TRIANGULATED RANDOM SURFACES AS A REPRESENTATION OF NON-CRITICAL STRINGS*

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The theory of strings should be the theory of random surfaces. I discuss the present understanding of this in view of the new results of Knizhnik, Polyakov and Zamolodchikov.

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1. Introduction

The success of the superstring theory may tend to make us forget how incomplete an understanding we have of the quantum theory of strings. This is true even for the bosonic string. This fact is highlighted by the recent result of Knizhnik, Polyakov and Zamolodchikov where they, using the techniques from conformal field theory, have been able to carry out the old program of quantizing the Liouville mode[1]. In fact they did much more. They solved a large class of conformal field theories coupled to 2D gravity. However, their solution becomes complex when the central charge $c > 1$. For the ordinary bosonic string it means that the dimensions of space-time have to be $\leqslant 1$. We are still left with the problem of understanding the bosonic string in *physical* dimensions, that is for $d > 1$. On the other hand the results obtained are very encouraging because they establish the first connection between the formal continuum expression for the partition function of strings and the rigorous defined sum over discretized random surfaces. Such a connection has been very useful in the case of field theory and random walks. The free field theory can be obtained as the scaling limit of a statistical theory of random walks [3, 4]. Interacting field theories can now be described as a theory of intersecting random walks [5] and many results, for instance the triviality in dimensions $d > 4$, are conveniently derived in this representation. For strings one expects the situation to be even better since the interaction between strings are severely restricted by pure geometry. From the point of view of random surfaces one does not expect any additional complications arising from the interaction.

In the following I will define the statistical theory of random surfaces and discuss how it is related to the continuum string theory.

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[1] It should be mentioned that the results relevant for the bosonic string can be found in earlier works [2].

2. *Definition of the regularized model*

Before defining the regularized random surface model it might be instructive to describe the corresponding random walk model [3, 4]. The (ill-defined) Feynman path integral is regularized by summing over all piecewise linear random walks. The (unnormalized) transition function for moving on a straight trajectory from $x \to y$ is

$$t_\beta(x \to y) = \exp(-\beta|x-y|^p) \tag{1}$$

and the regularized propagator is defined as

$$G_\beta(x, y) = \sum_{N=0}^{\infty} \int dx_1 \ldots dx_N t_\beta(x \to x_1) \ldots t_\beta(x_N \to y). \tag{2}$$

Since the integrand in (2) is a convolution the central limit theorem leads for $p \geqslant 1$ to

$$G_\beta(x, y) \sim \sum_N \left(\frac{c_0}{\beta^{d/2}}\right)^N \frac{\exp(-\beta|x-y|^2/(2\sigma_0 N))}{(2\pi\sigma_0 N)^{d/2}}, \tag{3}$$

where c_0 and σ_0 are the 0^{th} and 2^{nd} moment of t. From (3) it follows that $G_\beta(x, y)$ is well defined for $\beta > \beta_c = c_0^{2/d}$ and divergent for $\beta < b_c$. b_c is the critical point where we want to take the continuum limit. Right at the critical point all excitations are massless. The masses of the theory are introduced by the fine tuned approach to the critical point as follows: We define a *physical* length scale $a(\beta)$, going to zero at β_c in a well defined way

$$m_{\text{phys}}^2 a^2(\beta) = d/2 \log(\beta/\beta_c). \tag{4}$$

With this definition (3) can be converted into the proper time integral representation of the free propagator when $\beta \to \beta_c$. Note the universality: (almost) any transition function (1) leads to the same propagator when we approach the critical point. The universality is even larger. Instead of piecewise linear walks one could have considered random walks on a lattice.

In the case of random surfaces we want to imitate the same construction. The space-time points x and y are replaced by loops γ and γ' (in the case of closed strings). For the free string the "path" integral should be over all surfaces with trivial topology (i.e. no handles) and with boundary $\gamma \cup \gamma'$. This is trivially generalized to the case of n boundary loops $\gamma_1, \ldots, \gamma_n$. Interactions are introduced in a very natural way by summing over surfaces of different topology. The loop Green function for the (closed) string, in the formulation of Polyakov, is then given by

$$G(\gamma_1, \ldots, \gamma_n) = \int dg_{ab} \int_{\gamma_1 \cup \ldots \cup \gamma_n} \mathscr{D}x \exp\left(-\beta \int d^2\xi \sqrt{g}\, \partial_\alpha x \partial^\alpha x\right). \tag{5}$$

As for the ordinary path integral in field theory this is a formal expression that needs to be regularized. The choice is not so obvious as for random walks, because of the integration over metrics. The first attempt to regularize (5) as a statistical sum over random surfaces was by Durhuus, Fröhlich and Jonsson [6]. They chose to sum over all random surfaces on a hypercubic lattice and used the Nambu-Goto area action. On a lattice the action is then proportional to the number of plaquettes constituting the surface. Using some clever universality arguments they could essentially solve the model, but found somewhat disappointing that the theory was essentially a theory of free fields. I will later return to this result.

In [7] it was attempted to use triangulated surfaces in $\boldsymbol{R}^d$ with fixed internal connectivity and again the action was taken to be the Nambu-Goto area action. The area was defined as the sum of the areas of the individual triangles making up the surface. Afterwards it was shown, however, that this model only exhibits an acceptable behaviour in the limit where the dimension of space-time d was strictly infinity [3]. In general the partition function was not well defined because the area action does not suppress "spiky" configurations.

In [3], and independently in [8, 9], an alternative regularized model, closer in spirit to the formal expression (5) of Polyakov, was defined. The analog of the loop Green function (5) is

$$G_\beta(\gamma_1, \ldots, \gamma_n) = \sum_{T \in T} \varrho(T) \int \prod_{i \in T/\partial T} dx_i \exp\left(-\beta \sum_{\langle ij \rangle} (x_i - x_j)^2\right). \tag{6}$$

The boundary loops $\gamma_1, \ldots, \gamma_n$ are approximated by polygonal loops which constitute the boundary ∂T of T. The integration over metrics is replaced by summation over triangulations $T \in \mathscr{T}$, where $\mathscr{T}$ is a suitable class of triangulations with trivial topology. If the integral (5) is extended to include higher genus surfaces, the same can be done in (6) by simply enlarging the class of triangulations $\mathscr{T}$. Finally, the weight $\varrho(T)$ should be chosen appropriately (See [3, 15, 17] for a detailed discussion).

The idea behind the regularized version of the string loop Green functions given by (6) is that it represents an *explicit* invariant sum over all relevant surfaces. No reparametrizing invariance is to be taken care of by gauge fixing. It should be understood as a kind of Regge calculus. This point of view can be made slightly more transparent by assigning an internal length of one to each link in a triangulation. Thereby each vertex (by standard Regge calculus) will carry an internal curvature of

$$R_i = \frac{\pi}{3}(n_i - 6), \tag{7}$$

where n_i is the number of links joining the vertex. The area element of each vertex will be

$$A_i = \sqrt{g_i} = \tfrac{1}{3} n_i. \tag{8}$$

With these assignments the action

$$\sum_{\langle ij \rangle} (x_i - x_j)^2 \tag{9}$$

will, for a given triangulation, be the discretized analog of

$$\int d^2\xi \sqrt{g}\, \partial_\alpha x \partial^\alpha x. \tag{10}$$

However, there is no obvious weak coupling limit where this is correct. In this respect the discretization is different from, for instance, the lattice regularization of gauge theories. The reason for this difference is the summation over the large class of triangulations.

Let us mention the basic properties of the regularized model defined by (6) (see [3] for details). As for the random walk there exists a critical point $\beta_c > 0$, such that the sums defining the Green functions are convergent for $\beta > \beta_c$ and divergent for $\beta < \beta_c$. The critical point is independent of the given boundary, but depends of course of the chosen weigth $\varrho(T)$. It is also independent of the genus of the surface. In this way there exists a perturbation expansion in genus when $\beta > \beta_c$. It is easy to see that the corresponding series is divergent, and not even Borel summable. From this point of view our regularization is very different in nature from the lattice regularization of field theory. The latter can be used to define the theory non-perturbatively. For random surfaces that is not the case. For a fixed genus we have a perfectly well defined theory and we even have a formal perturbative expansion in genus. The sum makes no sense, however.

The proof of this is fairly simple. It is not difficult to prove that the integral in (6) can be bounded by

$$\beta^{-Nd/2}C_1^N < \int \prod_{i \in T/\partial T} dx_i \exp\left(-\beta \sum_{\langle ij \rangle} (x_i - x_j)^2\right) < \beta^{-Nd/2}C_2^N, \tag{11}$$

where C_1 and C_2 are > 0 and N is the number of points in the graph $T/\partial T$. Next we observe that each triangulation is uniquely associated with a dual ϕ^3 graph. Since the total number of ϕ^3 graphs grow as $N!$, where N is the number of vertices in the ϕ^3 graph, we conclude that the Green function is bounded from below with

$$\sum_N N!\beta^{-Nd/2}C_3^N < G_\beta(\gamma_1, \dots, \gamma_n) \tag{12}$$

for any reasonable choice of weight $\varrho(T)$ (see [3]) and the series defining G_β is not Borel summable. The divergence has its origin in the large number of triangulations with a high genus. The same result has recently been derived from the continuum formalism refering to (5) by estimating the volume of moduli space associated with the higher genus surfaces [10]. *We conclude that the regularization by random triangulations is perturbative in string interactions (loop corrections), but non-perturbative in geometry, which means the inclusion of the Liouville mode.*

As a consequence of the difficulty of defining the sum over all surfaces we will in the following always restrict ourselves to surfaces with trivial topology (genus zero).

We can now define the observables that should be of interest in the scaling limit $\beta \to \beta_c$. The mass gap $m(\beta)$ is defined by the exponential fall off of the two point function (the two-loop Green function where the loops are contracted to points):

$$G_\beta(x, y) \sim e^{-m(\beta)|x-y|} \quad \text{for} \quad |x-y| \to \infty \tag{13}$$

and the critical exponent ν for the mass gap is defined by

$$m(\beta) \sim (\beta - \beta_c)^\nu \quad \text{for} \quad \beta \to \beta_c. \tag{14}$$

The string tension $\sigma(\beta)$ is defined by the exponential decay of the one-loop Green function for a large square loop $\gamma_{L,L}$ of area L^2:

$$G_\beta(\gamma_{L,L}) \sim e^{-\sigma(\beta)L^2} \quad \text{for} \quad L \to \infty. \tag{15}$$

Finally the susceptibility χ is defined by integrating over the two-point function

$$\chi(\beta) = \int dx G_\beta(x, y) \tag{16}$$

and its critical exponent γ is determined by

$$\chi(\beta) \sim (\beta - \beta_c)^{-\gamma} \quad \text{for} \quad \beta \to \beta_c. \tag{17}$$

Since we get the integrated two-point function by differentiating the one-point function with respect to β we have the following behaviour of the one-point function $G_\beta(x_0)$ near β_c:

$$G_\beta(x_0) \sim \sum (\beta_c/\beta)^{dN/2} N^{\gamma-2} \sim (\beta - \beta_c)^{-(\gamma-1)}. \tag{18}$$

It can be proven that $\gamma \leqslant 1/2$ for any reasonable choice of $\varrho(T)$ [3, 15, 16], but other analytical results are not known for $d > 1$.

3. *Analytical results in unphysical dimensions*

The one-point function $G_\beta(x_0)$ has, by translational invariance, no dependence on x_0, and formally we can perform the integration over the coordinate variables x_i:

$$\begin{aligned} G_\beta(x_{i_0}) &= \sum_{T \in \mathcal{T}} \varrho(T) \int \prod_{i \in T/\{i_0\}} dx_i \exp\left(-\beta \sum_{\langle ij \rangle} (x_i - x_j)^2\right) \\ &= \sum_{T \in \mathcal{T}} (\det (C_{T,i_0}))^{-d/2} \varrho(T). \end{aligned} \tag{19}$$

Here i_0 refers to the index of the distinguished point x_{i_0}, and C_{T,i_0} denotes the coincidence matrix defined as follows: The entries of C_{T,i_0} are labeled by the vertices in the triangla-tion T different from i_0, and the matrix element $C_{i,j}$, $i, j \in T\, i_0$ equals -1 if i and j are nearest neighbours and equals the number of neighbours of i (i_0 included if neighbour) if $i = j$, and it is 0 otherwise. The determinant of this matrix is independent of the choice of point $i_0 \in T$.

Clearly this formula can be used to analytically continue $G_\beta(x_0)$ to any dimension d. The zero dimensional case is particularly simple since the determinant drops completely out, and the problem is reduced to counting planar triangulations or equivalently, planar ϕ^3 graphs [11]. This is possible if we choose a simple weight factor $\varrho(T) = 1$ for certain classes of triangulations or ϕ^3 graphs. As examples one can choose all planar ϕ^3 graphs or one can exclude tadpole or self-energy graphs. In all cases one finds $\gamma = -0.5$.

The model can also be solved for $d = -2$. This dimension is special because the power in (19) is simply 1. This allowed the authors of [9] to perform the sum over determinants in (19) by a clever combinatorical argument. The possibility of solving the model in -2 dimensions was made more transparent by the observation that using a Parisi-Sourlas dimensional reduction, the model could be mapped onto a zero-dimensional supersymmetric field theory [12]. In principle the problem was again reduced to pure counting of diagrams. Again it can be solved for the various classes of triangulations described above, and in all cases one finds $\gamma = -1$.

In $d = 1$, which we still have to call an unphysical dimension for the string, it is not possible to solve the model analytically if we use the Gaussian action. It can be done, however, if we replace the Gaussian action with an exponential action

$$\exp\Big(-\sum_{\langle i,j\rangle}(x_i-x_j)^2\Big) \to \exp\Big(-\sum_{\langle i,j\rangle}|x_i-x_j|\Big). \tag{20}$$

The reason for this is that the propagator of a free particle in one dimension is just the exponential function and by a transformation to the dual lattice we get just a sum over all planar ϕ^3 graphs in one dimension. This sum has been performed already in [13]. When used to extract γ from (18) and (19) we get [14] $\gamma = 0$. As there are no ultraviolet divergences in one dimension one would expect from universality arguments that we get the same critical exponents for the Feynman propagators as we get for the Gaussian propagators.

Finally the model can be solved in the limit $d \to -\infty$. From (19) it follows that the dominating triangulations in this limit are the ones with the largest determinants. It is not too difficult to show that these are the most regular ones compatible with the given topology [15, 16, 17]. It follows that the discretized Laplacian is a good approximation to the ordinary continuum Laplacian and we know the leading result for the continuum Laplacian is [18]

$$\gamma(d) = \frac{d\chi}{12} + O(1) \tag{21}$$

and there is agreement between the triangulated models and the continuum saddle point calculation [18] for $d \to -\infty$.

All these special results are in agreement with the formula recently derived in [1], using the techniques of conformal field theory:

$$\gamma(d) = \frac{d-1-\sqrt{(d-1)(d-25)}}{12}. \tag{22}$$

This coincidence for the critical exponent γ can be considered a proof that the two theories agree for $d \leqslant 1$ and it seems as if the model of triangulated random surfaces is indeed a well defined regularized version of the bosonic string. Unfortunately the continuum theory seems the break down for $d > 1$, which are precisely the physical dimensions for string theories.

It should be noted that the agreement between critical exponents can be extended to larger classes of models, describing matter fields coupled to 2D gravity, see [14] for a recent review, thereby providing further evidence that the dynamical random lattices provide a correct representation of 2D quantum gravity. Again it seems as if something drastic happens when the central charge of the matter fields coupled to 2D gravity is larger than one (the analog of $d > 1$ in the case of the bosonic string).

4. Analytical and numerical results for $d > 1$

The continuum theory of the bosonic string has unfortunately yet nothing to tell us about the most interesting region: $d > 1$. The regularized random surface models, which of course are perfectly well defined for $d > 1$ seem to tell us quite a lot, although they cannot be solved exactly except for $d > 1$.

In order to determine the critical exponents for the triangulated models one can use strong coupling expansions and Monte Carlo simulations. Further it is possible to perform a kind of large d expansion. When combining all these results a fairly consistent picture emerges although it is not clear to what extent each of the results can be trusted.

When d is large the most important triangulations can be determined as was the case in the opposite limit $d \to -\infty$. These triangulations have a connectivity which in a natural way classify them as *branched polymers* [15, 16, 17]. Although they are of course surfaces, their look more like thin tubes that at certain points can branch into additional tubes. The critical properties of such objects can be completely classified [15]. The exponent γ is either $1/2$, in which case the branched polymers can be considered as a simple "product" of two random walks with a Hausdorff dimension $d_H = 4$, or γ is negative. This case corresponds the "random walks" where the branching is more pronounced than the random walk itself. The Hausdorff dimension is in this case infinity because of the branching.

Although, as already stated, the numerical studies for finite embedding dimensions are somewhat inconclusive [16, 19, 20, 21, 22, 17], it is fair to say that the latest results are not inconsistent with the conjecture that the above picture for large d holds all the way down to $d = 1$. If this is true it means that in *physical dimensions* ($d > 1$) *the statistical theory of randomly triangulated surfaces always degenerates to a theory of branched polymers.*

An important theoretical argument in favour of the conjecture can be found in [23]. It was proven there that for any reasonable choice of weight $\varrho(T)$ for the triangulations the string tension, as defined above, can never scale. It will remain positive at the critical point β_c. This explains in an intuitive way the dominating role of the branched polymer configurations even at the critical point. If we assume that the mass scales (as will be the case at least if $\gamma > 1$) the scaling relations are fixed uniquely as one approaches the critical point. The fact that the bare string tension does not scale means that the physical string tension goes to infinity when $\beta \to \beta_c$, simply by dimensional reasons and the fluctuations including any surfaces having an area different from the minimal surface will be strongly suppressed. This leaves us with the class of surfaces consisting of a minimal surface, depending on the Green loop function in question, and singular, branched polymers growing out everywhere on this surface.

It should be stressed that this picture is in accordance with the somewhat discouraging results first discovered for the random surfaces on a hypercubic lattice [6]. Also for this model the string tension does not scale and for the same reasons as above the theory degenerates into a (essentially free field-) theory of branched polymers.

5. Discussion

We have arrived at a situation where a well defined regularized model of random surfaces is identical (at the critical point) to the formally defined continuum theory of bosonic strings for all unphysical dimensions ($d \leqslant 1$). For $d > 1$ the random surface models seem trivial, although no rigorous proof of this exists yet. It would be somewhat surprising if the formally defined continuum theory should agree with the regularized theory for $d \leqslant 1$ and develop into a completely different theory for $d > 1$. From this point of view it appears difficult to unite the systematic approach of Polyakov, where the Liouville field is included, and which is valid in dimensions different from 26, with the usual approach, valid only in 26 dimensions, where the Weyl group is factored out of the partition function from the beginning. The results from random surface theory suggest that the Liouville field couples so strongly above $d = 1$ that only a trivial theory is left.

A possible way out of the above situation is that there exists an unphysical branch (another critical point, which can only be reached by analytical continuation of the partition function) of the theories for $d > 1$. For $d = 26$ Weyl invariance could be recovered and one would get back the ordinary bosonic string theory. The existence of the tachyon would then be a manifestation of the fact that we have performed an analytical continuation of the partition function (there are of course no tachyons in the ordinary random surface theory). It would be interesting if these ideas could be substantiated.

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