CHIRAL ANOMALIES IN STRING THEORY*

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(*Received October 10, 1988*)

A brief review is presented of recent developments in the understanding of the cancellation of gauge and gravitational anomalies in string theory, in relation to modular invariance.

PACS numbers: 11.10.Lm

One of the remarkable advantages of string theory over field theory is that it can not only accommodate chiral fermions, but also knows how to avoid the chiral anomalies that are usually a consequence of their presence. The fact that chiral anomalies cancel in seemingly mysterious ways in certain ten-dimensional string theories was first observed in [1] and [2], and the "miracles" were first understood in terms of modular invariance in [3]. This understanding has become even more necessary with the recent proliferation of new chiral string theories in 10 and fewer dimensions. In this talk I will give a very brief review of the results of [3], and discuss a few more recent developments.

The basic idea of [3] was to isolate the chiral states of a string theory, and write down a formula for their Chern characters. If one factorizes the full one-loop partition function $\mathscr{P}_{\rm H}$ of a d-dimensional heterotic string into the contribution of the light-cone NSR-fermions ψ^i and the remainder, one gets an expression of the following form (Here τ is associated with left-movers, representing the bosonic sector of the heterotic string)

$$\mathscr{P}_{\rm H} = \sum_{i=1}^{4} \vartheta_i^{d/2-1}(0|\bar{\tau})P_i(\tau, \bar{\tau}). \tag{1}$$

It is easy to see that the chiral states contribute to the partition function only via the Jacobi ϑ-function ϑ_1. In fact $\vartheta_1(0|\bar{\tau})$ vanishes precisely for that reason: it receives equal but opposite contributions from each chirality. The *chiral partition function* is defined to be the factor of the zero-modes of the ϑ_1-functions, and can be calculated by taking in (1)

* Presented at the XXVIII Cracow School of Theoretical Physics, Zakopane, Poland, May 31 — June 10, 1988.

$\vartheta_1(\varepsilon|\bar{\tau})$ in the limit $\varepsilon \to 0$. The chiral partition function is thus simply

$$\mathscr{P}_{\text{chiral}} = [\vartheta_1'(0|\bar{\tau})]^{d/2-1} P_1(\tau, \bar{\tau}). \tag{2}$$

It can be shown that, as a consequence of world sheet supersymmetry, the chiral partition function depends only on τ, and not on $\bar{\tau}$. Hence all excitations of the chiral ground state cancel completely in the right-moving sector.

The chiral partition function has a simple interpretation. It can be expanded in terms of $q = e^{2i\pi\tau}$:

$$\mathscr{P}_{\text{chiral}} = \sum_{l=-1}^{\infty} d_l q^l. \tag{3}$$

The coefficient of q^l is the chiral multiplicity of states with left mass $\frac{1}{2}m_L^2 = l$ and right mass $m_R = 0$ (the chiral multiplicity is the number of states of one chirality minus the number of states of the other chirality). Note that only the $l = 0$ terms represent "physical" states because of the closed string condition $m_L = m_R$. All other terms represent states that cannot propagate to infinity, but that do contribute to loop diagrams.

To study anomalies one has to extend the chiral partition function to a function that contains more information, by replacing d_l in (3) by the Chern character of the gauge and Lorentz representation at the l^{th} level (similar *character-valued partition functions* were already used many years ago by Nahm [4], although for an entirely different purpose). Such a function will enable us to use the relation between anomalies and the index theorem to extract the anomaly. Consider a Weyl fermion in $2p+2$ in the representation $r = s^{\pm} \otimes t \otimes g$, where $s^{\pm}$ is a fundamental Lorentz spinor of positive or negative chirality, t a tensor representation of the transverse Lorentz-group and g a representation of some gauge group G. The contribution of such a fermion to the anomaly can be derived from

$$\pm \hat{A}(R) \operatorname{Tr} \exp\left(\frac{iR_t}{2\pi}\right) \operatorname{Tr} \exp\left(\frac{iF_g}{2\pi}\right), \tag{4}$$

where $\hat{A}(R)$ is the Dirac genus, and the other two factors are respectively the Chern characters of the Lorentz-representation and the gauge representation of the fermion (R_t and F_g are respectively the curvature and gauge two-forms in the representations t and g). The Dirac genus can be expressed in terms of the skew eigenvalues of R in the vector representation, i.e. if

$$\frac{R}{2\pi} = \operatorname{diag}(ix_1, -ix_1, \ldots, ix_{p+1}, -ix_{p+1})$$

then

$$\hat{A}(R) = \prod_{i=1}^{p+1} \frac{x_i/2}{\sinh(x_i/2)}.$$

To derive the anomaly one expands (4), and determines the coefficients of all terms of total order $p+2$ in F and R, summed over all fermions in the theory. The reason for using representations of the *transverse* Lorentz group is that this automatically takes care of all ghost contributions that one would otherwise have to include. Effectively, this means that we only consider anomalies of an SO(2p) subgroup of the Euclidean Lorentz-group SO(2p+2), i.e. we set $x_{p+1} = 0$. One can covariantize the result afterwards, if desired.

The *character-valued* chiral partition function is defined as follows

$$\mathscr{P}_{\text{chiral}}(F, R|\tau) = \sum_{l=-1}^{\infty} \sum_{r(l)} \varepsilon[r(l)] q^l \operatorname{Tr}\left[\exp\left(\frac{iF_g}{2\pi}\right) \exp\left(\frac{iR_t}{2\pi}\right)\right]_{r(l)},$$

where the first sum is over all levels, and the second one over all representations $r(l)$ at a given level. To count opposite chiralities with opposite signs, εr is $+1$ if r constructed from s^+ and -1 if it is constructed from s^-.

We define now the *anomaly generating function* $\mathscr{A}$ as follows

$$\mathscr{A}(F, R|\tau) = \hat{A}(R)\mathscr{P}_{\text{chiral}}(F, R|\tau).$$

This anomaly generating function contains far more information than is needed for the anomaly in the effective field theory. All we need is the coefficient function $C(\tau)$ of some term of order $p+2$ in F and R in the Taylor expansion of $\mathscr{A}$ (in 10 dimensions ($p = 4$) examples of $C(\tau)$ are the coefficient functions of $\operatorname{Tr} F^6$ or $\operatorname{Tr} F^4 \operatorname{Tr} R^2$); furthermore we are only interested in the massless fermions, whose contribution is the coefficient of q^0 in such a function $C(\tau)$, viewed as a function of q.

The character-valued chiral partition function consists of two parts, the contribution of the left-moving oscillators α_n^i, which build the transverse Lorentz representations t, and the contribution of all other left-moving world sheet degrees of freedom. The latter include in general some gauge current, coupling to the gauge representation g mentioned above. The transverse oscillator contributions are the same in any string theory, and depend only (in a trivial way) on the space-time dimension. For their contribution one gets without much effort the following factor in the chiral partition function (with $x_{p+1} = 0$)

$$\mathscr{P}_{\text{B}}(R) = \prod_{i=1}^{p} \frac{2i \sinh(x_i/2)}{\vartheta_1\left(\frac{ix_i}{2\pi}\middle|\tau\right)} \tag{5}$$

which combines nicely with the Dirac genus when substituted in the anomaly generating function. Notice that this function reduces to the usual light-cone partition function for $2p$ bosons in the limit $x_i \to 0$.

We have now obtained the complete R-dependence of the anomaly generating function. Unlike the R-dependence, the gauge dependence is by no means universal: it depends on the precise construction of the theory. In [3] the gauge dependence was computed for string theories constructed out of free fermions (see e.g. [5–7]) with arbitrary boundary conditions, and in [8] it was computed also for ten-dimensional string theories constructed with self-dual lattices (the result of [8] has a straightforward generalization to the covariant lattice construction of four-dimensional chiral string theories [9]). It was pointed out by Ginsparg, Moore and Vafa (see [10]) that (for non-abelian gauge groups) the gauge part of the character-valued partition function can be expressed in terms of the Weyl-Kac character formula [11]. This formula gives, by definition, precisely what was defined above as the character-valued partition function. The left-moving gauge currents in a heterotic string generate a Kac-Moody algebra, and the partition function is assembled out of the Weyl-Kac characters of the representations of that Kac-Moody algebra. For the special case of simply laced level-1 Kac-Moody algebras the result reduces to the one obtained explicitly for lattice theories. If the gauge group consist partly out of U(1) factors, one can write their contribution in terms of character-valued lattice partition functions, and use Kac-Moody characters for the remainder. (In general the Sugawara tensor of the gauge currents will not provide the complete central charge of the right-moving sector, so that there will be additional F-independent factors in the partition function, corresponding to the remaining central charge).

The crucial point in the argument is now that even though $\mathscr{A}$ may have a complicated structure, it must have a simple behaviour under modular transformations. This follows from the fact that the chiral partition function transforms as a modular function of weight $-p$:

$$\mathscr{P}_{\text{chiral}}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-p}\mathscr{P}_{\text{chiral}}(\tau). \tag{6}$$

From the explicit form of the gravitational factor (5) and from the transformations properties of the Weyl-Kac characters one sees that the character parameters F and R only affect the modular transformation by a universal exponential factor. Hence one can prove that (6) generalizes to

$$\mathscr{A}\left(\frac{F}{c\tau+d}, \frac{R}{cr+d}\,\middle|\,\frac{a\tau+b}{cr+d}\right)$$

$$= (c\tau+d)^{-p}\exp\left[\frac{ic}{32\pi^3(c\tau+d)}\left(\left(\sum_\alpha k_\alpha \operatorname{Tr} F_\alpha^2\right)-\operatorname{Tr}(R^2)\right)\right]\mathscr{A}(F, R|\tau). \tag{7}$$

Here $\operatorname{Tr}(R^2)$ is to be evaluated in the vector representation. The sum is over all factors in the gauge-group, and k_α is the Kac-Moody level of a factor. The trace $\operatorname{Tr}(F_\alpha^2)$ is to be evaluated on any representation, but with the representation matrices T^a normalized so that $\operatorname{Tr}(T^aT^b) = 2\delta^{ab}$. For U(1) factors one normalizes the charge in the same way, and chooses $k = 1$. The phase factor is usually referred to as the *modular anomaly.*

This formula generalizes the one obtained in [3] to higher level Kac-Moody algebras, and should therefore be valid for any string theory that has been constructed so far. The dependence on the Kac-Moody level agrees with what one finds in cases where the gauge group can be regarded as a sub-algebra of a level-1 algebra. The derivation of the modular transformation properties of the anomaly generating function given in [3] and [8] never assumes that there are gauge fields in the string spectrum that correspond to F. In particular it may happen that only a sub-algebra of the level-1 algebra is gauged. One can then re-express the result in terms of traces over the gauged sub-algebra, and in doing so one finds that $\mathrm{Tr}\, F^2$ is scaled by the embedding index k of the sub-algebra. But the embedding index is equal to the level of the Kac-Moody sub-algebra, so that one obtains precisely the factors k_a indicated in (7). A simple example where this happens is gauge symmetries in type-II theories [12].

It is easy to see that because of the rescalings of F and R in (7), all anomaly coefficient functions $C(\tau)$ transform as follows

$$C\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 C(\tau),$$

if we ignore the modular anomaly factor. Hence $C(\tau)$ is a modular function of weight two. Theorems on modular functions tell us then that it must be a τ-derivative of a weight zero modular function

$$C(\tau) = \frac{d}{d\tau} P(\tau).$$

It follows immediately that $C(\tau)$ does not have a constant term, so that there is no anomaly.

This argument is not valid for those terms that are affected by the modular anomaly. But those terms are proportional to $\sum k_\alpha \mathrm{Tr}\, F_\alpha^2 - \mathrm{Tr}\, R^2$, so that the remaining anomaly must contain these terms as an overall factor.

The remaining anomaly should be cancelled by the Green-Schwarz mechanism. This mechanism requires two terms to be present in the effective action. One is the familiar Chern-Simons terms, which is the same in any string theory, and in particular independent of the space-time dimensions. It can be derived from string tree diagrams [13], and yields vertices with one $B_{\mu\nu}$ and two gauge bosons or gravitons. The other term results from a loop diagram, is also linear in $B_{\mu\nu}$, but has a total of p external gauge-bosons and gravitons. Connecting these vertices with a B-propagator one obtains diagrams with the same number of external gauge bosons and gravitons as the polygon anomaly diagrams, which cancel the polygon anomalies up to local counterterms.

To verify this one can calculate this *anomaly cancelling term* by taking the zero--momentum limit of the string loop diagram. This calculation has been done in [14], and one obtains the following expression for the resulting effective action

$$S = \int d^{2p+2}x(-4gB) \wedge \left(\frac{-1}{64\pi^2} \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \tilde{\mathcal{A}}(F, R|\tau, \bar{\tau})\Big|_{2p\text{-forms}}\right). \tag{8}$$

The function $\mathcal{A}$ appearing here is almost the anomaly generating function, but not quite:

$$\tilde{\mathcal{A}}(F, R|\tau, \bar{\tau}) = \exp\left[\frac{1}{64\pi^3} \frac{1}{\mathrm{Im}\,\tau}\left(\mathrm{Tr} \sum k_\alpha F_\alpha^2 - \mathrm{Tr}\, R^2\right)\right] \mathcal{A}(F, R|\tau).$$

(So far, this result has only been derived for covariant lattice theories, but the generalization should be straighforward.). The extra non-holomorphic factor cancels the modular anomaly, which is clearly necessary since otherwise the restriction to the fundamental modular domain $\mathscr{F}$ in (8) would not be well-defined. However, this restoration of modular invariance can only be obtained at the price of giving up holomorphicity, a property which is not needed in (8).

One can now evaluate the modular integral in (8), because the integrand turns out to be a total $\bar{\tau}$ derivative. As a result one finds that factor of $B_{\mu\nu}$ in the anomaly cancelling term is precisely the same as the factor of $\mathrm{Tr} \sum k_\alpha F_\alpha^2 - \mathrm{Tr}\, R^2$ in the polygon anomaly, with the correct overall normalization to cancel it.

This verifies complete anomaly cancellation in the low-energy limit of string theory. Of course there is a more direct approach, which is not limited to momenta much smaller than the Planck scale, and that is to compute the polygon graphs directly in string theory. Such calculations have been presented in several papers [15]. Because in heterotic string theories there is just one diagram, it should vanish by itself. This makes the calculation somewhat unsatisfactory, because one is certain to find zero even if one misses the anomaly altogether.

The relation with the effective field theory approach sketched above, is that the effective field theory contributions come from two limits of the domain of integration of the one-loop graph: the fermion polygon loops in field theory correspond to the limit $\tau \to i\infty$ of the string loop diagram, whereas the B-exchange diagram comes from the coincidence limit of two graviton or gauge-boson vertices. In this limit the diagram can be factorized on a $B_{\mu\nu}$ pole. This interpretation of the field theory diagrams in terms of the string diagram has been verified in [16].

The anomaly generating function has been obtained in [3] by assembling it level-by--level. Although this is quite easy, there is another way of getting the result, which is more satisfactory from a mathematical point of view. Just as one can derive field theory anomalies from the index of the Dirac-operator, one can derive the generating function for string anomalies from the index of the Dirac-Ramond operator. This index can be calculated by means of path-integrals of supersymmetric sigma-models [17–19], just as one can derive the index theorem using supersymmetric quantum mechanics (see e.g. [20]). These should be yet another way to discuss anomalies in string theory, namely by studying the sigma--model in gauge and gravitational background fields on a torus, using the methods of [21].

The anomaly generating function discussed above has made its appearance simultaneously in mathematics and physics, although it is used to attack different kinds of problems. In the mathematics literature (see e.g. [22]) the gravitational part of this function is referred to as the *elliptic genus*. For a discussion of this subject that is accessible to physicist see [18].

In physics the elliptic genus has so far had few applications beside anomaly cancellation. One exception is [23], where it is used to compute holomorphic amplitudes in supersymmetric string theories. In such theories there are certain amplitudes with one more external line than the non-renormalization theorems would allow, which do not vanish at one loop, but have a holomorphic integrand. Using the integrals evaluated in [14], some of these amplitudes can be calculated exactly in a the low-energy limit, as was shown in [24, 25]. In [23] it was shown that these holomorphic amplitudes can be derived from the elliptic genus, by making a "triality" rotation that takes the Green-Schwarz fields S^a to NSR-fields ψ^μ in the PP-sector (i.e. with periodic boundary conditions along both homology cycles of the torus).

I conclude with a few remarks about four dimensional strings. In four dimensions one can have anomalies, but only if U(1) factors are present with a non-vanishing trace, since the anomaly has the form $(\sum k_\alpha \operatorname{Tr} F_\alpha^2 - \operatorname{Tr} R^2) \operatorname{Tr} F$. It has been shown in [26] that the presence of such a fermion loop anomaly (which of course is cancelled by the B-field) leads to a Fayet-Illiopoulos breaking of supersymmetry. A closely related phenomenon, caused by the anomaly cancelling term, is a Higgs-mechanism that generates a mass for the U(1) gauge boson [27]. The anomaly cancelling term in four-dimensions has just two external legs, the U(1) gauge bosons and the anti-symmetric tensor $B_{\mu\nu}$, which in four dimensions is equivalent to a scalar, and which becomes the longitudinal component of the massive vector boson. Such supersymmetric vacua with a fermion loop anomaly are not stable in four dimensions, but fortunately there are many vacua for which there is no such non-traceless U(1) factor. For example, if one has a (level-1) E_8 factor in the gauge group, one can easily see from the general form of the anomaly that no such U(1)'s can be present (the sum in $\sum k_\alpha \operatorname{Tr} F_\alpha^2$ is over all factors in the gauge group, but there is no non-trivial E_8 representation that can have a U(1) charge and still be massles). This class includes all the so-called (2, 2) string theories.

In conclusion, the present understanding of anomaly cancellation in string theory is fairly satisfactory, and covers all sensible string constructions. It is often useful to remember that the anomalies do not just cancel, but that the fermion loop contribution has a very specific form. This is often a useful check on fermion spectra, especially in some of the less transparent string constructions (indeed, there exist several papers with examples that are manifestly not anomaly free).

I would like to thank the organizers of the school for the invitation, Wolfgang Lerche, Bengt Nilsson and Nick Warner for several pleasant collaborations, J. Thierry-Mieg for a discussion that started my interest in string anomalies, C. Vafa and P. Ginsparg for bringing their observation regarding the Weyl-Kac character formula to my attention, P. Goddard for pointing out the existence of Ref. [4], and A. Tseytlin for a useful discussion.

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