QUANTIZING SOLITARY WAVES IN FIELD THEORY

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The partial success in interpreting solitary waves, quantized by semiclassical methods, as nucleons in a meson field has obscured fundamental questions of a semiclassical approach. A review of these problems is presented in this paper and one alternative non-semiclassical approach to quantization is discussed.

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1. *Introduction*

There has been a certain amount of success in interpreting quantized solitary wave solutions of a self interacting meson field theory as a nucleon surrounded by a meson field [1–5]. That this is a reasonable objective is seen from two points of view. Witten's conjecture that QCD reduces in a large N_c limit to a scalar field theory suggests that such a limit should produce objects that behave like confined quarks which thus constitute a nucleon [6, 7]. Another insight has been provided by Skyrme who has shown that nonlinear scalar field theories can give rise to soliton solutions which can be interpreted as classical, nucleon-like objects [8]. This idea has been developed to the point of fairly close agreement (around 30%) with experimental properties of the nucleon [9, 10].

The above ideas hinge on quantizing solitary wave solutions to nonlinear field equations by methods based on semiclassical approximation schemes. This paper points out some philosophical objections to this approach and discusses an alternative quantization method. There is no intent herein to claim originality, but rather to encourage alternative thinking regarding the quantization of nonlinear field theories and their interpretation.

2. *Standard approach*

In the path integral formulation of quantum field theory, the propagator is written as a functional integral of the Lagrangian density [11]:

$$G(t', x'|t, x) \equiv \mathrm{Tr}\,[\exp(-iHT/\hbar)] = \int \mathscr{D}[\phi(x, t)] \exp\{S[\phi(x, t)]i/\hbar\}, \tag{1}$$

where $\mathscr{D}[\phi(x, t)]$ is the measure and the action is given by

$$S[\phi(x, t)] = \int_0^T dt \int dx \mathscr{L}[\phi(x, t)], \tag{2}$$

with Lagrangian density $[\mathscr{L}[\phi(x, t)]$.

The above is subject to the boundary conditions

$$\phi(x, 0) = \phi(x, T) = \phi_0(x) \tag{3}$$

and an integration over $\phi_0(x)$ is understood in the expression for the trace. An additional boundary condition depends on the behavior of $\phi(x, t)$ at spatial infinity. Should the condition $\phi(\infty, t) = \phi(-\infty, t) = 0$ not hold true, as is the case for topological solitons, the solution space is divided into unconnected sectors and separate boundary conditions applied to each section.

In order to quantize around solitary wave solutions, the stationary phase approximation is applied to the action potential. In this approach

$$V[\phi] = \int dx[\tfrac{1}{2}(\nabla\phi)^2 - U(\phi)] \tag{4}$$

is expanded in a functional Taylor series around an exact nonlinear classical solution $\phi_0(x, t)$ which is an extremum of the potential [12, 14]:

$$V[\phi] = V(\phi_0) + \int dx \, \tfrac{1}{2} \left\{ \tilde{\phi} \left[-\nabla^2 + \left. \frac{d^2U(\phi)}{d\phi^2} \right|_{\phi_0} \right] \tilde{\phi} \right\} + \text{higher order terms}, \tag{5}$$

where

$$\tilde{\phi}(x) = \phi(x) - \phi_0(x) = \sum_{i=1} c_i(t)\eta_i(x). \tag{6}$$

Here, $\eta_i(x)$ are normal modes; small fluctuations around the classical solution which satisfy

$$\left[-\nabla^2 + \left. \frac{d^2U(\phi)}{d\phi^2} \right|_{\phi_0} \right] \eta_i(x) = \omega_i^2 \eta_i(x). \tag{7}$$

Higher order terms in the expansion of $V[\phi]$ are ignored. This is equivalent to choosing the path which gives the functional integral it's minimum value, the classical solution. The stationary phase approximation is therefore equivalent to a weak coupling condition where fluctuations around the classical solutions are small and harmonic.

The bound state energy levels that result from this approximation are

$$E = V[\phi_0] + \hbar \sum_{i=1} (n_i + \tfrac{1}{2})\omega_i + \text{higher order terms}. \tag{8}$$

This result agrees with results arrived at by other semiclassical methods [15, 16–20]. A review of these alternative approaches can be found in [21]. The interpretation of these results is that a single classical particle with energy $V[\phi_0]$ exists in addition to a series of linear normal mode states which are interpreted as a free meson field. Calculations for several field theories for which exact solitary wave classical solutions are known yield mass formulae which lend support to this interpretation.

Some minor difficulties must be overcome for an interpretation consistent with the above outlined procedure. Renormalization must be performed to rid the theory of divergent terms. There is also the problem of zero mode fluctuations. If ω_i is zero, which occurs under conditions of symmetry of the potential, the assumption that higher order terms can be ignored because they are small obviously breaks down. However, it is possible to reformulate the theory in terms of collective coordinates, representing the center of mass motion of the extended classical object, so that the problem of zero normal modes is avoided [13, 18, 21, 22].

3. *Objections*

A chief objection to the above scheme of quantization is that even though nonperturbative solutions have been utilized, the solution is a perturbative solution. The theory for the mesons remaining after the soliton contribution has been set aside is still a perturbative field theory with all of the associated problems including renormalization. The persistence of the interaction has been ignored as is evident since the in and out fields, the fields evaluated at infinity, are free noninteracting meson fields.

It is also not clear what happens for larger fluctuations around the classical solution. If either the fluctuations or the coupling constants are large, the semiclassical method fails because it depends on ignoring higher order terms in the expansion for the potential. The above approach makes no claims to validity in these regions even though it occasionally yields the exact results (see [23] for comments). Although it is appealing to interpret the soliton as a massive object, it is not reasonable to expect a free linear meson field to be generated from a strongly self-interacting field theory. In classical nonlinear differential equations (after which these field theories are modeled) self coupling results in totally new phenomena (solitary waves) which have entirely different properties than linear waves. In fact these equations do not have any linear solutions at all. This being the case, it seems more reasonable to expect a quantized nonlinear field theory to result in quantum objects which have properties totally unlike their linear quantum counterparts.

4. *Alternatives*

One alternative non-semiclassical quantization scheme has been developed by Burt [24]. In this method a propagator for a self-interacting field theory is constructed using the quantum superposition of states.

For a free scalar field theory the coefficients of a Fourier series expansion are interpreted as creation or annihilation operators acting on a Hilbert occupation space $|n\rangle$ where n labels the number of particles in the state. The probability amplitude for a particle created from the vacuum at x to be found in state $|\varepsilon\rangle$ is $\langle\varepsilon|\phi^{-}(x)|0\rangle$ where $\phi^{-}(x)$ is an exact solution to the field equation which contains a coefficient interpreted as a creation operator.

According to the rules of quantum mechanics, the time ordered sum of probability amplitudes over all intermediate states is the propagator;

$$G(x, x') = \frac{1}{2}\sum_{k,q} \langle 0|\phi_q^+(y)\phi_k^-(x)\theta(x_0-y_0)+\phi_q^+(x)\phi_k^-(y)\theta(y_0-x_0)\,|0\rangle. \tag{9}$$

If exact solutions $\phi^{\pm}(x) = a_q^{\pm}e^{\mp iqx}$ for the free meson theory

$$\Box^2\phi+m^2\phi = 0 \tag{10}$$

are substituted into this expression, the standard free meson Feynman propagator results with the interpretation of $a_q^{\pm}$ as creation/annihilation operators.

The nonlinear self-interacting field equation

$$\Box^2\phi+m^2\phi+\lambda\phi^{2p+1} = 0 \tag{11}$$

has exact solutions (25)

$$\phi^{\pm}(q, x) = \left\{\frac{1-\lambda(A^{\pm})^{2p}\exp(\mp 2ipq\cdot x)}{4(p+1)m^2}\right\}^{-1/p} A^+ \exp(\mp iq\cdot x), \tag{12}$$

where $q^2 = q_0 - \mathbf{q} = m^2$.

The fact that this equation (Eq. (11)) and its exact solution reduce to the free meson theory in the limit $\lambda \to 0$ suggests that the $A^{\pm}$ in the solution be interpreted as containing creation/annihilation operators [26]. With this assumption, the propagator for the self--interacting theory can be constructed with the identical probability amplitude sum over intermediate states in equation (9) [27].

Under this treatment the nonlinear field theory has given rise to a single self-interacting meson field. Advantages of this interpretation are several. In standard perturbation theory, nonlinear terms are assumed to be small and act only for short time periods. The in field and the out field are, at infinity, free, noninteracting fields. In the above interpretation the propagator represent sa self-interacting field, where the interaction is turned on at all times. The resulting theory is intrinsically nonperturbative and does not require renormalization [24]. Interactions with other fields can be added to the theory in which case perturbation theory is re-introduced for these new terms but the interpretation of the meson as a self-interacting intrinsically nonlinear object remains.

That this approach can lead to valid results in practice has been demonstrated in several calculations [28, 30]. A determination of nuclear data such as S_0 phase shifts due to a potential generated from this quantization scheme [29] has been more succesful thus far than the Skyrme model and as succesful as models based on meson exchange such as the Bonn model [31]. It is interesting to note in this connection that mass formula generated by the self-interacting propagator suggests an interpretation of the interaction as an exchange of a sum of mesons of different masses, where the number of masses involved depends on the total energy of collision.

5. *Conclusions*

Perturbation theory has been and continuous to be an extremely useful tool in quantum field theory. However, it is also evident that there are still some problems to overcome. It cannot be expected that perturbation theory will work well for self-interacting theories where the coupling is strong such as the self coupled meson theories discussed above. Solitary waves and other coherent phenomena resulting from nonlinear equations are evidence that at least some nonlinearities are nontrivial and cannot be dealt with using semiclassical approaches. Linear results cannot be expected to derive from nonlinear equations. Clearly more effort must be spent exploring alternative intrinsically nonlinear methods to quantum field theory.

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