystok Branch

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The scalar quantum field theory with $\lambda\Phi^4+g\Phi^6$ interaction in four dimensional space-time is studied to the first order of the optimized expansion. The results are very similar as for $\lambda\Phi^4$ theory studied in the same approximation. Renormalization can be explicitly performed; however, the renormalized theory is noninteracting or precarious. The autonomous phase of the theory, suggested recently by variational methods, is nonrenormalizable in the considered approximation.

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We consider a theory of the real scalar field in $n = 4$ dimensional Euclidean space-time with a classical action given by

$$S[\Phi] = \int d^nx[\tfrac{1}{2}\,\Phi(x)\,(-\partial^2+m^2)\Phi(x)+P(\Phi)], \tag{1}$$

where the self-interaction $P(\Phi) = \lambda\Phi^4+g\Phi^6$. The full information on Quantum Field Theory (QFT) is contained in the effective action $\Gamma(\varphi)$. The quantum action, as a functional of the background field φ, generates one-particle-irreducible (1PI) Green's functions [1]. Renormalization of the theory consists in reparametrization of the effective action in terms of renormalized fields

$$\varphi_{\rm R}(x) = Z^{-1/2}\varphi(x) \tag{2}$$

and renormalized parameters instead of bare ones. It is convenient to define a renormalized mass and coupling constants as 1PI vertices at vanishing external momenta. Such vertices can be obtained as derivatives of the effective potential (EP), defined as

$$V(\varphi) = -\frac{\Gamma[\varphi]\,|_{\varphi={\rm const}}}{\int d^nx}\,. \tag{3}$$

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** Mailing address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, 00-681 Warszawa, Poland.

Usually, the effective action is calculated in the loop expansion [1] and used to derive proper vertices (if the vacuum expectation value of the quantum fields vanishes, it is equivalent to the Feynman perturbation theory). The scalar QFT is perturbatively renormalizable, only if $P(\Phi) = \lambda\Phi^4$. Higher powers lead to nonrenormalizability, which shows up already in one loop approximation. However, a conjecture that perturbatively nonrenormalizable theories are meaningful when studied nonperturbatively has been considered since a long time [2]. A rigorous realization of the program "to renormalize nonrenormalizable" has been done for Fermi theory in $2+\varepsilon$ dimensions [3]. Many theories have been claimed equivalent to renormalizable ones [4], among them a scalar theory with Φ^6 interaction [5]. Further doubts on the reliability of the perturbative approach, which gives the renormalized parameters in $\lambda\Phi^4$ theory completely arbitrary, were raised after almost rigorous proofs that the renormalized coupling constant vanishes [6]. It is interesting to note that some nonperturbative methods point on triviality — the problem of "zero charge" in the leading logarithmic approximation [7], in the large N expansion [8], and in the Gaussian Effective Potential (GEP) approach [9]. In these approximations it is possible to escape triviality, choosing a bare coupling negative and infinitesimal, which makes the theory asymptotically free. This phase has been called precarious, since it becomes stable only after sending cutoff to infinity. The result hides an intrinsic instability, which shows up, if the temperature effects [8], or corrections to the GEP are taken into account [10]. Recently, an autonomous phase of $\lambda\Phi^4$ theory, which does not appear in the large N expansion, has been found in the GEP approach [11]. For an infinitesimal, but positive bare coupling, an infinite field renormalization has been performed in such a way that the renormalized GEP is finite [11] and stable [10]. The autonomous phase has been also found in the renormalized Φ^6 theory [12]. However, as for $\lambda\Phi^4$ theory, only the finiteness of the EP has been shown, which guarantees only that 1PI vertices at vanishing momenta are finite. We would like to point out that the theory is renormalized in some approximation, if the vertices at nonvanishing external momenta remain finite. Such vertices can be obtained only from the effective action. In the variational approach the effective action could be calculated using time dependent variational principle [13] for functional Schrödinger equation. Such an approach was applied to $\lambda\Phi^4$ theory [14], but corrections have been calculated only to the EP; therefore the discussion of the autonomous phase was still incomplete. Moreover, the Schrödinger representation is not convenient to study renormalization, since it needs the rigorous construction given recently by Symanzik [15]. In the Lagrangian formulation renormalization procedure becomes more clear; therefore we propose to calculate the effective action in a covariant way, using the optimized expansion (OE) [16]. The expansion for a scalar theory with polynomial interaction can be generated, modifying a clasical action (1) to

$$S_\varepsilon[\Phi] = \int d^n x[\tfrac{1}{2}\Phi[-\partial^2+\Omega^2]\Phi+\varepsilon[\tfrac{1}{2}(m^2-\Omega^2)\Phi^2+P(\Phi)]] \tag{4}$$

and applying the steepest descent method to the path integral representation for Green's function generating functional. The effective action is calculated as a series in the formal parameter ε, which is set equal to one at the end. The exact effective action does not depend on an arbitrary field $\Omega^2(x)$, but to any finite order the dependence on Ω^2 appears. The OE consists in choosing Ω^2 as the stationary point of the given order approximant for the effective action, which makes the approximation as insensitive as possible to small variations of the unphysical field.

The k-th order effective action can be obtained as a sum of 1PI vacuum Feynman diagrams with k interactions in the theory with propagator and vertices read off from the modified action (4) upon shifting a field $\Phi \to \Phi + \varphi$. We obtain the inverse propagator

$$G^{-1}(x, y) = (-\partial^2 + \Omega^2(x))\delta(x, y), \tag{5}$$

two-particle vertex $[\Omega^2 - m^2 - P''(\varphi)]$ and r-particle vertex given by the r-th derivative $P^{(r)}(\varphi)$. In the theory with $P(\Phi) = \lambda\Phi^4 + g\Phi^6$ interaction, the first order effective action is given by diagrams shown in Fig. 1 and we have

$$\begin{aligned}\Gamma(\varphi, \Omega^2) = \textstyle\int d^n x\{ &-\varphi(x)(-\partial^2 + m^2)\varphi(x) - \lambda\varphi^4(x) - g\varphi^6(x) \\ -\tfrac{1}{2}\textstyle\int d^n y[\delta(x-y)\,\mathrm{Ln}\,[G^{-1}(x, y)]] &+ \tfrac{1}{2}[\Omega^2(x) - m^2 - 12\lambda\varphi^2(x) - 30g\varphi^4(x)]G(x, x) \\ &-[3\lambda + 45g\varphi^2(x)]G^2(x, x) - 15gG^3(x, x)\}.\end{aligned} \tag{6}$$

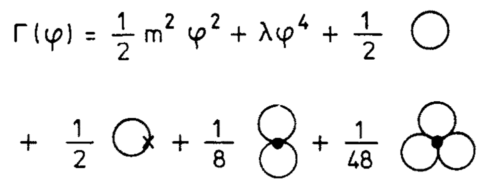

Fig. 1. The Feynman diagrams, which contribute to the effective action in $\lambda\Phi^4 + g\Phi^6$ QFT to the first order in the optimized expansion

The stationarity requirement

$$\begin{aligned}\frac{\delta\Gamma}{\delta\Omega^2(y)} = \int d^n x\{&\tfrac{1}{2}[\Omega^2(x) - m^2 - 12\lambda\varphi^2(x) - 30g\varphi^4(x)]G^2(x, y) \\ &-(6\lambda + 90g\varphi^2(x))G(x, x)G^2(x, y) - 45gG^2(x, x)G^2(x, y)\} = 0\end{aligned} \tag{7}$$

will be satisfied, if we choose $\Omega^2(x)$ to fulfil the gap equation

$$\Omega^2(x) - m^2 - 12\lambda\varphi^2(x) - 30g\varphi^4(x) - (12\lambda + 180g\varphi^2(x))G(x, x) - 90gG^2(x, x) = 0. \tag{8}$$

As in the $\lambda\varphi^4$ theory, this result is a covariant form of the effective action obtained in the time dependent Hartree approximation [13] and coincides with the result obtained by explicit summation of diagrams with nonoverlapping divergencies in Dyson-Schwinger equations [17].

It will be convenient to introduce a notation

$$I_1(\Omega) = \tfrac{1}{2}\int \frac{d^n k}{(2\pi)^n} \ln(k^2 + \Omega^2) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{\omega_k}{2} \tag{9a}$$

$$I_0(\Omega) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2+\Omega^2} = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{2\omega_k} \tag{9b}$$

$$I_{-1}(\Omega, p) = 2\int \frac{d^n k}{(2\pi)^n} \frac{1}{((p-k)^2+\Omega^2)(k^2+\Omega^2)} \tag{9c}$$

$$I_{-1}(\Omega) = I_{-1}(\Omega, 0) = 2\int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2+\Omega^2)^2} = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{2\omega_k^3}, \tag{9d}$$

where $\nu = n-1$ and $\omega_k = \Omega^2+k_1^2+k_2^2+ \dots +k_\nu^2$. The integrals $I_N(\Omega)$ in four dimensional space-time are divergent and are understood as regularized with momentum cutoff; the relations between them are quoted in Table I.

TABLE I

Relations between I_N integrals in four dimensions: $x = \Omega^2/m_R^2$

$$I_1(\Omega)-I_1(m_R) = \tfrac{1}{2}(\Omega^2-m_R^2)I_0(m_R)-\tfrac{1}{8}(\Omega^2-m_R^2)^2 I_{-1}(m_R)-\frac{1}{32\pi^2} m_R^4 L_3(x),$$

$$I_0(\Omega)-I_0(m_R) = (m_R^2-\Omega^2)I_{-1}(m_R)+\frac{1}{16\pi^2} m_R^2 L_2(x),$$

$$I_{-1}(\Omega)-I_{-1}(m_R) = \frac{1}{8\pi^2} L_1(x),$$

where

$$L_1(x) = \ln(x),$$
$$L_2(x) = x\ln(x)-(x-1),$$
$$L_3(x) = \tfrac{1}{2}x^2\ln(x)-\tfrac{1}{2}(x-1)-\tfrac{3}{4}(x-1)^2.$$

Taking the effective action for constant φ, the EP is calculated to be

$$V(\varphi, \Omega) = \tfrac{1}{2}m^2\varphi^2+\lambda\varphi^4+g\varphi^6+I_1(\Omega)+\tfrac{1}{2}(m^2+12\lambda\varphi^2-\Omega^2)I_0(\Omega)$$
$$+3\lambda[I_0(\Omega)]^2+15g\{\varphi^4 I_0(\Omega)+3\varphi^2[I_0(\Omega)]^2+[I_0(\Omega)]^3\} \tag{10}$$

with Ω^2 fixed by the gap equation, which becomes

$$\Omega^2-m^2-12\lambda[\varphi^2+I_0(\Omega)]-30g\{\varphi^4+6\varphi^2 I_0(\Omega)+3[I_0(\Omega)]^2\} = 0. \tag{11}$$

The result coincides with the GEP obtained in the variational approach [18]. This can be used to argue that objections [19] to renormalization of the GEP are unjustified. In the Lagrangian approach, used in the OE, the standard renormalization [20] is performed. One avoids problems of the renormalization in the Schrödinger representation [15], which could make results of the variational approach ill-defined [19].

The covariant formulation of the OE enables us to study the QFT at finite temperature in the same approximation. The EP at temperature T in the imaginary time formalism of statistical field theory is obtained by compactification of the time dimension with a lenght $1/T$ [21]. Therefore, the finite temperature EP in the given order of the OE [22] can be obtained replacing momentum integrals by their finite temperature counterparts; for the integrals (9a and b) we have

$$I_1^T(\Omega) = \tfrac{1}{2}\,T\sum_j \int \frac{d^\nu k}{(2\pi)^\nu}\ln(k^2+\Omega^2)$$

$$= \int \frac{d^\nu k}{(2\pi)^\nu}\left[\frac{\omega_k}{2} + T\ln(1-e^{-\omega_k/T})\right] = I_1(\Omega)+J_1^T(\Omega) \tag{12a}$$

$$I_0^T(\Omega) = T\sum_j \int \frac{d^\nu k}{(2\pi)^\nu}\frac{1}{k^2+\Omega^2} = \int \frac{d^\nu k}{(2\pi)^\nu}\frac{1}{\omega_k}\left[\tfrac{1}{2}+\frac{1}{e^{\omega_k/T}-1}\right] = I_0(\Omega)+J_0^T(\Omega),$$

where $k_0 = 2\pi jT$. $J_1^T(\Omega)$ and $J_0^T(\Omega)$ are finite in any space dimensions ν and vanish for $T = 0$. The first order OE result

$$V^T(\varphi,\Omega) = \tfrac{1}{2}\,m^2\varphi^2+\lambda\varphi^4+g\varphi^6+I_1^T(\Omega)+\tfrac{1}{2}\,(m^2+12\lambda\varphi^2-\Omega^2)I_0^T(\Omega)$$
$$+3\lambda[I_0^T(\Omega)]^2+15g\{\varphi^4 I_0^T(\Omega)+3\varphi^2[I_0^T(\Omega)]^2+[I_0^T(\Omega)]^3\} \tag{13}$$

with Ω^2 fixed by

$$\Omega^2-m^2-12\lambda[\varphi^2+I_0^T(\Omega)]-30g\{\varphi^4+6\varphi^2 I_0^T(\Omega)+3[I_0^T(\Omega)]^2\} = 0 \tag{14}$$

can be regarded as a justification of the finite temperature GEP, obtained by Roditi [23]. A justification of such replacing in the Hamiltonian approach has been done only recently, but the derivation is quite lenghty [24].

Now, we will discuss renormalization of the theory in the first order of the OE. The effective action (6), expressed in terms of renormalized fields (2) becomes

$$\Gamma(\varphi_R,\Omega^2) = \int d^n x\{-Z\varphi_R(x)(-\partial^2+m^2)\varphi_R(x)-\lambda Z^2\varphi_R^4(x)-gZ^3\varphi_R^6(x)$$
$$-\tfrac{1}{2}\int d^n y[\delta(x-y)\,\mathrm{Ln}\,[G^{-1}(x,y)]]+\tfrac{1}{2}\,[\Omega^2(x)-m^2-12\lambda Z\varphi_R^2(x)-30gZ^2\varphi_R^4(x)]G(x,x)$$
$$-[3\lambda+45gZ\varphi^2(x)]G^2(x,x)-15gG^3(x,x)\}, \tag{15}$$

and the gap equation (8) turns into

$$\Omega^2(x)-m^2-12\lambda Z\varphi_R^2(x)-30gZ^2\varphi_R^4(x)-(12\lambda+180gZ\varphi_R^2(x))G(x,x)-90gG^2(x,x) = 0. \tag{16}$$

This implicit expression enables us to calculate functional derivatives of the effective action with respect to the renormalized background field. When taken at constant field and Fourier transformed, they give 1PI vertices for arbitrary external momenta, expressed in terms of ordinary integrals in momentum space.

The true vacuum expectation value of the renormalized field φ_R should be constant and has to make the effective action stationary. To the first order OE, after using the gap equation (16), we obtain

$$\frac{\delta\Gamma}{\delta\varphi_R(x)} = -ZG^{-1}(x,x)\varphi_R(x)+8Z^2\varphi_R^3(x)\,[\lambda+3gZ\varphi_R^2(x)+15gG(x,x)]. \tag{17}$$

Therefore, the vacuum expectation value φ_R either vanishes and the reflection symmetry is unbroken, or φ_R fulfils equality

$$\Omega^2 = 8Z^2\varphi_R^2[\lambda+3gZ\varphi_R^2+15gG(x, x)] \tag{18}$$

and the symmetry is broken spontaneously.

The second derivative of the effective action becomes equal to

$$\frac{\delta^2\Gamma}{\delta\varphi_R(x)\delta\varphi_R(y)} = -ZG^{-1}(x, y)+12Z\varphi_R(x)\,[\lambda+5gZ\varphi_R^2(x)$$
$$+15gG(x, x)]\int d^n z G^2(x, z)\frac{\delta\Omega^2(z)}{d\varphi_R(y)}, \tag{19}$$

where $\dfrac{\delta\Omega^2(x)}{\delta\varphi_R(y)}$ fulfils an equality obtained by differentiation of the gap equation:

$$\frac{\delta\Omega^2(x)}{\delta\varphi_R(y)} - 24Z\varphi_R(y)\,[\lambda+5gZ\varphi_R^2(x)+15gG(x, x)]\delta(x, y)$$
$$-24[\lambda+15gZ\varphi_R^2(x)+15gG(x, x)]\int d^n z G^2(x, z)\frac{\delta\Omega^2(z)}{\delta\varphi_R(y)} = 0. \tag{20}$$

For constant φ_R the gap equation is satisfied by the constant Ω and the Fourier transform of the self-energy (19) is found to be

$$\Gamma^2(p, \varphi_R) = \int d^n x e^{ip(x-y)}\left.\frac{\delta^2\Gamma}{\delta\varphi_R(x)\delta\varphi_R(y)}\right|_{\varphi_R}$$
$$= -Z\left[p^2+\Omega^2-\frac{144Z\varphi_R^2(\lambda+5gZ\varphi_R^2+15gI_0)^2 I_{-1}(\Omega, p)}{[1+6(\lambda+15gZ\varphi_R^2+15gI_0)]I_{-1}(\Omega, p)}\right]. \tag{21}$$

If the symmetry in unbroken, the self-energy becomes

$$\Gamma^2(p) = -Z[p^2+\Omega^2|_{\varphi_R=0}], \tag{22}$$

and is finite at arbitrary external momentum, only if the field renormalization constant is finite. It is convenient to take $\Omega^2|_{\varphi_R=0}$ as the renormalized mass.

If the symmetry is broken, expanding the self-energy (21) into Taylor series for sall momenta and taking (18) into account, we obtain

$$\Gamma^2(p, \varphi_R) = -Z\left\{\Omega^2\left[1-\frac{18[\lambda+5gZ\varphi_R^2+15gI_0(\Omega)]^2 I_{-1}(\Omega)}{[1+6(\lambda+15gZ\varphi_R^2+15gI_0(\Omega))I_{-1}(\Omega)]}\right]\right.$$
$$\left.+\left[1+\frac{3[\lambda+5gZ\varphi_R^2+15gI_0(\Omega)]^2}{8\pi^2[1+6(\lambda+15gZ\varphi_R^2+15gI_0(\Omega))I_{-1}(\Omega)]^2}\right]p^2+O(p^4)\right\}. \tag{23}$$

The autonomous phase has been obtained [12], choosing renormalization flows

$$Z = I_{-1}(\mu) \tag{24a}$$

$$\lambda = \frac{1}{12 I_{-1}(\mu)} \tag{24b}$$

$$m^2 = \frac{3m_0^2}{2I_{-1}(\mu)} \tag{24c}$$

$$g = \frac{1}{\mu_0^2 I_{-1}^3(\mu)}, \tag{24d}$$

where m_0, μ and μ_0 are finite parameters with mass dimensions. This choice makes the EP finite, ensuring the finiteness of 1PI vertices at vanishing momenta. However, in the self-energy (23) only the momentum independent term becomes finite, but the coefficient at p^2 remains infinite. It is not possible to make both terms finite with infinite field renormalization. In $\lambda\Phi^4$ theory ($g = 0$) the conclusion remains the same and infinities appear in the self-energy at non-vanishing momentum in the autonomous phase, as observed by Kovner and Rosenstein [17]. The autonomous phase has been also criticized from different points of view by Soto [25]. Our result does not prove that the autonomous theory cannot be renormalized at all, it shows only that this is impossible in the first order OE, which is a covariant extension of the GEP on space-time dependent fields. We cannot exclude a possibility that renormalization could be done in a different approach; however, it is crucial to show that 1PI vertices remain finite at non-vanishing momenta.

In the discussed approximation the only way of renormalization is to take a field renormalization Z finite. In this case it is sufficient to find bare mass and coupling, which make the EP finite. It has been shown [9] that in the $\lambda\Phi^4$ theory reparametrization of the GEP for finite Z can be done in two ways, resulting in the renormalized theory which is either trivial or precarious. Now, we will show that in the first order OE the $\lambda\Phi^4 + g\Phi^6$ theory renormalized with finite Z is very similar. For simplicity, we take $Z = 1$ ($\varphi(x) = \varphi_R(x)$) and define renormalized parameters as derivatives of the EP at $\varphi = 0$. The second derivative

$$\left.\frac{d^2V}{d\varphi^2}\right|_{\varphi=0} = 2\left.\frac{dV}{d(\varphi^2)}\right|_{\varphi=0} = 2\left.\frac{\partial V}{\partial(\varphi^2)}\right|_{\varphi=0} = m^2 + 12\lambda I_0(\Omega) + 90g[I_0(\Omega)]^2\Big|_{\varphi=0} = \Omega^2\Big|_{\varphi=0} \tag{25}$$

can be used to define the renormalized mass as

$$m_R^2 = \left.\frac{d^2V}{d\varphi^2}\right|_{\varphi=0} = m^2 + 12\lambda I_0(m_R) + 90g[I_0(m_R)]^2. \tag{26}$$

To obtain higher derivatives, the implicit dependence $\Omega^2(\varphi)$ has to be taken into account. Differentiation of the gap equation (11) gives

$$\left.\frac{d\Omega^2}{d\varphi^2}\right|_{\varphi=0} = \frac{12\lambda_r}{1 + 6\lambda_r I_{-1}(m_R)} \tag{27}$$

and

$$\left.\frac{d^2\Omega^2}{d(\varphi^2)^2}\right|_{\varphi=0} = \frac{4}{[1+6\lambda_r I_{-1}(m_R)]I_{-1}(m_R)}\left\{15gI_{-1}(m_R) - \left[\frac{\lambda_r}{4\pi^2 m_R^2} + 15gI_{-1}^2(m_R)\right]\right.$$

$$\left.\times\frac{36\lambda_r}{1+6\lambda_r I_{-1}(m_R)} + \left[45gI_{-1}^3(m_R) + \frac{1+9\lambda_r I_{-1}(m_R)}{4\pi^2 m_R^2}\right]\frac{36\lambda_r^2}{[1+6\lambda_r I_{-1}(m_R)]^2}\right\} \tag{28}$$

where

$$\lambda_r = \lambda + 15gI_0(m_R). \tag{29}$$

After some calculation we obtain the renormalized four-vertex

$$\lambda_R = \frac{1}{4!}\left.\frac{d^4V}{d\varphi^4}\right|_{\varphi=0} = \frac{1}{2}\left.\frac{d^2V}{d(\varphi^2)^2}\right|_{\varphi=0} = \frac{1-12\lambda_r I_{-1}(m_R)}{1+6\lambda_r I_{-1}(m_R)}, \tag{30}$$

and the renormalized six-vertex

$$g_R = \frac{1}{6!}\left.\frac{d^6V}{d\varphi^6}\right|_{\varphi=0} = \frac{1}{6}\left.\frac{d^3V}{d(\varphi^2)^3}\right|_{\varphi=0}$$

$$= g(1-90g_r+1620g_r^2-3240g_r^3) + \frac{9\lambda_r^3}{\pi^2[1+6\lambda_r I_{-1}(m_R)]^3 m_R^2}, \tag{31}$$

where

$$g_r = \frac{\lambda_r I_{-1}(m_R)}{1+6\lambda_r I_{-1}(m_R)}. \tag{32}$$

Eq. (30) has two solutions, which in the limit of an infinite cutoff become

$$\lambda_r = -\tfrac{1}{2}\lambda_R, \tag{33}$$

or

$$\lambda_r = -\frac{1}{6I_{-1}(m_R)} - \frac{1}{12\lambda_R[I_{-1}(m_R)]^2}. \tag{34}$$

Similarily as in $\lambda\varphi^4$ theory [22], it can be shown that the finite solution (33) is pathological. It gives finite EP, suggesting that the renormalized theory is interacting. However, the EP at finite temperature bears no sign of interaction, since a temperature contribution does not depend on the background field and is the same as for free particle with a mass m_R.

If the infinitesimal solution (34) for the bare four-vertex is taken, the bare six-vertex has to be equal

$$g = -\frac{8G}{15[I_{-1}(m_R)]^3}, \tag{35}$$

where

$$G = \frac{g_R}{64\lambda_R^3} - \frac{1}{192\pi^2 m_R^2}. \tag{36}$$

The precarious EP can be written as

$$V(\varphi, \Omega^2) = G(\Omega^2 - m_R^2)^3 + \frac{1}{32\pi^2} m_R^4 L_3(x) + \tfrac{1}{2}\Omega^2\varphi^2 - \frac{1}{16\lambda_R}(m_R^2 - \Omega^2)^2, \tag{37}$$

with Ω fulfilling the gap equation

$$G(\Omega^2 - m_R^2)^2 + \frac{1}{16\pi^2} m_R^2 L_2(x) + \tfrac{1}{2}\varphi^2 - \frac{1}{8\lambda_R}(\Omega^2 - m_R^2) = 0, \tag{38}$$

where $x = \Omega^2/m_R^2$ and the relations from Table I have been used.

The gap equation has solutions for arbitrary φ, if $G < 0$. However, the theory is unstable, since the absolute minimum $V \to -\infty$ is at the end point $\Omega \to \infty$. A nontrivial theory is obtained only for $G \geqslant 0$. The gap equation has a real solution only for $\varphi < \varphi_{max}(\lambda_R, G)$ and the EP is determined only in this range. For $G = 0$ we obtain the precarious $\lambda\Phi^4$ theory, discussed by Stevenson [9]. For comparison, discussing the precarious $\lambda\Phi^4 + g\Phi^6$ theory, we take the same value of the parameter $\kappa = -4\pi^2/\lambda_R = 1.25$. For $\lambda_R < 0$ according to (36) the bare six-coupling g is negative, i.e. the theory is asymptotically free. For $G = 0.1$ the range of definiteness of the EP shrinks to the point $\varphi = 0$ and the precarious theory disappears, but even for smaller G it becomes metastable at $G_{cr} = 0.0012$, since the value of the EP at the endpoint $\Omega = 0$ becomes lower than the value obtained with the use of the gap equation. In the range of G in which the renormalized EP exists it has a minimum at origin, therefore the theory is symmetric.

At finite temperature the renormalized EP in the first order OE is given by

$$V^T(\varphi, \Omega^2) = G(\Omega^2 - m_R^2)^3 + \frac{1}{32\pi^2} m_R^4 L_3(x) + \tfrac{1}{2}\Omega^2\varphi^2 - \frac{1}{16\lambda_R}(m_R^2 - \Omega^2)^2 + J_1(\Omega), \tag{39}$$

where Ω^2 satisfies temperature dependent gap equation

$$G(\Omega^2 - m_R^2)^2 + \frac{1}{16\pi^2} m_R^2 L_2(x) + \tfrac{1}{2}\varphi^2 - \frac{1}{8\lambda_R}(\Omega^2 - m_R^2) - \tfrac{1}{2} J_0(\Omega) = 0. \tag{40}$$

In Fig. 2 we compare the EP in the units of m_R at temperature $T = 0$ and $T = 0.3$ for $\kappa = 1.25$ and $G = 0.001$. The figure can be compared with Fig. 2 of Ref. [9] obtained in $\lambda\Phi^4$ theory ($G = 0$). At $T = 0$ the critical value $\varphi_{max} = 0.041$, beyond which there is no solution to the gap equation, is smaller than $\varphi_{max} = 0.058$, obtained for $G = 0$. The temperature behavior is qualitatively the same as in $\lambda\Phi^4$ theory [22]. For increasing temperature φ_{max} decreases and becomes equal to zero for $T_2 = 0.38$. The critical temperature is even lower, since for $T_1 = 0.21$ the value of the EP at the end point becomes lower than the value obtained using the gap equation. These values have to be compared with $T_2 = 0.41$ and $T_1 = 0.35$, obtained for $G = 0$ [22]. The second derivative of the EP, which plays a role of the order parameter, jumps from a finite value to zero at critical temperature T_1 and the phase transition is of the first order. Beyond T_1 the EP is a constant potential of massless free theory. However, the conclusion is drawn using the endpoint behavior of the EP when the OE becomes unreliable. The lack of the solution to the gap equation, can be only a signal that the method breaks down for $\varphi > \varphi_{\max}$, and is unapplicable beyond

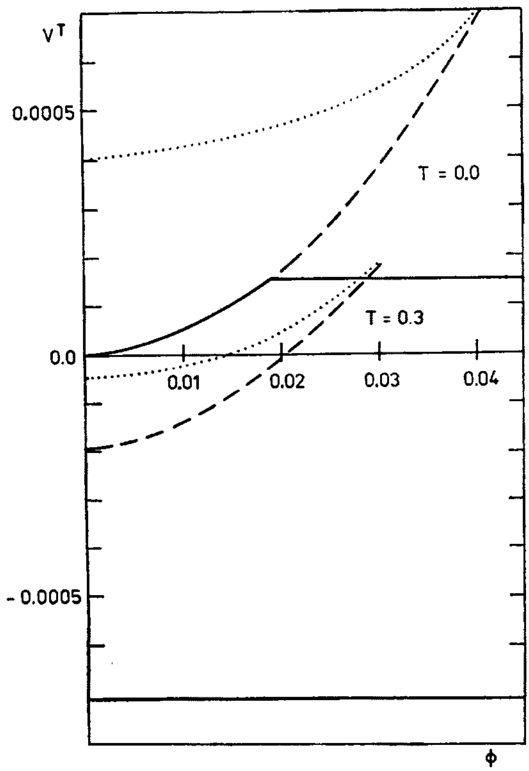

Fig. 2. The effective potential for $\lambda\Phi^4+g\Phi^6$ QFT for $\kappa = 1.25$ and $G = 0.001$ at the temperature $T = 0$ and $T = 0.3$. The critical temperature $T_2 = 0.21$. All variables are in the units of renormalized mass m_R

the critical temperature. In $\lambda\Phi^4$ theory such strange behavior above the critical temperature was atributed to internal instability due to negative bare four-coupling [8], which can be related to the perturbative instability. In the $\lambda\Phi^4+g\Phi^6$ theory the instability is due to the negative bare six-coupling. In this case perturbative nonrenormalizability appears already in one loop approximation — bare parameters can be chosen in such a way that the self--energy, the four- and six-vertex are finite at arbitrary external momenta, but the eight--vertex remains divergent. It has been claimed that the eight-vertex causes nonrenormalizability in the leading order of the large N expansion [26]. This is in disagreement with our results, since the first order of the OE for N-component scalar field with O(N) symmetric interaction becomes exact in the limit of large N. Renormalization can be performed, choosing a bare mass, four- and six-couplings in a similar way as for one field. The six--coupling is infinitesimal, which makes the EP (and the eight-vertex) finite. Therefore, the theory is renormalizable in the large N limit. For finite N the discussion of higher orders of the OE is necessary to check renormalizability and stability of the precarious phase of the theory.

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REFERENCES

[1] R. Jackiw, *Phys. Rev.* **D9**, 1686 (1974).
[2] P. J. Redmond, J. L. Utretsky, *Phys. Rev. Lett.* **1**, 145 (1958); N. N. Bogolubov, A. A. Logunov, D. V. Shirkov, *Zh. Eksp. Teor. Fiz.* **37**, 805 (1959); G. Parisi, *Nucl. Phys.* **B100**, 368 (1975); D. Gross, in *Methods in Field Theory*, ed. R. Balian and J. Zinn-Justin, North Holland, Amsterdam 1976.
[3] K. Gawędzki, A. Kupiainen, *Phys. Rev. Lett.* **55**, 363 (1985); *Nucl. Phys.* **B262**, 33 (1985).
[4] T. Eguchi, *Phys. Rev.* **D17**, 611 (1978); G. S. Guralnik, K. Tamvakis, *Nucl. Phys.* **B148**, 283 (1979).
[5] G. Rajasekaran, V. Srinivasan, *Pramãna* **9**, 239 (1977).
[6] M. Aizenman, *Phys. Rev. Lett.* **47**, 1 (1981); *Commun. Math. Phys.* **86**, 1 (1982); J. Fröhlich, *Nucl. Phys.* **B200**, 281 (1982); C. Aragão de Carvalho, C. S. Caracciolo, J. Fröhlich, *Nucl. Phys.* **B215**, 209 (1983).
[7] L. D. Landau, *Fundamental Problems* in *Collected Papers of L. D. Landau*, ed. D. Ter Haar, Pergamon Press, Oxford 1965.
[8] W. A. Bardeen, M. Moshe, *Phys. Rev.* **D28**, 1382 (1983); *Phys. Rev.* **D34**, 1229 (1986).
[9] P. M. Stevenson, *Phys. Rev.* **D32**, 1389 (1985) and references given therein.
[10] I. Yotsuyanagi, *Z. Phys.* **C35**, 453 (1987).
[11] P. M. Stevenson, R. Tarrach, *Phys. Lett.* **176**, 436 (1986).
[12] S. Paban, J. Taron, R. Tarrach, *Z. Phys.* **C34**, 85 (1987).
[13] R. Jackiw, A. Kerman, *Phys. Lett.* **71A**, 158 (1979); U. Kaulfuss, M. Altenbokum, *Phys. Rev.* **D35**, 609 (1987).
[14] S. Paban, R. Tarrach, *Phys. Lett.* **B197**, 383 (1987); *Phys. Lett.* **B199**, 596 (1987).
[15] K. Symanzik, *Nucl. Phys.* **B190** [FS3], 1 (1981).
[16] A. Okopińska, *Phys. Rev.* **D35**, 1835 (1987).
[17] A. Kovner, B. Rosenstein, Proc. of the International Workshop on Variational Calculations in Quantum Field Theory, Wangerooge 1987, ed. by L. Polley and D.E.L. Pottinger, World Scientific, Singapore 1988, p. 143.
[18] P. M. Stevenson, I. Roditi, *Phys. Rev.* **D33**, 2305 (1986).
[19] J. Wudka, *Phys. Rev.*, **D36** 1745 (1987); *Phys. Rev.* **D36**, 3828 (1987).
[20] J. Schwinger, *Phys. Rev.* **73**, 413 (1948); R. P. Feynman, *Phys. Rev.* **75**, 1430 (1948).
[21] L. Dolan, R. Jackiw, *Phys. Rev.* **D9**, 3320 (1974).
[22] A. Okopińska, *Phys. Rev.* **D36**, 2415 (1987).
[23] I. Roditi, *Phys. Lett.* **169B**, 264 (1985).
[24] G. A. Hajj, P. M. Stevenson, *Phys. Rev.* **D37**, 413 (1988).
[25] J. Soto, *Phys. Lett.* **D188**, 340 (1987); *Nucl. Phys.* **D316**, 141 (89).
[26] H. J. Schnitzer, *Nucl. Phys.* **B109**, 297 (1976).