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ASYMPTOTIC SOLUTION TO THE BREIT RELATIVISTIC EQUATION FOR COULOMB PARAFERMIONIUM*

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An analytic method of asymptotically solving the Breit relativistic equation is presented for a system of two spin-1/2 particles of equal masses bound in parastates by the Coulomb attraction.

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1. Introduction

Exact solutions to relativistic wave equations in quantum mechanics played an important role in the development of particle physics. It is enough to mention the exact solution to the Dirac equation for a spin-1/2 particle in the external Coulomb potential [1], that revealed the relativistic origin of fine-structure phenomena in atomic spectra and gave the impulse to the essential progress in quantum electrodynamics.

In spite of many efforts [2], the relativistic wave equations for a system of two spin-1/2 particles interacting through the Coulomb potential $V = -\alpha/r$ have resisted for a long time an exact treatment because, in this case, the Sommerfeld polynomial method of solving the second order differential equations [3] does not work. It is caused by a singularity at $r = -\alpha/E$ that, though regular, appears in the analytic extension of the respective radial equations in addition to the familiar regular singularity at $r = 0$ and irregular singularity at $r = \infty$. In the present note we report on an analytic method of asymptotically solving the Breit relativistic equation for a system of two spin-1/2 particles of equal masses bound in parastates by the Coulomb attraction.

As it is well known, the Breit relativistic equation [4],

$$[E-H_1(\vec{p}_1)-H_2(\vec{p}_2)-V(\vec{x}_1-\vec{x}_2)]\psi(\vec{x}_1, \vec{x}_2) = 0, \tag{1}$$

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where $H_i(\vec{p}_i) = \vec{\alpha}_i \cdot \vec{p}_i + \beta_i m_i$, offers the simplest wave equation for a system of two Dirac particles interacting through a "vector" potential $V(\vec{r})$, $\vec{r} = \vec{x}_1 - \vec{x}_2$. In contrast to the much more complicated Salpeter equation [5], it does not include the hole theory.Nevertheless, one usually believes that in the case of Coulomb potential $V = -\alpha/r$, $r = |\vec{r}|$, it is pretty well applicable, the deviations from the hole theory being manifested only when the nonstatic corrections depending on $\vec{\alpha}_i$ are added to the Coulomb potential [4]. In particular, the lowest nonstatic correction given by the Breit terms $(\alpha/r)\,\frac{1}{2}\,[\vec{\alpha}_1 \cdot \vec{\alpha}_2 + (\vec{\alpha}_1 \cdot \vec{r})\,(\vec{\alpha}_2 \cdot \vec{r})/r^2]$ introduces a large deviation from the hole theory, unless this correction is treated as a first-order perturbation only [4, 6].

The belief in the Breit relativistic equation with Coulomb potential is based on a perturbative treatment of Eq. (1), where one starts from the Schrödinger equation as the zero-order nonrelativistic approximation and evaluates from Eq. (1) consecutive relativistic corrections in powers of v/c with $v = |\vec{p}|/\mu$, $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ and $\mu = m_1 m_2/(m_1 + m_2)$ (in the centre-of-mass frame $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$). We shall see that such a perturbative procedure deforms the behaviour of the exact solution at $r = 0$, though it leads to the proper bound-state energy spectrum, at least in the orders $O(\alpha^2)$ and $O(\alpha^4)$ (we shall consider solely parastates).

2. *Radial equations for parastates*

Let us start from the subset of radial equations for parastates, following from Eq. (1) in the centre-of-mass frame when the case of a central potential $V = V(r)$ and equal masses $m_1 = m_2 \equiv m$ is considered. In the notation of the last Ref. [2] this subset has the form (cf. Eq. (A2) in Appendix A):

$$\begin{aligned}
\tfrac{1}{2}(E-V)\phi^0 + i\left(\frac{d}{dr} + \frac{2}{r}\right)\phi_{\mathrm{el}} + i\,\frac{1}{r}\,\phi_{\mathrm{long}} &= m\phi,\\
\tfrac{1}{2}(E-V)\phi &= m\phi^0,\\
\tfrac{1}{2}(E-V)\phi_{\mathrm{el}} + i\,\frac{d}{dr}\,\phi^0 &= 0,\\
\tfrac{1}{2}(E-V)\phi_{\mathrm{long}} - i\,\frac{j(j+1)}{r}\,\phi^0 &= 0,\\
\tfrac{1}{2}(E-V)\phi_{\mathrm{mag}} &= 0.
\end{aligned} \tag{2}$$

Among the four nonzero radial components of the wave function $\psi(r)$ appearing in Eq. (2) two, $\phi^0(r)$ and $\phi(r)$, correspond to the total spin $s = 0$ and are the "large-large" components (superposed with the "small-small' components), while two other, $\phi_{\mathrm{el}}(r)$ and $\phi_{\mathrm{long}}(r)$, correspond to $s = 1$ and are the "small-large" and "large-small" components superposed with each other (for the detailed definition of these components which is not relevant here cf. the last Ref. [2]). Thus, the subset (2) has the spectroscopic signature ${}^1j_j$, $j = 0, 1, 2, \ldots$, and the total parity $\eta(-1)^j$ (where $\eta = +1$ or -1 for a fermion-fermion system or a fermion-antifermion system, respectively, the latter, if bound, being called fermionium).

Eliminating from Eqs (2) the components ϕ, ϕ_{el} and ϕ_{long} we obtain the following second order differential equation for ϕ^0 (cf. Eq. (A8) in Appendix A):

$$\left[\tfrac{1}{4}(E-V)^2+\frac{1}{r}\frac{d^2}{dr^2}r-\frac{j(j+1)}{r^2}-m^2+\frac{1}{E-V}\frac{dV}{dr}\frac{d}{dr}\right]\phi^0 = 0. \tag{3}$$

Omitting the last term in Eq. (3), one gets the radial equation corresponding to the (two--body) Klein-Gordon equation for a system of two spin-0 particles of equal masses. Then, in the nonrelativistic approximation

$$\tfrac{1}{4}(E-V)^2-m^2 = [\tfrac{1}{2}(E-V)+m]\,[\tfrac{1}{2}(E-V)-m]$$
$$\cong 2m[\tfrac{1}{2}(E-V)-m], \tag{4}$$

one comes to the (two-body) Schrödinger equation. We can see that in the case of Coulomb potential $V = -\alpha/r$ both steps deform the behaviour of ϕ^0 at $r\to 0$ that for Eq. (3) is $r\phi^0 \sim r^\gamma$ with $\gamma = \sqrt{1+j(j+1)-(\alpha/2)^2}$, while for the Klein-Gordon equation and the Schrödinger equation it becomes $r\phi^0 \sim r^{\gamma_{KG}}$ with $\gamma_{KG} = \frac{1}{2}+\sqrt{(j+\frac{1}{2})^2-(\alpha/2)^2}$ and $r\phi^0 \sim r^{\gamma_S}$ with $\gamma_S = j+1$, respectively. In the case of Coulomb bound states the behaviour of ϕ^0 at $r\to\infty$ is $r\phi^0 \sim \exp(-ar)$ with $a = \sqrt{m^2-(E/2)^2} > 0$, both for Eq. (3) and the Klein--Gordon equation.

Note that in the first-order perturbative treatment of the last term in Eq. (3) only its Hermitian part contributes, if the unperturbed ϕ^0 can be taken real, as in the case of Coulomb bound states. This Hermitian part is

$$-\frac{\Delta V}{2(E-V)}-\frac{1}{2(E-V)^2}\left(\frac{dV}{dr}\right)^2 = -\frac{2\pi\alpha\delta(\vec{r})}{E+\alpha/r}-\tfrac{1}{2}\left[\frac{\alpha}{r^2(E+\alpha/r)}\right]^2$$
$$= -\tfrac{1}{2}\left[\frac{\alpha}{r^2(E+\alpha/r)}\right]^2 \tag{5}$$

if $V = -\alpha/r$. Such a form follows from the relation $(d/dr)^\dagger = -(d/dr+2/r)$. In the approximation of $V/E\to 0$ the last term in Eq. (3) tends to the operator $(1/E)(dV/dr)d/dr$ whose Hermitian part

$$-\frac{\Delta V}{2E} = -\frac{2\pi\alpha\delta(\vec{r})}{E} \tag{6}$$

is the Darwin term $\Delta V/(2m)^2 = (\pi\alpha/m^2)\delta(\vec{r})$ multiplied by $-m(2m/E) = -m+O(\alpha^2)$. In the first-order perturbative calculation the Darwin term contributes the correction

$$E_D = \frac{\pi\alpha}{m^2}\langle\delta(\vec{r})\rangle = \frac{\alpha^4 m}{16n^4}\frac{n}{j+\frac{1}{2}}\delta_{j0} \tag{7}$$

to the bound-state energy spectrum. Here, $\langle\rangle$ denote the Schrödinger bound-state expectation values. The Darwin correction (7) can be also obtained when the expression $-(1/m)\langle\rangle$ is evaluated for the exact last term in Eq. (3) with $V = -\alpha/r$ and then approximated up to the lowest power in α^2. In this case, e.g. for the ground state, one gets

$$\left\langle \frac{\alpha}{r(Er+\alpha)} \frac{d}{dr} \right\rangle_{n=1} = 4\left(\frac{\alpha m}{2}\right)^3 \frac{\alpha}{E} \int_0^\infty dr \frac{r}{r+\alpha/E} \left(-\frac{\alpha m}{2}\right) e^{-\alpha m r}$$

$$= -4\left(\frac{\alpha m}{2}\right)^4 \left(\frac{\alpha}{E}\right)^2 \left\{\left(\frac{\alpha^2 m}{E}\right)^{-1} + \left[C+\ln\frac{\alpha^2 m}{E} + \sum_{k=1}^{\infty} \frac{1}{kk!}\left(-\frac{\alpha^2 m}{E}\right)^k\right] e^{\frac{\alpha^2 m}{E}}\right\}$$

$$= -\frac{\alpha^4 m^2}{8} + O(\alpha^6 \ln \alpha^2) + O(\alpha^6), \tag{8}$$

where $C = 0.5772$ is the Euler constant and $E = 2m + O(\alpha^2)$. Thus, $-(1/m)\langle\rangle_{n=1} = \alpha^4 m/8 + O(\alpha^6 \ln \alpha^2) + O(\alpha^6)$ with $\alpha^4 m/8 = E_{Dn=1}$[1].

3. *Fine structure for parafermionium*

In the case of $V = -\alpha/r$ it is convenient to rewrite Eq. (3) in the form

$$\left[\frac{1}{x}\frac{d^2}{dx^2}x - \lambda^2 + \frac{1}{2x} - \frac{j(j+1)-(\alpha/2)^2}{x^2} + \frac{\alpha^2}{x(x+\alpha^2)}\frac{d}{dx}\right]\phi^0 = 0, \tag{9}$$

where $x \equiv \alpha E r$ and $\lambda \equiv a/\alpha E = (1/2\alpha)\sqrt{(2m/E)^2 - 1} > 0$ (for bound states). Here, $x\phi^0 \sim x^\gamma$ at $x \to 0$ and $x\phi^0 \sim \exp(-\lambda x)$ at $x \to \infty$. Note that for $x = \alpha^2$ we have $r = \alpha/E = \alpha/(2m) + O(\alpha^3)$ which is the "classical radius" of a pointlike limiting counterpart of our fermio-

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[1] Note that

$$\left\langle \frac{\alpha}{r(Er+\alpha)} \frac{d}{dr} \right\rangle = -\frac{1}{2}\left\langle \frac{\alpha^2}{r^2(Er+\alpha)^2} \right\rangle.$$

Here,

$$\left\langle \frac{\alpha}{Er^2} \frac{d}{dr} \right\rangle_{n=1} = -\frac{2m}{E}\frac{\alpha^4 m^2}{8} = -\frac{2m}{E} m E_{Dn=1}$$

is finite, while

$$-\frac{1}{2}\left\langle \frac{\alpha^2}{E^2 r^4} \right\rangle_{n=1} = -\lim_{\varepsilon\to 0} \frac{1}{2}\left\langle \frac{\alpha^2}{r^2(Er+\varepsilon)^2} \right\rangle_{n=1} = -\lim_{\varepsilon\to 0} 4\left(\frac{\alpha m}{2}\right)^4 \left(\frac{\alpha}{E}\right)^2$$

$$\times \left\{\left(\frac{\alpha m \varepsilon}{E}\right)^{-1} + \left[C+\ln\frac{\alpha m \varepsilon}{E} + \sum_{k=1}^{\infty} \frac{1}{kk!}\left(-\frac{\alpha m \varepsilon}{E}\right)^k\right] e^{\frac{\alpha m \varepsilon}{E}}\right\}$$

becomes divergent.

nium. For positronium it is equal to 1.4 fm. Since $\langle 1/x \rangle = O(\alpha)/\alpha = O(1)$, a range of r lying much outside this "classical radius" dominates the structure of any real fermionium.

It is not difficult to deduce from Eq. (9) its other equivalent form

$$\left[\frac{1}{x}\frac{d^2}{dx^2}x-\lambda^2+\frac{1}{2x}-\frac{j(j+1)-(\alpha/2)^2}{x^2}-\frac{3}{4}\frac{\alpha^4}{x^2(x+\alpha^2)^2}\right]\frac{\phi^0}{\sqrt{1+\alpha^2/x}} = 0. \tag{10}$$

Here, the truncated function $\phi^0/(\sqrt{E}\sqrt{1+\alpha^2/x})$ is normalized with the weight $\sqrt{E}\sqrt{1+\alpha^2/x} = \sqrt{E-V}$ like a (two-body) Klein-Gordon radial function, since ϕ^0 is normalized as a radial component of the (two-body) Dirac wave function $\psi(\vec{r})$. Notice that in the deduction of Eq. (10) the relation $r\delta(\vec{r}) = 0$ was used, where $\delta(\vec{r}) = -\Delta(1/r)/4\pi$.

It can be seen from Eq. (9) or (10) that the analytic extension of this equation in the complex x-plane has a regular singularity at $x = -\alpha^2$ in addition to the familiar regular singularity at $x = 0$ and irregular singularity at $x = \infty$. Thus, Eq. (9) or (10) cannot be reduced neither to the hypergeometric equation (which has three regular singularities) nor to the confluent hypergeometric equation (which has one regular and one irregular singularity). Because of its singularity at $x = -\alpha^2$ Eq. (9) or (10) cannot be solved in the whole physical range $x \geqslant 0$ by means of a universal series. But, its solution ϕ^0 may be looked for in the form of two separate series defined in the ranges $0 \leqslant x < \alpha^2$ and $x > \alpha^2$. Since the point $x = \alpha^2$ is regular for the analytic extension of Eq. (9) or (10), both series, if convergent, are defined and equal (with all their derivatives) in the limit of $x \to \alpha^2$. Unfortunately, in contrast to the hypergeometric equation with singularities at $x = 0$, $x = -\alpha^2$ and $x = \infty$, the singularity at $x = \infty$ in Eq. (9) or (10) is irregular, implying an awkward divergence at any x for the expansion of ϕ^0 at $x = \infty$ that can be considered only as an asymptotic expansion. On the other hand, by its extra singularity at $x = -\alpha^2$ Eq. (9) or (10) differs from the confluent hypergeometric equation with singularities at $x = 0$ and $x = \infty$.

By the substitution $x\phi^0 = x^\gamma \exp(-\lambda x) f$ we transform out from Eqs (9) and (10) the behaviour of ϕ^0 at $x \to 0$ and $x \to \infty$ getting, respectively, the equations

$$\left[\frac{d^2}{dx^2}+2\left(\frac{\gamma}{x}-\lambda\right)\frac{d}{dx}+\frac{\frac{1}{2}-2\lambda\gamma}{x}-\frac{\gamma-1}{x^2}+\frac{\alpha^2}{x(x+\alpha^2)}\left(\frac{d}{dx}+\frac{\gamma-1}{x}-\lambda\right)\right]f = 0 \tag{11}$$

and

$$\left[\frac{d^2}{dx^2}+2\left(\frac{\gamma}{x}-\lambda\right)\frac{d}{dx}+\frac{\frac{1}{2}-2\lambda\gamma}{x}-\frac{\gamma-1}{x^2}-\frac{3}{4}\frac{\alpha^4}{x^2(x+\alpha^2)^2}\right]\frac{f}{\sqrt{1+\alpha^2/x}} = 0. \tag{12}$$

Here, $f \sim 1$ at $x \to 0$ as $x\phi^0 \sim x^\gamma$, and $f \sim x^{1/(4\lambda)-\gamma}$ at $x \to \infty$. Defining ζ as the larger root of the quadratic algebraic equation

$$\zeta(\zeta-1+2\gamma)-\gamma+1 = 0 \tag{13}$$

that gives

$$\gamma+\zeta = \tfrac{1}{2}+\sqrt{(j+\tfrac{1}{2})^2-\left(\frac{\alpha}{2}\right)^2} \equiv \gamma_{KG}, \tag{14}$$

we obtain the following equations equivalent, respectively, to Eqs (11) and (12):

$$\left[\frac{d^2}{dx^2}+2\left(\frac{\gamma_{KG}}{x}-\lambda\right)\frac{d}{dx}+\frac{\frac{1}{2}-2\lambda\gamma_{KG}}{x}+\frac{\alpha^2}{x(x+\alpha^2)}\left(\frac{d}{dx}+\frac{\gamma_{KG}-1}{x}-\lambda\right)\right]x^{-\zeta}f=0 \quad (15)$$

and

$$\left[\frac{d^2}{dx^2}+2\left(\frac{\gamma_{KG}}{x}-\lambda\right)\frac{d}{dx}+\frac{\frac{1}{2}-2\lambda\gamma_{KG}}{x}-\frac{3}{4}\frac{\alpha^4}{x^2(x+\alpha^2)^2}\right]\frac{x^{-\zeta}f}{\sqrt{1+\alpha^2/x}}=0. \quad (16)$$

Here, $x^{-\zeta}f \sim x^{-\zeta}$ at $x \to 0$ and $x^{-\zeta}f \sim x^{1/(4\lambda)-\gamma_{KG}}$ at $x \to \infty$. If the last term in Eqs (15) and (16) is omitted, they become the confluent hypergeometric equation corresponding to the (two-body) Klein-Gordon equation with $V = -\alpha/r$.

Since $f \sim 1$ at $x \to 0$ (where $x = 0$ is a regular singularity), the function f can be represented in the range of $0 \leqslant x < \alpha^2$ by the Taylor series

$$f=\sum_{\nu=0}^{\infty} a_\nu x^\nu \quad \text{for} \quad 0 \leqslant x < \alpha^2. \quad (17)$$

On the other hand, since $f \sim x^{1/(4\lambda)-\gamma}$ at $x \to \infty$ (where $x = \infty$ is an irregular singularity), the function f can be asymptotically represented in the range of $x > \alpha^2$ by the formal Laurent series

$$f=x^{1/(4\lambda)-\gamma}\sum_{\nu=-\infty}^{0} b_\nu x^\nu = x^{1/(4\lambda)-\gamma-\nu_{max}}\sum_{\nu=-\infty}^{\nu_{max}} c_\nu x^\nu$$

$$= x^{\zeta+\delta}\sum_{\nu=-\infty}^{\nu_{max}} c_\nu x^\nu \quad \text{for} \quad x > \alpha^2, \quad (18)$$

where $\nu_{max} = 0, 1, 2, \ldots$ and δ is defined by $\lambda \equiv 1/[4(\nu_{max}+\gamma_{KG}+\delta)] > 0$ (for bound states). So, δ parametrizes the exact bound-state energy spectrum for parafermionium:

$$E \equiv \frac{2m}{\sqrt{1+(2\alpha\lambda)^2}} \equiv 2m\left[1+\left(\frac{\alpha/2}{\nu_{max}+\gamma_{KG}+\delta}\right)^2\right]^{-1/2}. \quad (19)$$

When δ is determined independently. Eq. (19) gives the bound-state energy spectrum for parafermionium. Note that Eq. (18) implies for ϕ^0 the asymptotic behaviour $x\phi^0 = x^{1/(4\lambda)}\exp(-\lambda x) = x^{\nu_{max}+\gamma_{KG}+\delta}\exp(-\lambda x)$ at $x \to \infty$ (while $\phi x^0 \sim x^\gamma$ at $x \to 0$).

If the last term in Eq. (9) is neglected, one gets the radial equation corresponding to the (two-body) Klein-Gordon equation with $V = -\alpha/r$. Then, for bound states λ becomes $\lambda_{KG} = 1/[4(\nu_{max}+\gamma_{KG})]$ with $\nu_{max} = 0, 1, 2, \ldots$ (cf. Eq. (15) with the last term neglected), leading to the familiar bound-state energy spectrum

$$E_{KG} \equiv \frac{2m}{\sqrt{1+(2\alpha\lambda_{KG})^2}} = 2m\left[1+\left(\frac{\alpha/2}{\nu_{max}+\gamma_{KG}}\right)^2\right]^{-1/2}. \quad (20)$$

Thus, δ can be calculated perturbatively in the following way:

$$\delta \equiv \frac{1}{4\lambda} - \frac{1}{4\lambda_{KG}} \equiv \frac{\alpha}{2}\left[\left(\frac{2m}{E_{KG}}\right)^2 - 1\right]^{-3/2} \frac{\delta E}{E_{KG}} + O[(\delta E)^2]$$

$$= \left(\frac{\alpha}{2}\right)^2 \delta_{j0} + O(\alpha^4), \tag{21}$$

where $E \equiv E_{KG} + \delta E$ and $\delta E = E_D + O(\alpha^6)$ with E_D given in Eq. (7). From Eqs (19) and (21) one can obtain the perturbative formula for the bound-state energy spectrum of parafermionium:

$$E = 2m - \frac{\alpha^2 m}{4n^2} + \frac{\alpha^4 m}{16n^4}\left[\frac{3}{4} - \frac{n(1-\delta_{j0})}{j+\frac{1}{2}}\right] + O(\alpha^6), \tag{22}$$

where $n \equiv v_{max} + j + 1 = 1, 2, 3, \ldots$ Of course, this formula takes into account the interaction given by the Coulomb potential $V = -\alpha/r$ only and so it is a fine-structure formula.

It is interesting to remark that up to the order $O(\alpha^4)$ the perturbative result (22) coincides in the case of $j = 0$ with the exact $j = 0$ bound-state energy spectrum [7]

$$E_{RS} = 2m\left[1 + \left(\frac{\alpha/2}{n}\right)^2\right]^{-1/2} \tag{23}$$

of the (two-body) Schrödinger relativistic equation

$$\left(E - 2\sqrt{\vec{p}^{\,2} + m^2} + \frac{\alpha}{r}\right)\psi_{RS}(\vec{r}) = 0. \tag{24}$$

Such an equation describes a system of two spin-0 particles of equal masses and (only) positive virtual energies, interacting through the Coulomb potential $V = -\alpha/r$. Unfortunately, the spectrum of Eq. (24) for $j > 0$ is not known yet.

The Salpeter equation including consequently the hole theory for a system of two spin-1/2 particles is, of course, still more difficult for exact treatment [5] than Eq. (24).

4. *Asymptotic expansion for parafermionium*

In the range of $x > \alpha^2$, inserting the formal Laurent series (18) into Eq. (11) multiplied by $x(x+\alpha^2)$, we derive the following recurrence formulae for the coefficients c_v:

$$\begin{aligned}&[\tfrac{1}{2} - 2\lambda(v + \gamma_{KG} + \delta)]c_v\\ &+\{(v+1)(v+2\gamma_{KG}+2\delta) + \delta(\delta-1) + \alpha^2[\tfrac{1}{2} - 2\lambda(v+\gamma_{KG}+\delta+\tfrac{3}{2})]\}c_{v+1}\\ &+\alpha^2[(v+2)(v+2+2\gamma_{KG}+2\delta) + \delta(\delta-1) + \gamma_{KG} + \delta - 1]c_{v+2} = 0.\end{aligned} \tag{25}$$

Putting in Eq. (25) $v = v_{max}$ we reproduce the spectral relation $\lambda = 1/[4(v_{max} + \gamma_{KG} + \delta)]$ (because $c_{v_{max}+1} = 0 = c_{v_{max}+2}$). Note that, if the formal Laurent series (18) representing f did not terminate for $v \geqslant 0$ at some $v_{max} = 0, 1, 2, \ldots$, it would behave as $x^{\zeta+\delta}\exp(2\lambda x)$ at $x \to \infty$ contradicting the bound-state behaviour of $x\phi^0 = x^\gamma \exp(-\lambda x) f$ at $x \to \infty$

characterized by exp $(-\lambda x)$. This can be seen from the recurrence formulae (25) considered for $\nu \to \infty$. Starting from $\nu = \nu_{max}-1$ we can solve the recurrence formulae (25) step by step (because $c_{\nu_{max}+1} = 0$), expressing all consecutive c_ν, $\nu = \nu_{max}-1, \nu_{max}-2, \ldots, 0, -1, -2, \ldots$, by $c_{\nu_{max}}$. Then, the resulting series (18) gives for $x > \alpha^2$ the asymptotic expansion of the solution f to Eq. (11). It means that

$$[x^{-\zeta-\delta}f(x) - \sum_{\nu=\nu_{min}}^{\nu_{max}} c_\nu x^\nu]x^{-\nu_{min}} \xrightarrow[x\to\infty]{} 0 \tag{26}$$

for any fixed $\nu_{min} = \nu_{max}, \nu_{max}-1, \ldots, 0, -1 -2\ldots$, though the formal series (18) is divergent at any $x > \alpha^2$ i.e., the sequence

$$\sum_{\nu=\nu_{min}}^{\nu_{max}} c_\nu x^\nu, \quad \nu_{min} = \nu_{max}, \nu_{max}-1, \ldots, 0, -1, -2, \ldots, \tag{27}$$

has no limit at $\nu_{min} \to -\infty$. This is implied by the recurrence formulae (25) considered for $\nu \to -\infty$. Evidently, the regular part of the formal series $\sum_{\nu=-\infty}^{\nu_{max}} c_\nu x^\nu$ is the polynomial given by the term $\nu_{min} = 0$ of the sequence (27), while its principal part contains nontrivially all negative powers $\nu < 0$.

The divergence of the asymptotic expansion of a solution to a linear ordinary differential equation of the second order is a generic phenomenon when, as in our case, $x = \infty$ is an irregular singularity of the equation [8]. A convergent expression for f could be perhaps found by an integral transformation. Then the formal series (18) would be the asymptotic expansion of a transform integral.

Now, in the range of $0 \leqslant x < \alpha^2$, inserting the Taylor series (17) into Eq. (11) multiplied by $x(x+\alpha^2)$, we obtain the following formulae for the coefficients a_ν

$$\begin{aligned}&[\tfrac{1}{2}-2\lambda(\nu+\gamma)]a_\nu \\ &+\{(\nu+1)(\nu+2\gamma)-\gamma+1+\alpha^2[\tfrac{1}{2}-2\lambda(\nu+\gamma+\tfrac{3}{2})]\}a_{\nu+1} \\ &+\alpha^2(\nu+2)(\nu+2+2\gamma)a_{\nu+2} = 0,\end{aligned} \tag{28}$$

where $\lambda = 1/[4(\nu_{max}+\gamma_{KG}+\delta)]$ as found in the range of $x > \alpha^2$. Starting from $\nu = -1$ we can solve the recurrence formulae step by step (because $a_{-1} = 0$), expressing all consecutive a_ν, $\nu = 1, 2, 3, \ldots$, by a_0. Then, the resulting series (17) gives for $0 \leqslant x < \alpha^2$ the solution f to Eq.(11). In fact, for $\nu \to \infty$ Eq. (28) takes the limiting form

$$-2\lambda a_\nu + (\nu+1)a_{\nu+1} + \alpha^2(\nu+2)a_{\nu+2} = 0 \tag{29}$$

which can be solved by

$$a_\nu = \xrightarrow[\nu\to\infty]{} \frac{(2\lambda)^\nu}{\nu!} a_0 \tag{30}$$

since then asymptotically at $\nu \to \infty$

$$\frac{\alpha^2(\nu+2)a_{\nu+2}}{-2\lambda a_\nu} = \frac{\alpha^2(\nu+2)a_{\nu+2}}{(\nu+1)a_{\nu+1}} = \frac{2\lambda}{\nu+1} \xrightarrow[\nu\to\infty]{} 0. \tag{31}$$

It implies that the series (17) is (quickly) convergent for any x from the range of $0 \leqslant x < \alpha^2$ because

$$\frac{a_{\nu+1}x^{\nu+1}}{a_\nu x^\nu} = \xrightarrow[\nu\to\infty]{} \frac{2\lambda x}{\nu+1} \xrightarrow[\nu\to\infty]{} 0. \tag{32}$$

5. *Hyperfine structure for parafermionium*

Turning back to the perturbative fine-structure formula (22) one should stress that the Breit terms, even if treated perturbatively in the first order, give the comparatively large orbital and hyperfine-structure corrections. For parafermionium such a correction is (cf. Eq. (B6) in Appendix B)

$$E_B = \frac{\alpha^4 m}{16n^4}\left[2-\frac{(3+\delta_{j0})n}{j+\frac{1}{2}}\right]+O(\alpha^6). \tag{33}$$

If added to Eq. (22), the correction (33) leads to the following perturbative spectral formula for parafermionium:

$$E+E_B = 2m-\frac{\alpha^2 m}{4n^2}+\frac{\alpha^4 m}{4n^4}\left(\frac{11}{16}-\frac{n}{j+\frac{1}{2}}\right)+O(\alpha^6). \tag{34}$$

It is a well known result [4] consistent with experimental data for positronium. Radiative corrections to Eq. (34) come in the order $O(\alpha^5)$ [9]. Annihilation corrections of the order $O(\alpha^4)$ vanish for parastates as being provided by virtual one-photon annihilation process.

We can see that the perturbative result (34) is in agreement with our asymptotic solution (18) when δ is evaluated as in Eq. (21), although the usual perturbative treatment of the Breit relativistic equation deforms (or rather ignores) our exact solution (17) in the range of $x < \alpha^2$.

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APPENDIX A

Radial equations following from Breit equation

The Breit relativistic equation for two spin-1/2 particles of equal masses $m_1 = m_2 \equiv m$ interacting through the "vector" potential $V(\vec{r})$ and a "scalar" potential $S(\vec{r})$ has in the centre-of-mass frame the form

$$[E-V-(\vec{\alpha}_1-\vec{\alpha}_2)\cdot\vec{p}-(\beta_1+\beta_2)(m+\tfrac{1}{2}S)]\psi(\vec{r}) = 0, \tag{A1}$$

where $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$ and $\vec{x}_1-\vec{x}_2 = \vec{r}$.

Assuming central potentials, $V = V(r)$ and $S = S(r)$ with $r = |\vec{r}|$, and using a multipole method of eliminating the angular variables described in the last Ref. [2], one can split Eq. (A1) into the following three independent subsets of radial equations [10]:
(i) subset ${}^1j_j$ with total parity $\eta(-1)^j$

$$
\begin{aligned}
&\tfrac{1}{2}(E-V)\varphi^0+i\left(\frac{d}{dr}+\frac{2}{r}\right)\phi_{\rm el}+i\,\frac{1}{r}\,\phi_{\rm long} = (m+\tfrac{1}{2}S)\phi,\\
&\tfrac{1}{2}(E-V)\phi = (m+\tfrac{1}{2}S)\phi^0,\\
&\tfrac{1}{2}(E-V)\phi_{\rm el}+i\,\frac{d}{dr}\,\phi^0 = 0,\\
&\tfrac{1}{2}(E-V)\phi_{\rm long}-i\,\frac{j(j+1)}{r}\,\phi^0 = 0,\\
&\tfrac{1}{2}(E-V)\phi_{\rm mag} = 0,
\end{aligned}
\tag{A2}
$$

(ii) subset ${}^3(j\pm 1)_j$ with total parity $\eta(-1)^{j+1}$

$$
\begin{aligned}
&\tfrac{1}{2}(E-V)\chi_{\rm el}+i\,\frac{d}{dr}\,\chi^0 = (m+\tfrac{1}{2}S)\chi^0_{\rm el},\\
&\tfrac{1}{2}(E-V)\chi_{\rm long}-i\,\frac{j(j+1)}{r}\,\chi^0 = (m+\tfrac{1}{2}S)\chi^0_{\rm long},\\
&\tfrac{1}{2}(E-V)\chi^0_{\rm el}+\frac{1}{r}\,\phi^0_{\rm mag} = (m+\tfrac{1}{2}S)\chi_{\rm el},\\
&\tfrac{1}{2}(E-V)\chi^0_{\rm long}-\left(\frac{d}{dr}+\frac{1}{r}\right)\phi^0_{\rm mag} = (m+\tfrac{1}{2}S)\chi_{\rm long},\\
&\tfrac{1}{2}(E-V)\chi^0+i\left(\frac{d}{dr}+\frac{2}{r}\right)\chi_{\rm el}+i\,\frac{1}{r}\,\chi_{\rm long} = 0,\\
&\tfrac{1}{2}(E-V)\phi^0_{\rm mag}+\frac{j(j+1)}{r}\,\chi^0_{\rm el}+\left(\frac{d}{dr}+\frac{1}{r}\right)\chi^0_{\rm long} = 0,
\end{aligned}
\tag{A3}
$$

(iii) subset ${}^3j_j$ with total parity $\eta(-1)^j (j>0)$

$$
\begin{aligned}
&\tfrac{1}{2}(E-V)\chi_{\rm mag} = (m+\tfrac{1}{2}S)\chi^0_{\rm mag},\\
&\tfrac{1}{2}(E-V)\chi^0_{\rm mag}+\frac{j(j+1)}{r}\,\phi^0_{\rm el}+\left(\frac{d}{dr}+\frac{1}{r}\right)\phi^0_{\rm long} = (m+\tfrac{1}{2}S)\chi_{\rm mag},
\end{aligned}
$$

$$\tfrac{1}{2}(E-V)\phi^0_{\rm el}+\frac{1}{r}\chi^0_{\rm mag} = 0,$$

$$\tfrac{1}{2}(E-V)\phi^0_{\rm long}-\left(\frac{d}{dr}+\frac{1}{r}\right)\chi^0_{\rm mag} = 0. \tag{A4}$$

Here, the spectroscopic signature ${}^{2s+1}l_j$ is conventionally determined by the spectroscopic signature of the "large-large" components involved in a given subset. The total parity, being a "good" quantum number, is the same for all components and so equal to the total parity $\eta(-1)^l$ of the "large-large" components in a given subset (here, $\eta = +1$ or -1 for a fermion-fermion system or a fermion-antifermion system, respectively). The total angular momentum $j = 0, 1, 2, \ldots$ is, of course, also a "good" quantum number.

The detailed definition of the 15 radial components appearing in the subsets (A2), (A3) and (A4) is given in the last Ref. [2] and is not relevant here. A 16-th radial component χ, appearing in the general case, identically vanishes in the case of $m_1 = m_2$. The norm squared of the state $\psi(\vec{r})$ is

$$\|\psi\|^2 = \|\phi\|^2+\|\phi^0\|^2+\|\chi\|^2+\|\chi^0\|^2+\|\vec{\chi}\|^2+\|\vec{\chi}^0\|^2+\|\vec{\phi}\|^2+\|\vec{\phi}^0\|^2, \tag{A5}$$

where $\vec{\chi} = (\chi_{\rm el}, \chi_{\rm long}, \chi_{\rm mag})$, etc., with

$$\|\vec{\chi}\|^2 = \|\chi_{\rm el}\|^2+\frac{\|\chi_{\rm long}\|^2+\|\chi_{\rm mag}\|^2}{j(j+1)} \tag{A6}$$

and $\chi_{\rm long} = 0 = \chi_{\rm mag}$ for $j = 0$. The scalar and vector components correspond to the spin $s = 0$ and $s = 1$, respectively. In the case of vector components the combinations

$$\sqrt{\frac{j}{2j+1}}\chi_{\rm el}-\sqrt{\frac{j+1}{2j+1}}\frac{\chi_{\rm long}}{\sqrt{j(j+1)}},\ \sqrt{\frac{j+1}{2j+1}}\chi_{\rm el}+\sqrt{\frac{j}{2j+1}}\frac{\chi_{\rm long}}{\sqrt{j(j+1)}},\ \frac{\chi_{\rm mag}}{\sqrt{j(j+1)}}, \tag{A7}$$

etc., correspond to the orbital angular momentum $l = j-1$, $l = j+1$ and $l = j$, respectively. The "large-large" components (superposed with the "small-small" components) are contained in the scalar components ϕ and ϕ^0 as well as in the vector components $\chi_{\rm el}$, $\chi_{\rm long}$, $\chi_{\rm mag}$ and $\chi^0_{\rm el}$, $\chi^0_{\rm long}$, $\chi^0_{\rm mag}$. Other scalar and vector components are superpositions of the "small-large" and "large-small" components.

Eliminating from the subset (A2) all components but ϕ^0 one gets the equation

$$\left[\tfrac{1}{4}(E-V)^2+\frac{1}{r}\frac{d^2}{dr^2}r-\frac{j(j+1)}{r^2}-(m+\tfrac{1}{2}S)^2+\frac{1}{E-V}\frac{dV}{dr}\frac{d}{dr}\right]\phi^0 = 0, \tag{A8}$$

where $s = 0$ and $l = j$ ($n\,^1j_j$ states). Similarly, from the subset (A4) one obtains the equation

$$\left[\tfrac{1}{4}(E-V)^2+\frac{1}{r}\frac{d^2}{dr^2}r-\frac{j(j+1)}{r^2}-(m+\tfrac{1}{2}S)^2+\frac{1}{E-V}\frac{dV}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)\right]\chi^0_{\rm mag} = 0, \tag{A9}$$

where $s = 1$ and $l = j > 0$ ($n\,^3j_j$ states). Finally, eliminating from the subset (A3) the components χ^0_{el}, χ^0_{long} and χ^0 one derives the system of three equations for χ_{el}, χ_{long} and ϕ^0_{mag}. In the case of $j = 0$, where $\chi_{long} = 0$ and $\phi^0_{mag} = 0$, it reduces to the single equation

$$\left[\tfrac{1}{4}(E-V)^2 + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{2}{r^2} - (m+\tfrac{1}{2}S)^2 + \frac{1}{E-V}\frac{dV}{dr}\left(\frac{d}{dr}+\frac{2}{r}\right)\right]\chi_{el} = 0. \quad \text{(A10)}$$

Here, $s = 1$ and $l = j+1 = 1$ ($n\,^3P_0$ states).

In the situation when only a "scalar" potential S is active, Eqs (A8), (A9) and (A10) become identical with radial equations corresponding to the (two-body) Klein-Gordon equation for a system of two spin-0 particles of equal masses interacting through this "scalar" potential. So, in such a situation the mathematical problem of solving the Breit relativistic equation is much simplified in comparison to the situation when a "vector" potential V is active, introducing into radial equations an additional singularity. The $^1S_0$ and $^3P_0$ states in the case of a linear scalar potential $S = \kappa^2 r$ acting alone were discussed in Ref. [11].

APPENDIX B

Correction from Breit terms

When one adds to the static potential V in the Breit relativistic equation (A1) the Breit terms

$$-V'\tfrac{1}{2}\left[\vec{\alpha}_1\cdot\vec{\alpha}_2 + \frac{1}{r^2}(\vec{\alpha}_1\cdot\vec{r})(\vec{\alpha}_2\cdot\vec{r})\right] \quad \text{(B1)}$$

representing the lowest nonstatic correction, one can derive in the case of central potentials, $V = V(r)$, $V' = V'(r)$ and $S = S(r)$, the following radial equations for parastates [6]:

$$\left[\tfrac{1}{4}(E-V-2V')(E-V) + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{j(j+1)}{r^2}\frac{E-V}{E-V+V'}\right.$$

$$\left.-(m+\tfrac{1}{2}S)^2\frac{E-V}{E-V+2V'} + \frac{1}{E-V}\frac{dV}{dr}\frac{d}{dr}\right]\phi^0 = 0. \quad \text{(B2)}$$

To avoid large deviations of the Breit relativistic equation from the hole theory, the Breit terms must be treated as a first-order perturbation only [4, 6]. Thus, in Eq. (B2) only terms linear in V' should be taken into account. Then, Eq. (B2) transits into the equation

$$\left[\tfrac{1}{4}(E-V)^2 + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{j(j+1)}{r^2} - (m+\tfrac{1}{2}S)^2 + \frac{1}{E-V}\frac{dV}{dr}\frac{d}{dr} + I_B\right]\phi^0 = 0, \quad \text{(B3)}$$

where

$$I_B = \left[-\tfrac{1}{2}(E-V) + \frac{j(j+1)}{r^2(E-V)} + \frac{2(m+\tfrac{1}{2}S)^2}{E-V}\right]V'. \quad \text{(B4)}$$

In the electromagnetic case of $V = V' = -\alpha/r$ and $S = 0$ one gets for Schrödinger bound-state expectation values of I_B the formula

$$\langle I_B \rangle = -\left\langle \frac{\alpha^3 m}{4n^2 r} \right\rangle + \left\langle \frac{\alpha^2}{r^2} \right\rangle - \left\langle \frac{j(j+1)\alpha}{2mr^3} \right\rangle (1-\delta_{j0}) + O(\alpha^6)$$

$$= -\frac{\alpha^4 m^2}{8n^4} + \frac{\alpha^4 m^2}{4m^3(j+\frac{1}{2})} - \frac{\alpha^4 m^2(1-\delta_{j0})}{16n^3(j+\frac{1}{2})} + O(\alpha^6). \quad \text{(B5)}$$

Hence, for parafermionium the energy correction from Breit terms is given by

$$E_B = -\langle I_B \rangle \frac{1}{m}[1+O(\alpha^2)] = \frac{\alpha^4 m}{16n^4}\left[2 - \frac{(3+\delta_{j0})n}{j+\frac{1}{2}}\right] + O(\alpha^6). \quad \text{(B6)}$$

If the potentials $V = V'$ were (uncorrectly) treated on the same footing from the very beginning, Eq. (B2) could be exactly rewritten in the form (B3) with I_B replaced by

$$I_B^{\text{exact}} = \left[-\tfrac{1}{2}(E-V) + \frac{j(j+1)}{r^2 E} + \frac{2(m+\frac{1}{2}S)}{E+V}\right] V. \quad \text{(B7)}$$

Then, in the electromagnetic case of $V = V' = -\alpha/r$ and $S = 0$ one would obtain perturbatively in the first order

$$E_B^{\text{exact}} = -\langle I_B^{\text{exact}} \rangle \frac{1}{m}[1+O(\alpha^2)] = \frac{\alpha^4 m}{16n^4}\left[2 + \frac{(1-\delta_{j0})n}{j+\frac{1}{2}}\right] + O(\alpha^6), \quad \text{(B8)}$$

the energy correction for parafermionium deviating largely from the previous outcome (B6). In fact

$$E_B^{\text{exact}} - E_B = \frac{\alpha^4 m}{4n^4} \frac{n}{j+\frac{1}{2}} + O(\alpha^6). \quad \text{(B9)}$$

In this case, the perturbative spectral formula for parafermionium would become

$$E + E_B^{\text{exact}} = 2m - \frac{\alpha^2 m}{4n^2} + \frac{\alpha^4 m}{16n^4}\frac{11}{4} + O(\alpha^6) \quad \text{(B10)}$$

in place of the usual perturbative result (34). As can be seen from Eq. (B7), in this case there would appear an additional regular singularity at $r = \alpha/E$ (lying in the physical range of $r > 0$), while the previous regular singularity at $r = 0$ would turn irregular.

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