HIGHER ORDER CANONICAL FORMALISM FOR THE SCALAR FIELD THEORY

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We quantize canonically the scalar field in the theory with higher derivatives. We show that the results are consistent with those obtained in the theory without higher derivatives.

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1. *Introduction*

Theories whose Lagrangians contain higher order derivatives [1] can be rewritten with lower derivatives provided new degrees of freedom are introduced [2]. An alternative way of dealing with these theories consists in working in enlarged phase space [3] (in the sense of the Hamiltonian formalism). In this case, velocities are also taken as canonical variables.

The purpose of this paper is to show the consistency of this method by quantizing the scalar field in the theory described in terms of higher derivatives. Contrary to the usual case, the theory is now constrained. We shall see that developing the canonical quantization in a consistent way, by using the Dirac formalism [4], the usual quantum commutators are obtained.

Our paper is organized as follows: In Section 2 we present main points of the higher order formalism. The quantization of the scalar field is carried out in Section 3. In Section 4 we analyze the equivalence between the usual and higher order formalism using canonical transformations. Section 5 contains concluding remarks.

2. *The higher order formalism*

Let us consider the following Lagrangian density

$$\mathscr{L} = \mathscr{L}(\varphi, \partial_\mu \varphi, \partial_\mu \partial_\nu \varphi). \tag{2.1}$$

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Cases with derivatives higher than second order can be dealt with in a straightforward way by comparing with what is presented in this section.

The Euler-Lagrange equation of motion is obtained by the usual variational principle. Namely

$$\delta I = \delta \int_{t_1}^{t_2} dt \int d^3x \mathscr{L} = 0, \tag{2.2}$$

where one assumes that at the instants t_1 and t_2 the system is characterized by

$$\delta\varphi(t_1) = \delta\varphi(t_2) = 0, \quad \delta\dot{\varphi}(t_1) = \delta\dot{\varphi}(t_2) = 0. \tag{2.3}$$

The equation of motion is then found to be

$$\frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} + \partial_\mu \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \varphi)} = 0. \tag{2.4}$$

A natural way of introducing canonical momenta in this formalism is to consider variations in the action with only one of the end points held fixed, for instance

$$\delta\varphi(t_1) = 0, \quad \delta\dot{\varphi}(t_1) = 0. \tag{2.5}$$

Variation is to be calculated for the actual motion [5]. The canonical momenta π and s, conjugated to φ and $\dot{\varphi}$, respectively, are given by

$$\delta I = \int d^3x (\pi \delta\varphi + s \delta\dot{\varphi}), \tag{2.6}$$

where $\delta\varphi$ and $\delta\dot{\varphi}$ denote variations at $t_2 = t$. In the case of Lagrangian densities like (2.1) we have

$$\delta I = \int d^3x \left[\left(\frac{\partial \mathscr{L}}{\partial \dot{\varphi}} - \frac{\partial}{\partial t} \frac{\partial \mathscr{L}}{\partial \ddot{\varphi}} - 2\partial_i \frac{\partial \mathscr{L}}{\partial(\partial_i \dot{\varphi})} \right) \delta\varphi + \frac{\partial \mathscr{L}}{\partial \ddot{\varphi}} \delta\dot{\varphi} \right] \tag{2.7}$$

By comparing the last two expressions one obtains

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} - \frac{\partial}{\partial t} \frac{\partial \mathscr{L}}{\partial \ddot{\varphi}} - 2\partial_i \frac{\partial \mathscr{L}}{\partial(\partial_i \dot{\varphi})}, \tag{2.8}$$

$$s = \frac{\partial \mathscr{L}}{\partial \ddot{\varphi}}. \tag{2.9}$$

The next step is to define the canonical Hamiltonian. This is done by following the standard way, that is

$$H_c = \int d^3x (\pi \dot{\varphi} + s \ddot{\varphi} - \mathscr{L}). \tag{2.10}$$

Considering a general variation of H_c and using (2.8) and (2.9) we find that the canonical Hamiltonian is a functional of φ, $\dot{\varphi}$, π and s. (These will play the role of canonical variables in the present formalism.) We can thus obtain the Hamilton equations of motion.

$$\dot{\varphi}(x) = \frac{\delta H_c}{\delta \pi(x)}, \qquad \ddot{\varphi}(x) = \frac{\delta H_c}{\delta s(x)},$$

$$\dot{\pi}(x) = -\frac{\delta H_c}{\delta \varphi(x)}, \qquad \dot{s}(x) = -\frac{\delta H_c}{\delta \dot{\varphi}(x)}. \tag{2.11}$$

Let $A[\varphi, \dot{\varphi}, \pi, s]$ be some dynamical quantity. The total time derivative of A is

$$\dot{A} = \int d^3x \left(\frac{\delta A}{\delta \varphi} \dot{\varphi} + \frac{\delta A}{\delta \dot{\varphi}} \ddot{\varphi} + \frac{\delta A}{\delta \pi} \dot{\pi} + \frac{\delta A}{\delta s} \dot{s} \right). \tag{2.12}$$

Using the Hamilton equations of motion (2.11) we have

$$\dot{A} = \{A, H_c\}, \tag{2.13}$$

where

$$\{A, H_c\} = \int d^3x \left(\frac{\delta A}{\delta \varphi} \frac{\delta H_c}{\delta \pi} + \frac{\delta A}{\delta \dot{\varphi}} \frac{\delta H_c}{\delta s} - \frac{\delta A}{\delta \pi} \frac{\delta H_c}{\delta \varphi} - \frac{\delta A}{\delta s} \frac{\delta H_c}{\delta \dot{\varphi}} \right) \tag{2.14}$$

is the Poisson bracket of A and H_c. For any two functionals A and B, the Poisson brackets will be defined analogously to (2.14).

Expressions (2.11) and (2.13) show the consistency of the canonical Hamiltonian defined by (2.10). An interesting point to be emphasized here is that Eq. (2.8) is not equivalent to $\delta L/\delta \dot{\varphi}$, where L is the Lagrangian of the system. These two expressions are equivalent only when higher order derivatives are not present. If we want to interpret H_c as defined by (2.10) as the generator of time translations via Poisson brackets defined by (2.14), Eq. (2.8) is the appropriate way of defining $\pi(x)$.

The fundamental non vanishing Poisson brackets are

$$\{\varphi(x), \pi(x')\}_{x_0 = x_0'} = \delta^3(\vec{x} - \vec{x}'),$$

$$\{\dot{\varphi}(x), s(x')\}_{x_0 = x_0'} = \delta^3(\vec{x} - \vec{x}'). \tag{2.15}$$

Other brackets as $\{\varphi, \dot{\varphi}\}$, $\{\pi, s\}$ etc. are null at the level of Poisson brackets.

At this stage, the next step is to incorporate constraints into the theory. This is done by following the standard procedure [4, 6]. We will ilustrate this procedure with the example analysed in the next section.

3. *Scalar field theory with higher derivatives*

The scalar field theory is usually described by a Lagrangian density of the form

$$\mathcal{L}' = \tfrac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + V(\varphi). \tag{3.1}$$

To make the theory renormalizable, we take $V(\varphi)$ in the form

$$V(\varphi) = \tfrac{1}{2}\, m^2\varphi^2 + \tfrac{1}{4}\,\lambda\varphi^4. \tag{3.2}$$

Instead of (3.1), let us consider

$$\mathscr{L} = -\tfrac{1}{2}\,\varphi \,\Box\, \varphi + V(\varphi). \tag{3.3}$$

It is obvious that $\mathscr{L}$ and $\mathscr{L}'$ lead to the same classical equations of motion. We shall see that both lead to the same quantum results too. Let us develop the canonical formalism starting from (3.3). Using equations (2.8) and (2.9) we obtain the following canonically conjugate momenta

$$\pi = \tfrac{1}{2}\,\dot{\varphi}, \quad s = -\tfrac{1}{2}\,\varphi. \tag{3.4}$$

Both equations are constraints, which we rewrite as

$$X_1 = \pi - \tfrac{1}{2}\,\dot{\varphi} \approx 0, \quad X_2 = s + \tfrac{1}{2}\,\varphi \approx 0. \tag{3.5}$$

The symbol $\approx$ means weakly equal [4, 6]. The Poisson bracket of X_1 and X_2 is

$$\{X_1(x), X_2(x')\}_{x_0=x_0'} = -\delta^3(\vec{x}-\vec{x}'). \tag{3.6}$$

The primary constraints X_1 and X_2 are of second class. Thus, there are no secondary constraints [4, 6]. The matrix C involving the Poisson brackets of the constraints X_1 and X_2 is

$$C(x, x') = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \delta^3(\vec{x}-\vec{x}'). \tag{3.7}$$

The Dirac bracket definition between any two quantities A and B is [4, 6]

$$\{A(x), B(x')\}^{D}_{x_0=x_0'} = \{A(x), B(x')\}_{x_0=x_0'}$$
$$- \int d^3y\, d^3z \{A(x), X_i(y)\}_{x_0=y_0} C^{-1}_{ij}(y, z)\, \{X_j(z), B(x')\}_{z_0=x_0'}. \tag{3.8}$$

We thus obtain the following fundamental Dirac brackets of the theory

$$\begin{aligned}
\{\varphi(x), \pi(x')\}^{D}_{x_0=x_0'} &= \tfrac{1}{2}\,\delta^3(\vec{x}-\vec{x}'),\\
\{\dot{\varphi}(x), s(x')\}^{D}_{x_0=x_0'} &= \tfrac{1}{2}\,\delta^3(\vec{x}-\vec{x}'),\\
\{\varphi(x), \dot{\varphi}(x')\}^{D}_{x_0=x_0'} &= \delta^3(\vec{x}-\vec{x}'),\\
\{s(x), \pi(x')\}^{D}_{x_0=x_0'} &= -\tfrac{1}{4}\,\delta^3(\vec{x}-\vec{x}').
\end{aligned} \tag{3.9}$$

The remaining brackets vanish. In all these brackets, the constraints can be used in a strong way. It is easy to see that all of them lead to one of the following brackets

$$\{\varphi(x), \dot{\varphi}(x')\}^{D}_{x_0=x_0'} = \delta^3(\vec{x}-\vec{x}'),$$
$$\{\varphi(x), \varphi(x')\}^{D}_{x_0=x_0'} = 0 = \{\dot{\varphi}(x), \dot{\varphi}(x')\}^{D}_{x_0=x_0'}. \tag{3.10}$$

The canonical quantization is carried out in a straightforward way with the $i\hbar$ prescription for the above Dirac brackets. Thus

$$[\varphi(x), \dot{\varphi}(x')]_{x_0=x_0'} = i\hbar\delta^3(\vec{x}-\vec{x}'),$$
$$[\varphi(x), \varphi(x')]_{x_0=x_0'} = 0 = [\dot{\varphi}(x), \dot{\varphi}(x')]_{x_0=x_0'}, \tag{3.11}$$

which are well known results for the scalar field.

4. Further analysis of the equivalence of the two cases

The theory described by the Lagrangian density (3.1) leads to a Hamiltonian formalism in a phase space with two field canonical variables. For the Lagrangian density (3.3), we have a phase space with four field canonical variables and two constraint relations. Hence the degrees of freedom in both cases are the same, as it must be because, as a matter of fact, both Lagrangian densities describe the same theory. Equations (3.11) confirm this point of view.

It may be instructive to develop the case of the Lagrangian density given by (3.1), working also in a phase space with four field canonical variables. In this case we also have constraints and they are

$$X_1' = \pi' - \dot{\varphi} \approx 0, \quad X_2' = s' \approx 0, \tag{4.1}$$

where π' is the canonical momentum conjugate to φ and s' to $\dot{\varphi}$. The constraints given by (4.1) are second class. Following the same steps as before, one obtains the Dirac brackets

$$\{\varphi(x), \pi'(x')\}^{\mathrm{D}}_{x_0=x_0'} = \delta^3(\vec{x}-\vec{x}'),$$
$$\{\varphi(x), \dot{\varphi}(x')\}^{\mathrm{D}}_{x_0=x_0'} = \delta^3(\vec{x}-\vec{x}'); \tag{4.2}$$

other brackets vanish. Again the result is consistent with what we have obtained previously.

We now construct the total Hamiltonian [4, 6] for these two cases. Namely

$$H = H_c + \int d^3x\lambda_i X_i, \quad H' = H_c' + \int d^3x\lambda_i' X_i', \tag{4.3}$$

where λ_i and λ_i' ($i = 1, 2$) are the Lagrange multiplier fields. By imposing the consistency condition that constraints cannot evolve in time, that is to say

$$\dot{X}_i = \{X_i, H\} \approx 0, \quad \dot{X}_i' = \{X_i', H'\} \approx 0, \tag{4.4}$$

one obtains λ_i and λ_i' in terms of the respective canonical variables (We do not obtain new constraints because X_i and X_i' are second class). The result is

$$\lambda_1 = \pi - \tfrac{1}{2}\dot{\varphi} \approx 0,$$

$$\lambda_2 = -(s+\tfrac{1}{2}\varphi)\frac{\partial^2 V}{\partial\varphi^2} - \nabla^2(s+\tfrac{1}{2}\varphi) \approx 0,$$

$$\lambda_1' = \pi' - \dot{\varphi} \approx 0, \quad \lambda_2' = -s'\frac{\partial^2 V}{\partial\varphi^2} - \nabla^2 s' \approx 0. \tag{4.5}$$

Inserting back this result into (4.3), we obtain

$$H = \int d^3x \left[\pi\dot{\varphi} + s\left(\nabla^2\varphi + \frac{\partial V}{\partial \varphi}\right) + \tfrac{1}{2}\varphi \frac{\partial V}{\partial \varphi} - V \right], \tag{4.6}$$

$$H' = \int d^3x \left[(\pi' - \tfrac{1}{2}\dot{\varphi})\dot{\varphi} + (s' - \tfrac{1}{2}\varphi)\left(\nabla^2\varphi + \frac{\partial V}{\partial \varphi}\right) + \tfrac{1}{2}\varphi \frac{\partial V}{\partial \varphi} - V \right]. \tag{4.7}$$

The (canonical) transformation

$$\pi' = \pi + \tfrac{1}{2}\dot{\varphi}, \qquad s' = s + \tfrac{1}{2}\varphi, \tag{4.8}$$

which is compatible with the constraint relations (3.5) and (4.1), leads to $H' = H$. Thus, in the 4-dimensional phase space, the theories related to $\mathscr{L}$ and $\mathscr{L}'$ are connected by a canonical transformation.

5. *Conclusion*

We have studied the higher order canonical formalism for the scalar field theory using the theory of constrained systems in an enlarged phase space. We have shown the consistency of the method by obtaining the well known quantum commutators for the scalar field. We have also shown that in this higher dimensional phase space there is a canonical transformation linking the usual formalism to the higher order one.

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