# A COMMENT ON THE BERRY AND AHARONOV-ANANDAN EFFECTS

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We generalize the notion of the Aharonov-Anandan phase and discuss its relation to the Berry matrix defined by Wilczek and Zee and by Giler et al.

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## 1

Much attention has been paid recently to the notion of the so-called Berry Phase (BP) [1]. Its applications cover a very wide range of quantum-mechanical phenomena including anomalies, superfluidity and many others.

In the original Berry paper the BP was studied within the framework of adiabatic approximation. However, as it was noticed by Aharonov and Anandan [2], one can relax the assumption about the validity of adiabatic approximation still retaining nontrivial phase factor called afterwards the Aharonov-Anandan Phase (AAP).

It has been shown [3–5] that BP as well as AAP result from the geometry of the manifold of rays in the Hilbert space of states. Namely, if the space of states is $N+1$ dimensional then they can be viewed as resulting from the U(1) part of the Riemannian connection of Fubini-Study metric on $CP^N$ [4–5].

In Ref. [5] a more detailed analysis of the geometrical properties of BP has been given. In particular, the generalization of BP to the case of degenerate energy eigenvalues has been discussed; in this case the BP is replaced by a unitary matrix — the Berry Matrix (BM). It can be proved that the BM results from the parallel transport over the manifold which may always be chosen to be a Grassman manifold (see below).

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The aim of this note is to discuss the relationship between the BM and the corresponding generalization of AAP — the Aharonov-Anandan Matrix (AAM). We show that both notions are in fact equivalent. We also generalize the results of Page [4] concerning the relationship between AAP and the Riemannian structure on $CP^N$.

## 2

Let us recall the notion of AAP. Given any time-dependent unit[1] state vector $|\psi(t)\rangle$ which undergoes a cyclic motion for $0 \leqslant t \leqslant T$, $|\psi(T)\rangle = e^{i\phi}|\psi(0)\rangle$ we define the AAP by the equation

$$\beta = \phi + i \int_0^T dt \langle\psi(t)| \frac{d}{dt} |\psi(t)\rangle.$$

This definition is invariant under the change of phase $|\psi(t)\rangle \to e^{i\lambda(t)}|\psi(t)\rangle$ proving that $\beta$ results from the projective structure of the space of states.

The notion of AAP may be generalized as follows. Let the time-dependent orthonormal frame $\{|\psi_i(t)\rangle\}_{i=1,\dots,d}$ undergoes a cyclic motion for $0 \leqslant t \leqslant T$ in the sense that

$$|\psi_i(T)\rangle = U_{ij}|\psi_j(0)\rangle,$$

where $U$ is an unitary $d \times d$ matrix. Then we define the Aharonov-Andan Matrix (AAM) as follows

$$B = U^T \mathcal{T}(\exp(-\int_0^T dt A(t))).$$

Here $\mathcal{T}$ denotes the chronological product and the matrix $A(t)$ is defined by

$$A_{mn}(t) \equiv \langle\psi_m(t)| \frac{d}{dt} |\psi_n(t)\rangle.$$

Under the redefinition $|\psi_i(t)\rangle \to V_{ij}(t)|\psi_j(t)\rangle$ with an arbitrary unitary $V(t)$ the matrix $B$ transforms as follows

$$B_{ij} \to \bar{V}_{ik}(0) B_{kl} V^T_{lj}(0).$$

Therefore the matrix $B$ represents an unitary operator $\hat{B}$ acting in the subspace spanned by the vectors $|\psi_i(0)\rangle$ which depends only on the "flow" of the time-dependent subspace spanned by the vectors $|\psi_i(t)\rangle$ but not on the specific choice of the orthonormal frame $\{|\psi_i(t)\rangle\}_{i=1,\dots,d}$.

Alternatively, one might define the matrix

$$B' = \mathcal{T}(\exp(-\int_0^T dt A(t))) \cdot U^T$$

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[1] it is only important to demand that the norm of $|\psi(t)\rangle$ is time-independent.

which transforms according to the rule

$$B'_{ij} \to \bar{V}_{ik}(T) B_{kl} V^{T}_{lj}(T).$$

It is easy to check that $B$ represents the same operator $\hat{B}$ expressed in the basis $\{|\psi_i(T)\rangle\}$.

### *3*

It has been noticed in Ref. [5] that the BM does not depend on dynamics; the values of energies are completely irrelevant[2]. What really counts is the behaviour of the energy eigenspaces as a function of time. For this reason the BM is of purely geometric origin. The AAM also depends only on the geometry of the manifold of subspaces in the space of states. Therefore it is not surprising that both notions are related. We shall show that the BM is nothing but AAM calculated for the cyclic "flow" of the energy eigenspace corresponding to the energy level under consideration. To this end we use the formalism of Wilczek and Zee [6]. Let $\{|n_i(t)\rangle\}_{i=1,\dots,d}$, be an orthonormal basis in the eigenspace corresponding to the energy $E_n(t)$. We look for the solution to the Schroedinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)\,|\psi(t)\rangle$$

in the form

$$|\psi(t)\rangle = \exp\Big(-i\int_0^t d\tau E_n(\tau)\Big) \sum_{i=1}^{d} a_i(t)\,|n_i(t)\rangle.$$

Multiplying Eq. (2) by $\langle n_j(t)|$ one obtains

$$\dot{a}_j(t) = -A_{ji}(t) a_i(t), \qquad A_{ji}(t) \equiv \langle n_j(t)|\frac{d}{dt}|n_i(t)\rangle$$

or

$$a_j(T) = \mathscr{T}\Big(\exp\Big(-\int_0^T dt A(t)\Big)\Big)_{ji} a_i(0).$$

Therefore

$$|\psi(T)\rangle = \exp\Big(-i\int_0^T dt E_n(t)\Big)\mathscr{T}\Big(\exp\Big(-\int_0^T dt A(t)\Big)\Big)_{ij} U_{ik} a_j(0)\,|n_k(0)\rangle$$

and we conclude that $U^T\mathscr{T}(\exp(-\int_0^T dt A(t)))$ is nothing but the BM. Analogously, $\mathscr{T}(\exp(-\int dt A(t)))U^T$ is the BM expressed in the basis $\{|n_i(T)\rangle\}$.

### *4*

Conversely, the AAM for any cyclic quantum process can always be expressed as a BM for a suitably chosen hamiltonian. This is in fact trivial. Assume we have calculated the AAM for some cyclic process described by a time-dependent orthonormal frame

[2] Only the accuracy of adiabatic approximation depends on the behaviour of energies.

$\{|\psi_i(t)\rangle\}$. Let $\Pi(t)$ be the projector onto the subspace spanned by $\{|\psi_i(t)\rangle\}$ and let $\Pi_\perp(t)$ be its orthonormal complement in the space of states; then obviously $\Pi(T) = \Pi(0)$, $\Pi_\perp(T) = \Pi_\perp(0)$. Define the hamiltonian

$$H(t) \equiv E(t)\Pi(t)+E_\perp(t)\Pi_\perp(t)$$

where $E(t)$, $E_\perp(t)$ are arbitrary real functions such that $E(0) = E(T)$, $E_\perp(0) = E_\perp(T)$, $E(t) \neq E_\perp(t)$. Using the result of the previous point we conclude easily that the AAM under consideration is nothing but the BM coresponding to the energy $E(t)$.

One can combine the above result with the ones contained in Ref. [5] to get insight into the structure of AAM. Let us assume that the space of states is finite-dimensional, its dimension being $N$. We consider the cyclic motion of a $d$-dimensional frame $\{|\psi_i(t)\rangle\}$ and interprete the AAM as the BM in a way described above. Then, according to the results of Ref. [5]:

($i$) the motion under consideration is represented by a closed curve $\gamma$ in the symmetric space $\Gamma = \mathrm{U}(N)/\mathrm{U}(d)\times \mathrm{U}(N-d)$ — the Grassman manifold;

($ii$) the AAM may be expressed in terms of parallel transport along the curve $\gamma$, the connection under consideration is the $\mathrm{U}(d)$ part of the Riemannian connection on $\Gamma$.

We will not repeat here the arguments leading to ($i$) and ($ii$) — for details we refer to [5]. Let us only note that in the case of AAP ($d = 1$) we obtain the result given by Page [4].

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