Vol. B20 (1989) ACTA PHYSICA POLONICA No 6

AN INTRODUCTION TO STRING THEORY*

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(*Received November 24, 1988*)

CONTENTS

PACS numbers:

* Presented at the XXVIII Cracow School of Theoretical Physics, Zakopane, Poland, May 31 — June 10, 1988.

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1. *Introduction*

Let us begin by giving very briefly, an account of some of the aims of string theory.

One of the most important motivations for studying string theory is that it may provide a unified theory of all known forces. Certain string theories contain massless spin-two, spin-3/2 and spin-one particles. A spin-two particle implies that the theory, at least in the low-energy limit, contains Einstein's theory of gravity. It turns out that the spin-ones are identifiable as Yang-Mills fields while the spin 3/2 describe the gravitino of supergravity. As such the string contains all the correct particles to potentially unify in one theory all the known forces of Nature, that is gravity the strong nuclear force and the electroweak force and the so far unobserved supergravity. Further, some string theories contain spin-$\frac{1}{2}$ and spin-zero particles which one can try to identify with the quarks and leptons and Higgs of the standard model. In a realistic string theory one must be able to identify the spin ones with the $SU(3) \times SU(2) \times U(1)$ of the standard model. Also the low mass spin $\frac{1}{2}$'s must be in the correct representations of the groups to be identifiable with quarks and leptons. Of course, these are only the beginning of a long list of constraints that a string theory must obey in order that it is not in conflict with experimental data.

Despite the many promising signs, it is fair to say that there is as yet no truly convincing way of making contact between string theory and phenomenology. To descend from the Planck scale to the scale of the standard model is a non-trivial task which may contain a number of surprises.

There are reasons to think that string theory provides a consistent quantum theory which includes gravity although this has not as yet been proved. It is also thought that the infinite number of higher spins in string theory propagate causally. String theory is therefore the most promising candidate for a consistent theory of quantum gravity.

Yang-Mills gauge symmetries, Einstein general co-ordinate transformations and local supersymmetry arise naturally in string theory. In fact, the string contains an infinite number of particles, and although only a few lower spins are massless, it would appear that there are further gauge invariances, which are important for the consistency of the string, associated with these massive particles. It would be of interest to better understand the gauge symmetries of the string. There are also indications that string theory may be able to shed light on what physical concepts are important at the Planck scale. In particular, it is possible that it may provide an alternative description of space-time.

String theory makes deep connections with mathematics. Indeed, constructions in string theory have already led to new developments in group theory and in particular Kac--Moody algebras. It is hoped, however, that it may provide insights into many other disciplines.

String theory may also be useful in helping to explain confinement in QCD and with phase transitions in three dimensions such as for the Ising model. Our understanding of second order phase transitions has benefitted from the development of string theory.

One may summarize the above, but saying that quantum string theory is a relatively new (about 20 years old) concept in physics, which has potentially many important applications, involving beautiful mathematics. It is also fun.

A certain number of features of string theory are shared by the point particle which is after all obtained in the limit when the string collapses. We will therefore first discuss the point particle.

1.1. The point particle

The point particle sweeps out a line in space-time. Let us choose τ, called the proper time, to parametrize the line which is swept out. The particle moves so as to be an extremum of the length of the world line, that is of the action

$$A = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}}, \tag{1.1}$$

where $\dot{x}^\mu = dx^\mu/d\tau$. The choice of parametrization of the world line is of no physical significance and indeed the action of Eq. (1.1) is invariant under reparametrizations

$$\tau \to \tau + f(\tau) \text{ implying } \quad x^\mu = f(\tau)\dot{x}^\mu. \tag{1.2}$$

We may write the above action in an alternative, but classically equivalent way, namely

$$A = \tfrac{1}{2} \int d\tau \{V^{-1} \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} - m^2 V\}. \tag{1.3}$$

This action is also reparametrization invariant; under $\tau \to \tau + f(\tau)$

$$\delta x^\mu = f\dot{x}^\mu, \quad \delta V = f\dot{V} + \dot{f}V. \tag{1.4}$$

We recognize V as the vierbein on the one-dimensional world sheet, the metric being V^2. It is often useful to introduce explicitly the momentum p^μ and write the action in the equivalent form

$$A = \int d\tau \{\dot{x}^\mu p^\nu \eta_{\mu\nu} - V(p^\mu p^\nu \eta_{\mu\nu} + m^2)\}.$$

The equations of motion of the action of Eq. (1.4) are

$$\frac{d}{d\tau}(V^{-1}\dot{x}^\mu) = 0,$$

$$V^2 m^2 + \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} = 0. \tag{1.5}$$

Substituting V from its equation of motion into the action of Eq. (1.3), we recover the action of Eq. (1.1).

To analyze the point particle, we may start from either of the above equations and use Hamiltonian or Lagrange formulations. Let us first take the action of Eq. (1.1) and use the Hamiltonian formulation. Taking τ as our evolution parameter, the corresponding canonical momentum is given by

$$p^\mu = \frac{\delta A}{\delta \dot{x}^\mu(\tau)} = \frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}}}. \tag{1.6}$$

We then find that the momenta satisfy the constraint

$$p^\mu p_\mu + m^2 = 0. \tag{1.7}$$

This is a consequence of the reparametrization invariance of the action. The Hamiltonian vanishes

$$H = p^\mu \dot{x}_\mu - L, \tag{1.8}$$

also as a result of reparametrization invariance. The method of dealing with such a constrained system was given in Ref. [1] and we now apply this method in outline to the point particle.

The reader who wishes to read further details is encouraged to consult Ref. [2]. We take the Hamiltonian to be proportional to the constraints, i.e.

$$H = v(\tau)(p_\mu^2 + m^2), \tag{1.9}$$

where $v(\tau)$ is an arbitrary function of τ. One may verify that in this case there are no further constraints and that H generates time translations or reparametrizations in the sense that

$$\frac{dx^\mu}{d\tau} = \{x^\mu, H\} = 2v(\tau)p^\mu. \tag{1.10}$$

The fundamental Poisson brackets vanish except for

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu}. \tag{1.11}$$

To quantize the theory we make the usual transition, according to the Dirac rule, from Poisson brackets to commutators, with an appropriate factor of $i\hbar$. These commutators are represented by the replacements

$$x^\mu \to x^\mu; \quad p^\mu \to -i\hbar \frac{\partial}{\partial x^\mu}. \tag{1.12}$$

The constraints then become

$$\hat{\phi} = (-\partial^2 + m^2), \tag{1.13}$$

which is no longer an algebraic condition but a differential operator. To proceed further, we consider the state to be described by a field $\psi(x^\mu, \tau)$ and we impose the constraint

$$\hat{\phi}\psi = (-\partial^2 + m^2)\psi = 0. \tag{1.14}$$

We also impose the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial \tau} = H\psi. \tag{1.15}$$

The right-hand side of this equation vanishes and we find that ψ is independent of τ. We recognize the usual formulation of a second-quantized spin-zero particle namely the τ dependence has disappeared and we are left with the Klein-Gordon equation.

Let us consider the action of Eq. (1.3) and take a Lagrangian viewpoint. We may use the reparametrization invariance of Eq. (1.4) to choose $V = 1$. Although, we may not implement this choice naively in the action, we may use it on the equations of motion which become

$$\ddot{x}^\mu = 0, \tag{1.16}$$

$$\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} + m^2 = 0. \tag{1.17}$$

We may regard the first equation as the equation of motion and the second one as a constraint. The constraint is none other than the condition that the "energy-momentum" tensor of the one-dimensional system in the absence of gravity should vanish. We may read off the "energy-momentum" tensor from the coefficient of h where $V = 1+h$ is substituted into Eq. (1.3). Of course, the result agrees with Eq. (1.17).

To quantize the system we proceed much as before. We must, to a large extent, slip back to the Hamiltonian approach as only there can we apply the Dirac rule. We identify the momentum conjugate to x^μ, in the gauge $V = 1$, to be $p^\mu = \dot{x}^\mu$ and turn the Poisson brackets of Eq. (1.11) into commutators which are represented by Eq. (1.12). The constraint of Eq. (1.17) now becomes that of (1.13) and we impose it on the wave function $\psi(x^\mu)$ as in Eq. (1.14).

Finally, we wish to carry out yet another approach to the point particle, namely the BRST approach as this will be particularly important in the extension to the string. The reader may wish to first read the Appendix where the BRST formulation of Yang-Mills theory is given. We begin with the action of Eq. (1.3) and fix the gauge in the standard BRST fashion. We choose the gauge

$$V = 1 \quad \text{or} \quad \ln V = 0 \tag{1.18}$$

and so add the gauge fixing term

$$A^{gf} = \int d\tau \lambda \ln V. \tag{1.19}$$

The BRST transformations of the original fields are found by the substitution $f(\tau) \to \Lambda c(\tau)$ where Λ is an anticommuting BRST parameter and $c(\tau)$ is the ghost field:

$$\delta x^\mu = (\Lambda c)\dot{x}^\mu, \quad \delta V = \frac{d}{d\tau}((\Lambda c)V) \tag{1.20}$$

We choose c to be Hermitian and then Λ is antihermitian in order that Λc is real. The standard rule for taking the complex conjugate of two anticommuting variables being

$$(\Lambda c)^* = c^*\Lambda^* = -\Lambda^* c^* = \Lambda c. \tag{1.21}$$

The transformation law of the ghost is given by

$$\delta c = (\Lambda c)\dot{c}. \tag{1.22}$$

The commutator of two infinitesimal reparametrizations f_1 and f_2 which yields a third parameter

$$(-f_1\dot{f}_2 + f_2\dot{f}_1). \tag{1.23}$$

Making the above substitution for f and removing one Λ, we identify the result as δc. This procedure is identical to that followed in Yang-Mills (see the Appendix). We then introduce the antighost b which transforms in the standard way with the multiplier λ namely:

$$\delta\lambda = 0, \qquad \delta b = \Lambda\lambda. \tag{1.24}$$

Although the above may seem like a cook-book recipe, we have arrived at the desired result, namely a set of nilpotent transformations

$$\begin{aligned}\delta_{\Lambda_1}\delta_{\Lambda_2}c &= \delta_{\Lambda_1}\{(\Lambda_2 c)\dot{c}\} \\ &= \Lambda_2((\Lambda_1 c)\dot{c})\dot{c} + \Lambda_2 c\frac{d}{d\tau}\{(\Lambda_1 c)\dot{c}\} = 0.\end{aligned} \tag{1.25}$$

Finally, we can write down a BRST invariant action of the form

$$A^{\text{total}} = A^{\text{orig}} + A^{\text{gf}} + A^{\text{gh}}, \tag{1.26}$$

where A^{orig} is the action of Eq. (1.43) and A^{gf} is that of Eq. (1.19). Due to its original reparametrization invariance A^{orig} is automatically BRST invariant. We find by cancelling the variations of A^{gf} that

$$A^{\text{gh}} = -\int d\tau b\nabla_\tau c \tag{1.27}$$

where $\nabla_\tau c = \dot{c} + d/d\tau(\ln v)c$. An alternative way to arrive at the above result is to note that under a BRST transformation

$$\delta\{b(\ln V-1)\} = \Lambda(\lambda(\ln V-1) - b\nabla_\tau c) \tag{1.28}$$

and use the nilpotency of δ.

In the functional integral, we can do the λ integral which sets $V = 1$ and substituting in the above result, we find the resulting BRST action to be

$$A^{\text{total}} = \int d\tau(\tfrac{1}{2}\dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu} - \tfrac{1}{2}m^2 - b\dot{c}). \tag{1.29}$$

This result is BRST invariant, however, for δb we must substitute the value of λ by its equation of motion, i.e.,

$$\delta b = \Lambda\left(\frac{(\dot{x}^\mu)^2}{2} + \frac{m^2}{2} + \frac{d}{d\tau}(bc)\right) \tag{1.30}$$

the other variations being unchanged.

The action of Eq. (1.29) being BRST invariant, we can in the standard way deduce associated Noether current Q which in this one dimensional case is also the charge. We find that it is given by

$$Q = c((p^\mu)^2 + m^2), \tag{1.31}$$

where $p^\mu = \dot{x}^\mu$ is the momentum conjugate to x^μ. We take the definition of momenta for anticommuting variables to be left differentiation of the action by the co-ordinate. Hence the momentum for the co-ordinate c is given by

$$\frac{\vec{\partial}}{\partial \dot{c}} A = b. \tag{1.32}$$

We could also take b as our co-ordinate and then $-c$ would be the momentum.

In general, given a system with co-ordinates q_A and corresponding momenta p_A, some of which may be anticommuting, we define the Poisson bracket of two functions f and g of q_A and p_B as

$$\{f, g\}_{\mathrm{PB}} = \sum_A \left\{ f \frac{\overleftarrow{\partial}}{\partial q_A} \frac{\vec{\partial}}{\partial p_A} g - (-1)^{fg} g \frac{\overleftarrow{\partial}}{\partial g_A} \frac{\vec{\partial}}{\partial p_A} f \right\}, \tag{1.33}$$

where

$$(-1)^{fg} = \begin{cases} -1 \text{ if both } f \text{ and } g \text{ are Grassmann odd} \\ +1 \text{ otherwise.} \end{cases}$$

It satisfies

$$\{f, g\}_{\mathrm{PB}} = -(-1)^{fg} \{g, f\}_{\mathrm{PB}} \tag{1.34}$$

and a suitable Jacobi identity. The non-zero Poisson brackets for the co-ordinates and momenta are

$$\{x^\mu, p^\nu\}_{\mathrm{PB}} = \eta^{\mu\nu}, \quad \{c, b\}_{\mathrm{PB}} = 1. \tag{1.35}$$

The Hamiltonian associated to the action of Eq. (1.29) is

$$H = (p_\mu^2 + m^2). \tag{1.36}$$

The BRST charge is the generator of transformations in the sense that

$$\delta \bullet = \{\bullet, \Lambda Q\}, \tag{1.37}$$

where • is any field. The reader may verify that, on-shell, these transformations agree with those previously given and are an invariance of the Hamiltonian equations of motion. The nilpotency of Q is encoded in the equation $\{Q, Q\}_{\mathrm{PB}} = 0$.

To quantize the system we apply the Dirac rule

$$\{\ ,\ \} \to \frac{1}{i\hbar} [\ ,\ ] \tag{1.38}$$

to the Poisson brackets regardless of whether they are for Grassmann odd or even variables. We find that we must demand

$$[x^\mu, p^\mu] = i\hbar; \quad \{c, b\} = i\hbar. \tag{1.39}$$

We may therefore use the representation

$$x^\mu \to x^\mu, \qquad c \to c$$

$$p^\mu \to -i\hbar \frac{\partial}{\partial x^\mu}, \qquad b \to i \frac{\partial}{\partial c}. \tag{1.40}$$

In checking the appearance of i's, it is important to remember that b is antihermitian. The BRST charge now becomes the operator

$$Q = c(-\partial^2 + m^2) \tag{1.41}$$

and it is obviously nilpotent, i.e., $Q^2 = 0$ as $c^2 = 0$.

We now consider functionals of the co-ordinates, i.e., $\Psi(x^\mu, c)$ which as c is anticommuting are of the form

$$\Psi = \psi(x^\mu) + c\phi(x^\mu). \tag{1.42}$$

In the BRST formalism, physical states are taken to satisfy the condition

$$Q\Psi = 0. \tag{1.43}$$

We realize that this is equivalent to

$$(-\partial^2 + m^2)\psi = 0 \tag{1.44}$$

that is the usual Klein-Gordon result. The field ϕ, in fact, never enters the dynamics and in this sense it is gauge away. We postpone, until we discuss the string case, the appropriate way of realizing this statement.

The method of BRST quantization was originated [3] in the context of Yang-Mills theory which is still the prototype example of how to proceed. The systematic use of the BRST charge, for general systems with first class constraints, was carried out in Ref. [4]. For some reviews of this procedure, see Ref. [5]. It must be stated, however, that the BRST method, as with any quantization method, is more like an art than a science. Its justification is that the final result, namely a nilpotent set of transformations and an invariant action. These usually define a quantum theory which is unitary and whose physical observables are independent of how the gauge was fixed. Another point in its favour is that it allows one to demonstrate the renormalizability of Yang-Mills theory for a general class of gauges.

The reader is encouraged to consult the point particle case when puzzled by aspects of the string discussion. The point algebra is much simpler and has conceptually many points in common with the string case.

Using various formalisms, we have arrived at the standard Klein-Gordon equation and so the usual starting point for the second quantized spin-zero particle has been used for many years. However, there are a number of points in the above discussion which do not seem entirely correct. In particular, invariance of the action, implies that the parameters of reparametrization transformations vanish at the end point of the τ integration. This restriction results in an inability to reach the gauge $V = 1$ from a general V, in fact the best one can do is get $\dot{V} = 0$ [6]. Derivative gauge fixing conditions imply a different ghost action to that considered above and was given in Ref. [7]. Upon quantization one recovered [8] a second-quantized field theory which contains a τ dependence and has a corresponding local invariance. In perturbation theory, it agrees with the more familiar formulation due to a Parisi-Sourlas symmetry [9]. Having alerted the reader, we will not discuss these points further. but encourage the reader to consult the above references.

2. *The classical bosonic string*

In this chapter, we will discuss the classical bosonic string. As the string moves through space-time, it sweeps out a two-dimensional world sheet. By extending the analogy with the point particle, we will take the string action to be the area of the world sheet sweep out [10]. From this action we will derive the equations of motion for the string.

2.1. The action and equations of motion

Strings may be open or closed as in Fig. 2.1. We take the length along the string to be parametrized by σ and its passage in time parametrized by τ. Hence the world sheet has co-ordinates ξ^α, $\alpha = 1, 2$ with $\xi^\alpha = (\tau, \sigma)$.

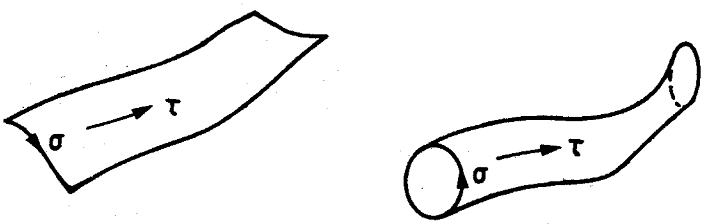

Fig. 2.1. World sheets of the free open and closed strings

In the open case we take the range of σ to be from 0 to π while in the closed case we take $-\pi < \sigma \leqslant \pi$. It is natural for the closed string to take the boundary condition $x^\mu(-\pi) = x^\mu(\pi)$ as $\sigma = -\pi$ and $\sigma = \pi$ are the same point on the string. We will discuss the boundary condition for the open string below.

The action for both the open and closed string is the area sweep-out which is given by

$$A = -\frac{1}{2\pi\alpha'}\int d^2\xi\,\sqrt{-\det\{\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}\}}, \tag{2.1}$$

where α' is a constant which has the dimensions of $[\text{mass}]^{-2}$ as A and ξ^α, are dimensionless. We notice that the action explicitly involves the Minkowski metric $\eta_{\mu\nu}$ which is given by $\eta_{\mu\nu} = (-1, +1, +1, \ldots, +1)$ in our conventions. That this is the area sweep-out follows from the fact that if we move from ξ^α to $\xi^\alpha + d\xi^\alpha$ on the world sheet, the corresponding $(\text{distance})^2$ moved in space-time is $-d\xi^\alpha\partial_\alpha x^\mu d\xi^\beta\partial_\beta x^\nu\eta_{\mu\nu}$. Hence we interpret $-\partial_\alpha x^\mu\partial_\beta x^\nu\eta_{\mu\nu}$ as the induced metric on the world sheet.

The minus sign under the square root is such that for causal propagations of the string ensure that the quantity under the square root is positive. We refer the reader to the review of Joel Scherk for a clear discussion of this point.

The factors of $(2\pi\alpha')^{-1}$ outside the action of Eq. (2.1) will lead to such messy factors in the subsequent equations. To simplify expressions one can, as in the old days, take the scale of energy to be such that $2\alpha' = 1$ or scale $x^\mu \to \sqrt{2\alpha'}\, x^\mu$ in all expressions.

The action of Eq. (2.1) is invariant under the following symmetries.

(*i*) Reparametrization invariance:

$$\xi^\alpha \to \xi^{\alpha'} = f(\xi^\alpha) \tag{2.2}$$

under which the fields $x^\mu(\xi)$ are scalars, i.e.,

$$x'^\mu(\xi') = x^\mu(\xi). \tag{2.3}$$

(*ii*) Poincaré symmetry:

$$x'^\mu(\xi) = \Lambda^\mu{}_\nu x^\nu(\xi) + a^\mu, \tag{2.4}$$

where as usual

$$\eta_{\mu\nu}\Lambda^\mu{}_\varrho\Lambda^\nu{}_\tau = \eta_{\varrho\tau}. \tag{2.5}$$

The infinitesimal version of these transformations comes from taking $\delta x^\mu \equiv x'^\mu(\xi) - x^\mu(\xi)$ for $\delta\xi^\alpha \equiv \xi'^\alpha - \xi^\alpha$ small. Reparametrization invariance means that the particular co-ordinate system used to parametrize the world sheet is of no physical significance.

From the two-dimensional viewpoint, the Poincaré invariance is an internal symmetry. We shall see that the bosonic string is consistent only in certain dimensions higher than four. Although it is not apparent why one should assume that Poincaré invariance is true in the dimensions beyond the four, we will, in this first treatment, take this to be the case. An action with a square root is not particularly easy to handle especially when one comes to quantization. Fortunately, one can rewrite the action in the form [11]

$$A = -\frac{1}{4\pi\alpha'}\int d^2\xi \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}, \tag{2.6}$$

where $g_{\alpha\beta}$ is the two-dimensional metric on the world sheet. As usual $g^{\alpha\beta}$ is its inverse and $g \equiv \det g_{\alpha\beta}$. In this form the action is a set of scalar fields x^μ coupled to gravity $g_{\alpha\beta}$, its equivalence, at the classical level, to the previous action is shown below. It is invariant under reparametrization invariance; the transformations being those of Eq. (2.3) and

$$g'_{\alpha\beta}(\xi') = \frac{\partial\xi^\gamma}{\partial\xi'^\alpha}\frac{\partial\xi^\delta}{\partial\xi'^\beta}\, g_{\gamma\delta}(\xi). \tag{2.7}$$

The Poincaré transformation of Eq. (2.4), as well as

$$g'_{\alpha\beta}(\xi) = g_{\alpha\beta}(\xi) \tag{2.8}$$

also lead to an invariance of the action of Eq. (2.6). However, the alternative action also has a Weyl symmetry which is realized by the transformations

$$g'_{\alpha\beta}(\xi) = \Lambda(\xi) g_{\alpha\beta}; \quad x^{\mu'}(\xi) = x^\mu(\xi). \tag{2.9}$$

The action of Eq. (2.1) is also Weyl invariant in the sense that the fields (i.e., x^μ) are inert. To verify the reparametrization invariance of the Nambu action we use the formula

$$\frac{\partial x'^\mu}{\partial \xi'^\alpha}\frac{\partial x'^\nu}{\partial \xi'^\beta} = \frac{\partial \xi^\gamma}{\partial \xi'^\alpha}\frac{\partial \xi^\delta}{\partial \xi'^\beta}\partial_\gamma x^\mu \partial_\delta x^\nu. \tag{2.10}$$

Since det AB = det A det B for any two matrices A and B, we find that

$$\sqrt{-\det \frac{\partial x'^\mu}{\partial \xi'^\alpha}\frac{\partial x'^\nu}{\partial \xi'^\beta}\eta_{\mu\nu}} = J\sqrt{-\det \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta}\eta_{\mu\nu}}, \tag{2.11}$$

where

$$J = \det \frac{\partial \xi^\alpha}{\partial \xi'^\beta}. \tag{2.12}$$

The final step is to compute the variation of the measure, we have

$$d^2\xi' = J^{-1}d^2\xi. \tag{2.13}$$

As a result, the invariance of the action is apparent. The analogous computation for the action of Eq. (2.6) is a textbook exercise in general relativity.

In verifying the above statements for infinitesimal variations, it is useful to use the equations

$$\delta g = g g^{\alpha\beta}\delta g_{\alpha\beta}, \quad \delta g^{\alpha\beta} = -g^{\alpha\beta}\delta g_{\delta\gamma}g^{\gamma\delta}, \tag{2.14}$$

which express the variation of g and $g^{\alpha\beta}$ induced from the variation of $g_{\alpha\beta}$.

We now have two choices in deriving the equations of motion. We can adopt a Lagrangian or Hamiltonian viewpoint or we can work with the action of Eq. (2.1) or that of Eq. (2.6). The Hamiltonian treatment of the action of Eq. (2.1) can be found in the same notation as that here in the review of Ref. [12]. Here, let us perform a Lagrangian treatment of the action of Eq. (2.6). The results are of course the same, no matter which formalism one uses.

We recall that the variational principle states that a system moves from a fixed configuration at the initial time τ_1 to a fixed configuration at a final time τ_2 in such a way as to be an extremum of the action. In our case this means that

$$0 = \delta A = \int_a^\pi d\sigma \int_{\tau_1}^{\tau_2} d\tau \left\{\frac{\delta A}{\delta g_{\alpha\beta}}\delta g_{\alpha\beta} + \frac{\delta A}{\delta(\partial_\sigma x^\mu)}\delta(\partial_\sigma x^\mu) + \frac{\delta A}{\delta(\partial_\tau x^\mu)}\delta(\partial_\tau x^\mu)\right\} \tag{2.15}$$

for arbitrary $\delta g_{\alpha\beta}$ and δx^μ provided that these quantities vanish at τ_1 and τ_2. In the above equation $a = 0$ and $-\pi$ for the open and closed strings respectively. Integrating the last two terms by parts, according to the above instruction, we find that

$$0 = \int_a^\pi d\sigma \int_{\tau_1}^{\tau_2} d\tau \left\{ \frac{\delta A}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} + \left[-\partial\sigma \frac{\delta A}{\delta(\partial_\sigma x^\mu)} - \partial\tau \frac{\delta A}{\delta(\partial_\tau x^\mu)} \right] \delta x^\mu \right\}$$

$$+ \int_{\tau_1}^{\tau_2} d\tau \frac{\delta A}{\delta(\partial_\sigma x^\mu)} \delta x^\mu \Big|_{\sigma=a}^{\sigma=\pi}. \tag{2.16}$$

Consequently, we have for arbitrary $\delta g_{\alpha\beta}$ but $\delta x^\mu = 0$, the equation

$$\frac{\delta A}{\delta g_{\alpha\beta}} = 0. \tag{2.17}$$

Similarly for $\delta g_{\alpha\beta} = 0$ and δx^μ arbitrary but vanishing at $\sigma = \pi$ and a, we find that

$$\partial_\alpha \Pi^{\alpha\mu} = 0, \tag{2.18}$$

where

$$\Pi^{\alpha\mu} = \partial_\alpha \frac{\delta A}{\delta(\partial_\alpha x^\mu)}. \tag{2.19}$$

Evaluating Eqs. (2.17) and (2.18) we find respectively

$$0 = \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \tfrac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu \eta_{\mu\nu} \tag{2.20}$$

and

$$\partial_\alpha(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta x^\mu) = 0. \tag{2.21}$$

By continuity we may also take Eq. (2.18) and therefore Eq. (2.21) to hold at the boundary $\sigma = 0$ and a. Hence for arbitrary δx^μ we find, corresponding to the last term in Eq. (2.10), the boundary condition

$$\Pi^{\mu 1}\Big|_a^\pi \equiv \frac{\delta A}{\delta(\partial_\sigma x^\mu)}\Big|_a^\pi = 0. \tag{2.22}$$

For the closed string as $x^\mu(-\pi) = x^\mu(\pi)$, this vanishes automatically. However, for the open string we conclude that

$$\partial_\sigma(\sqrt{-g}\, g^{1\beta} \partial_\beta x^\mu) = 0 \tag{2.33}$$

at both $\sigma = 0$ and $\sigma = \pi$. We must impose this condition at the two ends of the string separately as, in general, the two ends are not causally connected.

The above equations are rather complicated, but simplify considerably when we make a convenient gauge choice which relies on the following theorem.

Theorem

On any orientated Riemann surface, we may locally choose a coordinate system such that the metric takes the form

$$g_{\alpha\beta}(\xi) = \eta_{\alpha\beta} e^{\phi(\xi)} \tag{2.24}$$

or in terms of the line element

$$ds^2 = e^{\phi} d\xi^\alpha d\xi^\beta \eta_{\alpha\beta}, \tag{2.25}$$

where $\eta_{\alpha\beta}$ is the flat metric, i.e., $\eta_{\alpha\beta} = (-1, +1)$.

One may expect such a theorem on a count of degrees of freedom. The metric $g_{\alpha\beta}$ has three degrees of freedom while that of Eq. (2.24) has only one corresponding to the field $\phi^{(\xi)}$, the two degrees of freedom less being removed by the two ξ^α reparametrization transformations. The proof of the theorem, which is constructive by nature, can be found in many places, but in particular in the book by Spencer and Schriffer.

In the mathematical literature a Riemann surface is any two-dimensional surface with a Euclidean metric. The string world sheet is a two-dimensional surface, but with a Minkowski metric. Nevertheless, many results, including the above theorem, may be taken across.

Consequently, we may choose the metric $g_{\alpha\beta}$ which appears in the above formulation to be of the form of Eq. (2.24) with $\eta_{\alpha\beta}$ the Minkowski metric. It is, of course, illegal to impose a gauge choice in the action before carrying out the variation, however, we can impose it on the equations of motion. As such Eq. (2.2) becomes the wave equation namely

$$\partial_\alpha \partial^\alpha x^\mu = 0 \tag{2.26}$$

while Eq. (2.20) can be written as

$$T_{\alpha\beta} = 0, \tag{2.27}$$

where

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \tfrac{1}{2} \eta_{\alpha\beta} \partial_\gamma x^\mu \partial^\gamma x^\nu \eta_{\mu\nu}. \tag{2.28}$$

The use of the symbol $T_{\alpha\beta}$ is deliberate as we recognize it as the energy-momentum tensor of the theory of free scalar fields x^μ with action

$$\int d^2\xi \{-\tfrac{1}{2} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \eta^{\alpha\beta}\}. \tag{2.29}$$

As a consequence of Weyl invariance, the field ϕ does not appear in the equations of motion; or put another way $\sqrt{-g}\, g^{\alpha\beta}$ is Weyl invariant and so independent of ϕ. This invariance ensures that $T_{\alpha\beta}$ is automatically traceless. The two non-zero components are

$$T_{00} = T_{11} = \tfrac{1}{2} (\dot{x}^\mu \dot{x}^\nu + x^{\mu'} x^{\nu'}) \eta_{\mu\nu} \tag{2.30}$$

where $\dot{x}^\mu = \partial x^\mu / \partial \tau$, $x^{\mu'} = \partial x^\mu / \partial \sigma$ and

$$T_{01} = T_{10} = \dot{x}^\mu x^{\mu'}. \tag{2.31}$$

In this gauge the boundary condition for the *open string* also takes on a particularly simple form:

$$x^{\mu'}(\sigma) = 0 \quad \text{at } \sigma = 0 \quad \text{and } \sigma = \pi. \tag{2.32}$$

It is convenient in the case of the open string to extend the range of σ from 0 to π to $-\pi < \sigma < \pi$. This is achieved by setting

$$x^{\mu}(\sigma) = \begin{cases} x^{\mu}(\sigma) & \text{for} \quad 0 < \sigma < \pi \\ x^{\mu}(-\sigma) & \text{for} \quad -\pi < \sigma < 0 \end{cases} \tag{2.33}$$

or $x^{\mu}(\sigma) = +x^{\mu}(-\sigma)$ for $0 < \sigma < \pi$. The boundary condition at $\sigma = 0$ of Eq. (2.32) is automatically incorporated into this identification. We can also consider further extending the range of σ to be from $-\infty$ to ∞ by taking $x^{\mu}(\sigma) = x^{\mu}(\sigma+2\pi)$. Such an extension is allowed as the boundary conditions $x^{\mu}(\pi) = x^{\mu}(-\pi)$ and $x^{\mu'}(\pi) = x^{\mu'}(-\pi) = 0$ ensure that $x^{\mu}(\sigma)$ is continuous and possess a first derivative at $\sigma = \pm\pi$. Higher derivatives are ensured by virtue of the equation of motion. Conversely taking $x^{\mu}(\sigma) = x^{\mu}(\sigma+2\pi)$ and Eq. (2.33) ensures that all the boundary conditions are obeyed. This mathematical extension allows us to rewrite the two constraints of Eqs. (2.30) and (2.31) as one equation, namely

$$(P^{\mu}(\sigma))^2 = 0; \qquad -\pi < \sigma < \pi, \tag{2.34}$$

where

$$P^{\mu} = (\dot{x}^{\mu} + x^{\mu'})\frac{1}{\sqrt{2\alpha'}}, \tag{2.35}$$

the two equations (2.30) and (2.31) emerging as the parts of Eq. (2.34) which are even and odd under $\sigma \to -\sigma$.

When quantizing the string, we will find it convenient to use normal modes or Fourier transformed quantities. We define

$$x^{\mu}(\sigma) = \sum_{n=0}^{\infty} x_n^{\mu} \cos n\sigma = x_0^{\mu} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} x_n^{\mu} \frac{e^{in\sigma}}{2}, \tag{2.36}$$

where $x_{-n}^{\mu} = x_n^{\mu}$. This expansion and the latter condition are consistent with the above extension of the range of σ and the boundary conditions of Eq. (2.32).

In terms of the normal modes the equation of motion of Eq. (2.26) becomes

$$\ddot{x}_n^{\mu} + n^2 x_n^{\mu} = 0,$$

which we recognize for $n \geqslant 1$ as those corresponding to an infinite set of simple harmonic oscillators, in this case, however, we also have the constraints of Eq. (2.27). For $n = 0$ we have

$$x_0^{\mu}(\tau) = x_0^{\mu}(0) + c\tau,$$

where c is a constant. It will be useful, when we quantize, to introduce the analogue of harmonic oscillators for this system

$$P^\mu \equiv (\dot{x}^\mu + x^{\mu'}) \frac{1}{\sqrt{2\alpha'}} \equiv \sum_n \alpha_n^\mu e^{-in\sigma}$$

or

$$\alpha_n^\mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma e^{in\sigma} P^\mu(\sigma).$$

To express x^μ in terms of α_n^μ we integrate the equation

$$\frac{2x^{\mu'}}{\sqrt{2\alpha'}} = P^\mu(\tau, \sigma) - P^\mu(\tau, -\sigma)$$

to find

$$x^\mu(\tau, \sigma) = x_0^\mu + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{in} (\alpha_n^\mu - \alpha_{-n}^\mu) e^{in\sigma},$$

where we have recognized $x_0^\mu(\tau)$ as the integration constant. The equations of motion when expressed in terms of the α_n^μ take the simple form

$$\dot{\alpha}_n^\mu = -in\alpha_n^\mu \quad \text{or} \quad \alpha_n^\mu(\tau) = e^{-in\tau}\alpha_n^\mu(0).$$

Let us compute the Poisson brackets. For the co-ordinates and momenta we have as usual

$$\{x^\mu, x^\nu\} = 0 = \{p^\mu, p^\nu\},$$

$$\{x^\mu(\sigma), P^\nu(\sigma')\} = i\hbar\delta(\sigma - \sigma').$$

In the gauge in which we are working, the momentum p^μ not to be confused with P^μ above, is given by

$$p^\mu = \frac{1}{\pi} \frac{\dot{x}^\mu}{2\alpha'} = \frac{\delta A}{\partial(\partial_\tau x^\mu)}.$$

From the fundamental Poisson bracket above, we deduce that

$$\{P^\mu(\sigma), P^\nu(\sigma')\} = 2\pi\delta'(\sigma - \sigma') \equiv 2\pi \frac{d}{d\sigma} \delta(\sigma - \sigma')$$

as we may write

$$P^\mu(\sigma) = \sqrt{2\alpha'}\, \pi p^\mu + x^{\mu'}/\sqrt{2\alpha'}.$$

Consequently, we have

$$\{\alpha_n^\mu, \alpha_m^\nu\} = -i\delta_{n+m,0}\eta^{\mu\nu}.$$

Taking the Fourier transform of the constraints we have, for the open string

$$L_n = 0 \quad \forall n, \tag{2.37}$$

where

$$L_n \equiv \frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma e^{in\sigma}(P^\mu)^2. \tag{2.38}$$

For the *closed* string it will turn out to be convenient to work with the two quantities

$$(P^\mu)^2 \equiv \frac{1}{\alpha'}(T_{00}+T_{01}), \tag{2.39}$$

and

$$(\bar{P}^\mu)^2 \equiv \frac{1}{\alpha'}(T_{00}-T_{01}), \tag{2.40}$$

where

$$P^\mu \equiv \frac{1}{\sqrt{2\alpha'}}(\dot{x}^\mu + x^{\mu\prime}), \tag{2.41}$$

$$\bar{P}^\mu \equiv \frac{1}{\sqrt{2\alpha'}}(\dot{x}^\mu - x^{\mu\prime}). \tag{2.42}$$

As we shall see, working with $(P^\mu)^2$ and $(\bar{P}^\mu)^2$ rather than $T_{00}T_{01}$ corresponds to working in light-cone co-ordinates. Defining the normal modes for the closed string, corresponding to the boundary $x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma+2\pi)$ we have

$$x^\mu(\tau, \sigma) \equiv \sum_{n=-\infty}^{\infty} e^{in\sigma}x_n^\mu(\tau), \tag{2.43}$$

where $x_{-n}^\mu = (x_n^\mu)^*$ to ensure that $x^\mu(\tau, \sigma)$ is real. The constraints can be written as

$$L_n = \bar{L}_n = 0 \quad \forall n, \tag{2.44}$$

where

$$L_n \equiv \frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma e^{in\sigma}(P^\mu)^2, \tag{2.45}$$

$$\bar{L}_n \equiv \frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma e^{-in\sigma}(\bar{P}^\mu)^2. \tag{2.46}$$

We introduce the oscillators α_n^μ and $\tilde{\alpha}_n^\mu$ by

$$P^\mu \equiv \sum_n \alpha_n^\mu e^{-in\sigma}, \qquad \bar{P}^\mu \equiv \sum_n \bar{\alpha}_n^\mu e^{in\sigma},$$

and one can verify that all their Poisson brackets are zero except for

$$\{\alpha_n^\mu, \alpha_m^\nu\} = -i\delta_{n+m,0}\eta^{\mu\nu},$$

$$\{\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu\} = -i\delta_{n+m,0}\eta^{\mu\nu}.$$

Using Eq. (2.43) in Eqs. (2.41) and (2.42) we find that

$$\alpha_n^\mu = \frac{1}{\sqrt{2\alpha'}}(\dot{x}_{-n}^\mu - inx_{-n}^\mu),$$

$$\bar{\alpha}_n^\mu = \frac{1}{\sqrt{2\alpha'}}(\dot{x}_n^\mu - inx_n^\mu).$$

The equation of motion $\ddot{x}_n^\mu = x_n^\mu n^2$ implies that

$$\dot{\alpha}_n^\mu = -in\alpha_n^\mu; \qquad \dot{\bar{\alpha}}_n^\mu = -in\bar{\alpha}_n^\mu,$$

and hence

$$\alpha_n^\mu(\tau) = e^{-in\tau}\alpha_n^\mu(0); \qquad \bar{\alpha}_n^\mu = \bar{\alpha}_n^\mu(0)e^{-in\tau}.$$

From Eqs. (2.41) and (2.42) we may solve for the oscillators

$$x^{\mu'} = \sqrt{\frac{\alpha'}{2}}\sum_n (e^{-in\sigma}\alpha_n^\mu(\tau) + \bar{\alpha}_n^\mu(\tau)e^{in\sigma}).$$

Integrating with respect to σ, we have

$$x^\mu(\tau, \sigma) = x_0^\mu(\tau) + i\sqrt{\frac{\alpha'}{2}}\sum_n \left\{\frac{\alpha_n^\mu(\tau)}{n}e^{-in\sigma} + \frac{\bar{\alpha}_{-n}^\mu}{n}e^{-in\sigma}\right\},$$

where $x_0^\mu(\tau)$ satisfies $\ddot{x}_0^\mu(\tau) = 0$ and hence it is of the form $x_0^\mu(\tau) = q^\mu + \tau p^\mu$ where q^μ and p^μ are independent of τ. Substituting in the above equation we have

$$x^\mu(\tau, \sigma) = q^\mu + \tau p^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_n \{\alpha_n^\mu(0)e^{-in(\tau+\sigma)} + \bar{\alpha}_{-n}^\mu e^{In(\tau-\sigma)}\}.$$

We may write this in the form

$$x^\mu(\tau, \sigma) = x_L^\mu(\tau+\sigma) + x_R^\mu(\tau-\sigma),$$

where

$$x_L^\mu(\tau+\sigma) = \frac{q^\mu}{2} + \frac{(\tau+\sigma)}{2} p^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu(0) e^{-in(\tau+\sigma)}$$

$$x_R^\mu(\tau+\sigma) = \frac{q^\mu}{2} + \frac{(\tau-\sigma)}{2} p^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \bar{\alpha}_n^\mu(0) e^{in(\tau-\sigma)}.$$

Clearly it is a solution of the wave equation.

2.2. The energy-momentum tensor and angular momentum

From the two-dimensional viewpoint, the Poincaré symmetry is an internal symmetry and has correspondingly conserved two-dimensional currents. Either by the Noether technique or other standard methods, we find the current for translations Π_α^μ and Lorentz rotations $M_\alpha^{\mu\nu}$ are given by

$$\Pi^{\mu\alpha} = -\frac{\delta A}{\delta(\partial_\alpha x^\mu)}, \tag{2.47}$$

$$M^{\mu\nu\alpha} = -x^\mu \frac{\delta A}{\delta(\partial_\alpha x^\nu)} - (\mu \leftrightarrow \nu) = x^\mu \Pi^{\nu\alpha} - x^\nu \Pi^{\mu\alpha}. \tag{2.48}$$

Using the equations of motion, we find these are conserved, i.e.,

$$\partial^\alpha \Pi_\alpha^\mu = 0; \quad \partial^\alpha M_\alpha^{\mu\nu} = 0. \tag{2.49}$$

Taking the metric of Eq. (2.34), we find that

$$\Pi_\alpha^\mu = \frac{1}{2\pi\alpha'} \partial_\alpha x^\mu, \quad M_\alpha^{\mu\nu} = \frac{1}{2\pi\alpha'} (x^\mu \partial_\alpha x^\nu - x^\nu \partial_\alpha x^\mu). \tag{2.50}$$

By Stokes' theorem if C is a closed curve on the world sheet then,

$$\int_C \Pi_\tau^\mu d\sigma + \Pi_\sigma^\mu d\tau = \int d\sigma d\tau \partial^\alpha \Pi_\alpha^\mu = 0. \tag{2.51}$$

Hence, as usual, we may interpret

$$\int_C \Pi_\tau^\mu d\sigma + \Pi_\sigma^\mu d\tau \tag{2.52}$$

as the flow of energy momentum through any open curve C on the world sheet. In particular, the flow across an infinitesimal curve from (τ, σ) to $(\tau + d\tau, \sigma + d\sigma)$ is $\Pi_\tau^\mu d\sigma + \Pi_\sigma^\mu d\tau$. For the open string, the boundary condition of Eq. (2.22) implies that the flow of energy and momentum through the ends of the string is zero.

The total conserved energy-momentum of the string is given by integrating on any curve C from one boundary of the world sheet to the other. By the above, the result is independent of the curve chosen. One such curve being τ = constant which yields the result

$$\Pi^{\mu} = \int_{a}^{\pi} \Pi^{\mu}_{\tau} d\sigma \tag{2.53}$$

where $a = 0$ and $-\pi$ for the open and closed strings respectively.

A similar analysis applies to the Lorentz current, the total momentum of the string being

$$J^{\mu\nu} = \int_{a}^{\pi} M^{\mu\nu}_{\tau} d\sigma. \tag{2.54}$$

2.3. A classical solution

Although the string in the above gauge satisfies the simple wave equation (2.26), it is also subject to the constraints of Eq. (2.27). To get a feeling for this system, it is very educational to find an explicit solution. To be specific, let us consider the open string. We can choose to correlate the time on the string τ with the time of the tangent space-time by making the identification

$$x^{0} = \tau d, \tag{2.55}$$

where d is a constant. We then find that the constraint of Eq. (2.7) implies that

$$-1 + \sum_{i=1}^{D-1} \frac{\partial x^{i}}{\partial t} \cdot \frac{\partial x^{i}}{\partial t} + \frac{1}{d^{2}} x^{\mu\prime} x^{\mu\prime} = 0. \tag{2.56}$$

However, for the endpoints of the open string $x^{\mu\prime} = 0$ and so the above equation states that the ends of the strings move with the speed of light.

Clearly, a static string cannot be a solution. The next simplest possibility is to have the string rotate about its mid point. We can take this point to be the origin of the space-time and the rotation to be in the 1–2 plane. Let us choose the string to be of length L then the motion is given by

$$x^{0} = d\tau, \qquad x^{1} = \frac{L}{1} \sin\tau\cos\sigma, \qquad x^{2} = \frac{L}{2}\cos\tau\cos\sigma,$$

$$x^{3} = x^{4} = \ldots = x^{D-1} = 0. \tag{2.57}$$

This is a solution as it solves both the constraints of Eqs. (2.20) and (2.31) and the wave equation (2.26) provided $d = \frac{1}{2} L$.

The total energy E and angular momentum $J^{\mu\nu}$ can be found by substituting the above solution into Eq. (2.53) and (2.54) for these quantities. The result is

$$E \equiv \Pi^{0} = \frac{L}{4\alpha'} \tag{2.58}$$

and the only non-zero component of $J^{\mu\nu}$ is

$$J^{12} = -\frac{L}{4}\frac{L}{4\alpha'}. \tag{2.59}$$

We observe that the total angular momentum $J = |J^{12}|$ is related to the energy by

$$J = \alpha' E. \tag{2.60}$$

For this reason α' was called the Regge slope.

One could envisage more complicated solutions in which the string not only rotates but vibrates along its length.

3. The free quantum string

The initial reaction of many people when told that the fundamental entities of Nature may be string-like is the response: why strings? This attitude is a reflection of the unexciting or perhaps all too familiar nature of string-like objects that we encounter in everyday life. Indeed, in the previous chapter on the classical string we did not, at first sight, encounter any properties that might lead one to expect that strings would seem preferred over many other extended objects.

In fact, it is only when one quantizes strings that one becomes aware that strings are in some way magical objects. In some sense, they take to quantum mechanics as a duck to water. We shall see that just as classically the string can be viewed as an infinite collection of point particles, when quantized it can be thought of as an infinite set of quantum particles, each corresponding to an irreducible representation of the Poincaré group and so to a given spin. The range in spins is from zero to infinity, there being in general more than one particle of a given spin.

Furthermore, it will emerge that some of these particles are massless. These massless particles have, for the open bosonic string, spin one, while for the closed bosonic string spin two and spin zero. These bosonic strings which have only space-time bosons contain the bosonic massless particles of most interest to theoretical physicists, namely the spin two of gravity and the spin one of Yang-Mills theories. In fact, in the limit in which the string shrinks to a point only, the massless states survive and one can verify that in the interacting strings the spin two and spin one describe Einstein gravity and Yang-Mills theories respectively [13, 14].

The spectra for superstring theories [15–17] are even more exciting in the sense that some of these theories have massless particles with spins two, 3/2, one and 1/2 and zero and hence can potentially also contain the known fermions, i.e., the quarks and leptons.

In general, there are considerable problems with theories which describe particless of spin 3/2 and above, in that they would not appear, even classically, to propagate these particles causally and they lead to an inconsistent theory when quantized. While, causal propagation occurs in some supergravity theories [18], these theories do not share the spectacular ultra-violet properties of rigid supersymmetric theories and are most unlikely to be renormalizable [19]. There are, however, reasons to believe that string theories are consistent in the sense that they are quantum mechanically consistent and they propagate causally.

The major obstacle to quantizing the string is that one must take into account the constraints, discussed in Section 2, which result from the reparametrization-invariance of the string. One procedure known as the light-cone theory solves the constraints and one is left with an unconstrained set of variables which may be straightforwardly quantized [2]. The disadvantage of this approach is that one loses manifest Lorentz invariance and if one is not careful, loses Lorentz invariance altogether.

The original quantization, called the old covariant method, of the free string works with the constraints and is Lorentz covariant. We will see that the spectrum of the quantum string emerges as a direct consequence of constraints.

The old covariant approach was developed before the discovery of BRST symmetries. This method has more recently also been used for string theory and it seems particularly powerful in this context. The gauge invariance upon which the formalism is applied is the two-dimensional reparametrization symmetry which is as usual fixed and has corresponding ghosts and antighosts introduced. Despite the power of this formalism, it is sometimes difficult to see the trees for the wood and even familiar manoeuvres may appear unfamiliar. We will therefore give first the old covariant method. Before doing this, however, let us give a heuristic argument which determines the particle spectrum of the string. Consider the classical solution given in Eq. (2.57) which was the rotating string of length L. The string in this configuration has an energy E and total angular momentum $J = |J^{12}|$ given by

$$E = \frac{1}{4\alpha'} L, \qquad J = \frac{L^2}{16\alpha'}. \tag{3.1}$$

We observe that the energy E and angular momentum J are related by

$$J = \alpha' E^2. \tag{3.2}$$

When we quantize the system, the angular momentum J is quantized, i.e., $J = n\hbar$ and as a result we have the relation

$$\alpha' E^2 = J = n\hbar. \tag{3.3}$$

We may regard E as the energy retained in the configuration and so interpret it as the rest mass. As a result, we find that we have a series of rest masses related to the spin by

$$\alpha' m_n^2 = n\hbar. \tag{3.4}$$

Thus the quantum string has an infinite number of particles whose (mass)^2 is proportional to their spin J.

Corresponding to the string vibrating as it rotates, we can find further particles whose relation to their spin is more complicated than that of Eq. (6.4). Despite the heuristic nature of the above argument, we will find, as usual for such arguments, the results are essentially correct.

3.1. The old covariant method

To quantize the classical string, we follow Dirac and demand that the Poisson brackets be rewritten as commutators according to the rule

$$\{A, B\} = C \tag{3.5}$$

becomes

$$[A, B] = i\hbar C, \tag{3.6}$$

where in (3.5) A, B and C are classical variables while in (3.6) they are quantum operators. As is well known, to apply this rule to all operators leads to inconsistencies, but it can always be applied to the co-ordinates and momenta.

For the string the momentum is given by

$$p^\mu = \frac{\delta A}{\delta(\partial_\tau x^\mu)}, \tag{3.7}$$

which in the gauge of Eq. (2.24) becomes $p^\mu = (+1/2\pi\alpha')\dot{x}^\mu$. The classical Poisson brackets for the string are given by

$$\begin{aligned} \{x^\mu(\sigma), x^\nu(\sigma')\} &= 0 = \{p^\mu(\sigma), p^\nu(\sigma')\}, \\ \{x^\mu(\sigma), p^\nu(\sigma')\} &= \eta^{\mu\nu}\delta(\sigma-\sigma'). \end{aligned} \tag{3.8}$$

We therefore impose on the quantum theory the equal time commutation relations

$$\begin{aligned} [x^\mu(\sigma), x^\nu(\sigma')] &= 0 = [p^\mu(\sigma), p^\nu(\sigma')], \\ [x^\mu(\sigma), p^\nu(\sigma')] &= i\hbar\delta(\sigma-\sigma')\eta^{\mu\nu}. \end{aligned} \tag{3.9}$$

The Schrödinger representation of these commutators corresponds to taking

$$x^\mu(\sigma) \to x^\mu(\sigma); \quad p^\mu(\sigma) \to -i\hbar \frac{\delta}{\delta x^\mu(\sigma)}. \tag{3.10}$$

We now find, as for the point particle, that the constraints of the classical theory given in Section 2, i.e., $T_{\alpha\beta} = 0$, become differential conditions which we must implement on a wave functional, $\psi[x^\mu(\sigma)]$. Unlike the point particle, the quantum operators L_n do not commute and we must be careful only to implement a compatible set on the wave functional. The correct procedure is to adopt the constraints [20]

$$\begin{aligned} L_n\psi &= 0, \quad n \geqslant 1 \\ (L_0 - 1)\psi &= 0. \end{aligned} \tag{3.11}$$

These equations are called the *physical state conditions*. Since $L_n^+ = L_{-n}$, the expectation value of L_n, $\langle\psi|L_n|\psi\rangle$ vanishes for all $n \neq 0$.

The situation here is like that which occurs in the Gupta-Bleuther formulation of electrodynamics where only half of the classical constraint $\partial^\mu A_\mu = 0$ is implemented on the wave function, i.e., $\partial^\mu A_\mu^+ \cdot \psi = 0$ where A_μ^+ is the positive frequency part of A_μ. To impose $\partial_\mu A^\mu \psi = 0$ at the quantum level would imply ψ itself was zero.

We now motivate and explain the meaning of the physical state conditions of Eq. (3.9). To do this, one must develop a precise formalism which in particular takes care of any normal ordering constant. The functional representation of Eq. (9) does not do this and we now give the oscillator formalism [21] for the open string and then for the closed string.

The oscillator formalism for the open string is based on the normal mode expansion of the string of Eq. (2.36). We define as for the classical string the oscillators α_n^μ by taking the Fourier components of the operator P^μ of Eq. (2.35):

$$P^\mu(\sigma) = i\hbar\sqrt{2\alpha'}\,\pi\frac{\delta}{\delta x^\mu(\sigma)} + \frac{x^{\mu'}(\sigma)}{\sqrt{2\alpha'}}$$

$$= \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\sigma}. \tag{3.12}$$

Using Eq. (2.36) we find that

$$\frac{\delta}{\delta x^\mu(\sigma)} = \sum_{n=-\infty}^{\infty} \frac{\partial x_n^\nu}{\partial x^\mu(\sigma)}\frac{\partial}{\partial x_n^\nu} = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{in\sigma}\frac{\partial}{\partial x_n^\mu}, \tag{3.13}$$

and as a result we may make the identification

$$\alpha_n^\mu = -i\left(\sqrt{2\alpha'}\frac{\partial}{\partial x_n^\mu} + \frac{n}{\sqrt{2\alpha'}}\frac{x_n^\mu}{2}\right). \tag{3.14}$$

From the reality of $P^\mu(\sigma)$, or by inspection, we conclude that

$$\alpha_n^{\mu\dagger} = \alpha_{-n}^\mu. \tag{3.15}$$

The α's are easily seen to satisfy the commutation relations

$$[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0}\eta^{\mu\nu} \tag{3.16}$$

in agreement with the Dirac rule of Eqs. (3.5) and (3.6) (see previous chapter). The Virasoro operators L_n of Eq. (2.38) can be expressed in terms of α_n^μ, by substituting Eq. (3.12), to yield the result

$$L_n = \tfrac{1}{2} : \sum_{m=-\infty}^{\infty} \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu} : \tag{3.17}$$

The dots indicate, as is standard in quantum mechanics, that we normally order the operators by putting α_{-n}^μ to the left of α_n^μ for $n > 0$. This difficulty only occurs for L_0.

The classical Poisson brackets for L_n are of the form

$$\{L_n, L_m\} = -i(n-m)L_{n+m}, \tag{3.18}$$

however, the above normal ordering of L_0 results in an additional modification to the quantum analogue of Eq. (3.18) over and above the implementation of the rule of Eqs (3.5) and (3.6). The result is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12} n(n^2-1)\delta_{n+m,0}, \tag{3.19}$$

where D is the dimension of the space-time.

The last term is called the central term [22] and it can be found by evaluating

$$\begin{aligned} \langle|[L_n, L_{-n}]|\rangle &= \langle|L_n L_{-n}|\rangle \\ &= \tfrac{1}{4} \sum_{m=1}^{n-1} \sum_{p=1}^{n-1} \langle|\alpha_m^\mu \alpha_{n-m}^\mu \alpha_{n-p}^{\nu\dagger} \alpha_p^{\nu\dagger}|\rangle \\ &= \frac{D}{2} \sum_{m=1}^{n-1} m(n-m) = \frac{D}{12} n(n^2-1), \end{aligned}$$

where $\alpha_n^\mu|\rangle = 0$ for $n \geqslant 1$, and we used the expression of Eq. (3.17) for L_n. The reader will easily see that for $n = 1$, the central term must vanish.

As suggested above, imposing $L_n\psi = 0$ for all n implies, as a result of the central term, that ψ itself is zero. That only L_0 has a constant term when implemented on ψ is a reflection of the fact that only this operator has a normal-ordering ambiguity. Imposing $L_n\psi = L_{-n}\psi = 0$ for more than two values of n implies $\psi = 0$. However, we can get a non-zero ψ by imposing $L_n\psi = 0$, $n \geqslant 1$ and in addition either $L_{-1}\psi = 0$ which implies $L_0\psi = 0$, or $(L_0 + a)\psi = 0$; $a = 0$. We will see that only the latter possibility is acceptable. We find in particular that

$$L_0 = \tfrac{1}{2}\alpha_0^\mu \alpha_{\mu 0} + \sum_{m=1}^{\infty} \alpha_m^{\mu\dagger} \alpha_{m\mu}, \tag{3.21}$$

$$L_1 = L_{-1}^{+} = \alpha_0^\mu \alpha_{1\mu} + \sum_{m=1}^{\infty} \alpha_m^{\mu\dagger} \alpha_{m+1\mu}, \tag{3.22}$$

where

$$\alpha_0^\mu = -i\sqrt{2\alpha'} \frac{\partial}{\partial x_0^\mu} \equiv \sqrt{2\alpha'}\, p_0^\mu, \tag{3.23}$$

i.e., α_0^μ is proportional to the momentum.

We can now analyze the content of the physical state conditions. Given *any* functional $\psi[x^\mu(\sigma)]$ we may express it in terms of the occupation number basis by using the creation operators $\alpha_n^{\mu\dagger}$, $n \geqslant 1$

$$\begin{aligned} \psi[x^\mu(\sigma)] &= \{\phi(x^\mu) + iA_\mu^1(x^\mu)\alpha_1^{\mu\dagger} \\ &+ iA_\mu^2(x^\mu)\alpha_2^{\mu\dagger} + h_{\mu\nu}\alpha_1^{\mu\dagger}\alpha_1^{\nu\dagger} + \ldots\} \langle x^\mu(\sigma)|0\rangle, \end{aligned} \tag{3.24}$$

where $x^\mu = x_0^\mu$. The vacuum satisfies the equation

$$\alpha_n^\mu < x^\mu(\sigma)\,|0\rangle = 0 \qquad n \geqslant 1. \tag{3.25}$$

The vacuum of Eq. (3.25) is of the form

$$\langle x^\mu(\sigma)\,|0\rangle = \prod_{n=1}^{\infty} c_n \exp\left(-\frac{n}{2\alpha'}\, x_n^\mu x_{n\mu}\right). \tag{3.26}$$

The action of the $\alpha_n^{\mu\dagger}$ on $\langle x^\mu(\sigma)|0\rangle$ produces the well-known complete set of Hermite polynomials. In terms of component fields, we find that the physical state conditions have

$$(\alpha'\partial^2+1)\phi = 0 = (\alpha'\partial^2-(l-1))A_\mu^l = (\alpha'\partial^2-1)h_{\mu\nu} \tag{3.27}$$

as well as

$$\begin{aligned}\partial^\mu A_\mu^1 = 0 &= \sqrt{2\alpha'}\,\partial^\mu A_\mu^2+2h_\nu^\nu \\ &= -\sqrt{2\alpha'}\,\partial^\mu h_{\mu\nu}+A_\nu^2.\end{aligned} \tag{3.28}$$

As a result, the string contains a tachyon, $\phi(x^\mu)$, a massless spin one, A_μ, a massive "spin two" and an infinite tower of states of even increasing mass. For a given mass $m_n^2 = \alpha'_{n-1}$, the highest spin is related to the contribution $\alpha_{-1}^{\mu_1}, \ldots, \alpha_{-1}^{\mu_n}$, $t_{\mu_1,\ldots,\mu_n}$ in ψ and has spin $J = n$.

A general string functional of $x^\mu(\sigma)$ contains an infinite number of possible negative norm states associated with the oscillator $\alpha_n^{(0)}$ which obeys the relation $[\alpha_n^0, \alpha_{-n}^0] = -n$. One such example being the state $\alpha_{-n}^0|\rangle$ which has norm $-n$. Physical states must, however, have positive norms and so one must ask if the physical state conditions eliminate all the negative norm states. One of the miracles of string theory is contained in the following theorem.

THEOREM [23]

For the space-tme dimension $D \leqslant 26$ the states ψ which satisfy the equations $L_n\psi = 0 n \geqslant 1$ $(L_0-1)\psi = 0$ have positive norm.

It is this theorem which provides the justification for the physical state conditions adopted earlier. In fact, for $D < 26$ the bosonic string is inconsistent for another reason associated with the singularity structure in one-loop graphs, i.e., it has a cut rather than pole structure. The reader may find it a good exercise to find, for $D > 26$ the lowest negative norm state (hint: it occurs with the massive states with the lowest positive mass).

The norms of the physical states are not, however, all positive definite. Consider for example the state

$$|s\rangle = L_{-1}|\Omega\rangle \tag{3.29}$$

with $L_0|\Omega\rangle = 0$, $L_n|\Omega\rangle = 0$, $n \geqslant 1$. It is easily verified that $|s\rangle$ is a physical state, that is, satisfies Eq. (3.11), and so

$$\langle\chi|s\rangle = \langle\chi|L_{-1}|\Omega\rangle = 0 \tag{3.30}$$

for any other physical state $|\chi\rangle$ including $|s\rangle$ itself. At the next level we may consider the state

$$|s'\rangle = (L_{-2}+bL_{-1}^2)\,|\Omega'\rangle, \tag{3.31}$$

where now $L_n|\Omega'\rangle = 0$, $n \geqslant 1$ $(L_0+1)\,|\Omega'\rangle = 0$. We demand $|s'\rangle$ to be physical and in particular

$$L_1|s'\rangle = 0 = (3L_{-1}-2bL_{-1})\,|\Omega'\rangle, \tag{3.32}$$

which implies that $b = +3/2$, while $L_2|s'\rangle = 0$ implies that $D = 26$. Consequently, if $D = 26$ and $b = 3/2$ we find that $|s'\rangle$ is a physical state and again

$$\langle s'|\chi\rangle = 0 \tag{3.33}$$

for any physical state $|\chi\rangle$.

The existence of zero norm states of the form

$$L_{-n_1} \dots L_{-n_2}|\Omega''\rangle, \tag{3.34}$$

with $L_n|\Omega''\rangle = 0$, $n \geqslant 1$, $L_0|\Omega''\rangle = (\Sigma_i n_i - 1)|\Omega''\rangle$ can be read off from the Kac determinant. The reader may verify by explicit computation that at the next level such a state only exists if $D = 28$.

The no-ghost theorem asserts that for $D = 26$ the physical states have positive norm which include states of zero norm, as found above, and states with positive definite norm. To decompose a given physical state into these two types of states, we choose a Lorentz frame in which the momentum of the state is labelled by p^μ and specify a vector $\bar{k}^\mu$ such that $(\bar{k}^\mu)^2 = 0$ and $\bar{k}_\mu p^\mu \neq 0$. Then $|\psi\rangle$ can be uniquely decomposed as $|\psi\rangle = |\chi_1\rangle + |\chi_2\rangle$ where both $|\chi_1\rangle$ and $|\chi_2\rangle$ are each separately physical states and in addition $K_n|\chi_1\rangle = 0$, $n > 0$ where $K_n = \alpha^\mu \bar{k}_\mu$ and $\langle$any physical state $|\chi_2\rangle = 0$. It follows that $\langle\chi_2|\chi_2\rangle = 0$, but it can also be shown that $\langle\chi_1|\chi_1\rangle > 0$. Furthermore [24], any zero norm physical state $|\chi_2\rangle$ can be written in the form $a|s\rangle + b|s'\rangle$ where $|s\rangle$ and $|s'\rangle$ are given in Eqs (3.29) and (3.31) above. In this sense the discussion of zero norm (i.e., null) states given above is complete.

At least one zero norm physical state is to be expected since we have in the spectrum a massless spin one state which is well known to have such a state. In the Lorentz frame, where the spin one has momentum $k^\mu = (k, 0, 0, k)$, the four photon states can be written as $\varepsilon_\mu a^{\mu\dagger}(k)|1\rangle$ and a particular basis is given by the states

$$a^{i\dagger}(k)\,|k\rangle, \quad i = 1, 2; \quad (a^0(k)^\dagger - a^3(k)^\dagger)\,|k\rangle, \tag{3.35}$$

and

$$(a^0(k)^\dagger + a^3(k)^\dagger)\,|k\rangle. \tag{3.36}$$

The last state is unphysical as it does not obey $k^\mu\varepsilon_\mu = 0$. The first three states are physical, but only the first two have positive definite norm. The third state has polarization vector k^μ and so has zero norm and also has zero scalar product with any physical state. The effect of this is to leave us with two states in accord with the theory of the irreducible representations of the Poincaré group.

In QED one known that, on-shell, the above zero norm physical state does not couple to states with positive definite norm states in the sense that, given any scattering amplitude for fermions and photons, all the photons of which are physical positive norm states except for one of which is a zero norm physical state, then the process will vanish. This is a consequence of the U(1) gauge invariance and in particular its Ward identity. Were it to fail, then unitarity would be violated. In string theory, however, we must verify that such decoupling of null states does indeed take place. This requirement has important consequences for interacting string theory. The number $d(n)$ of positive definite states at a given mass level $n[\alpha' m^2 = (n-1)]$ is given by the formula

$$\prod_{n=1}^{\infty} d(n)x^n = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{D-2}} = 1+(D-2)x+ \dots \tag{3.37}$$

This is in agreement with the above explicit computations, we have one tachyon at level zero and one spin one at level one which has on shell $(D-2)$ positive definite degrees of freedom. The reader may analyze the field content at level two and check the result with the above formula. The power $D-2$ is most easily seen in the light-cone formulation where one has solved the constraints leaving one with D centre-of-mass x_0^μ co-ordinates and $(D-2)$ co-ordinates x_n^i; $i = 1, \dots, D-1$ [25]. The most general functional of x_0^μ and x_n^i is built up with oscillators α_n^i.

Of course, the particles are labelled by the irreducible representations of the Poincaré group which are labelled by representations of $SO(D-1)$ if massive and $SO(D-2)$ if massless. The massive spin two mentioned above being the symmetric traceless representation of $SO(D-1)$.

It remains to express the total momentum and angular momentum in terms of oscillators. We find, substituting into Eq. (2.53), that

$$\Pi^\mu = \int_0^\pi \frac{1}{2\pi\alpha'}\partial_\tau x^\mu d\sigma = \frac{1}{4\pi\alpha'}\int_{-\pi}^{\pi}\partial_\tau x^\mu d\sigma$$

$$= \frac{1}{2\pi\sqrt{2\alpha'}}\int_{-\pi}^{\pi} P^\mu d\sigma = \frac{1}{2\pi\sqrt{2\alpha'}}\int_{-\pi}^{\pi}\sum_n \alpha_n^\mu e^{in\sigma} d\sigma$$

$$= \frac{\alpha_0^\mu}{\sqrt{2\alpha'}} = -i\frac{\partial}{\partial x_0^\mu}.$$

We have used Eq. (2.33), (2.35) and (3.12). As expected, it is the centre-of-mass momentum of the string.

To compute the total angular momentum we require x^μ in terms of oscillators given in the previous section. Substituting in Eq. (2.54) we find

$$J^{\mu\nu} = \frac{1}{2\pi\alpha'} : \int_0^\pi (x^\mu \partial_\tau x^\nu - x^\nu \partial_\tau x^\mu) d\sigma :$$

$$= -\tfrac{1}{2} : \int_{-\pi}^{\pi} (x^\mu P^\nu - x^\nu P^\mu) d\sigma :$$

$$= \frac{1}{\sqrt{2\alpha'}} (x^\mu \alpha_0^\nu - x_0^\nu \alpha_0^\mu) + \frac{2}{i} \sum_{n=1}^{\infty} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu).$$

The first term is the usual orbital expressed for angular momentum, while the second term takes account of the Lorentz indices carried by the fields. This completes the discussion for the open string.

The *closed string* proceeds very similarly. After quantization, we first define oscillators α_n^μ and $\tilde{\alpha}_n^\mu$ by

$$P^\mu = \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\sigma},$$

$$\tilde{P}^\mu = \sum_{n=-\infty}^{\infty} \bar{\alpha}_n^\mu e^{+in\sigma}. \qquad (3.38)$$

They obey the relations

$$\alpha_n^{\mu\dagger} = \alpha_{-n}^\mu, \qquad \bar{\alpha}_n^{\mu\dagger} = \bar{\alpha}_{-n}^\mu.$$

The commutation relations are

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}; \qquad [\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0}, \qquad (3.39)$$

$$[\alpha_n^\mu, \tilde{\alpha}_m^\nu] = 0.$$

We may express α_n^μ and $\tilde{\alpha}_n^\mu$ in terms of the normal co-ordinates x_n^μ to find

$$\alpha_n^\mu = i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial x_n^\mu} - \frac{inx_{-n}^\mu}{\sqrt{2\alpha'}},$$

$$\bar{\alpha}_n^\mu = -i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial x_{-n}^\mu} - \frac{inx_n^\mu}{\sqrt{2\alpha'}}.$$

We recognize that $\alpha_n^\mu = \tilde{\alpha}_n^\mu = \dfrac{\sqrt{2\alpha'}}{2} p_0^\mu$ as is expected as there is only one centre-of-mass co-ordinate.

The Virasoro generators are expressed in terms of the oscillators by

$$L_n = \tfrac{1}{2} : \sum_{m=-\infty}^{\infty} \alpha_m^\mu \alpha_{n-m}^\nu : \eta_{\mu\nu},$$

$$\bar{L}_n = \tfrac{1}{2} : \sum_{m=-\infty}^{\infty} \bar{\alpha}_m^\mu \bar{\alpha}_{n-m}^\nu : \eta_{\mu\nu}. \tag{3.40}$$

They obey the algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12} n(n^2-1)\delta_{n+m,0},$$

$$[L_n, \bar{L}_m] = 0,$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{D}{12} n(n^2-1)\delta_{n+m,0}. \tag{3.41}$$

The physical state conditions for the closed string are given by

$$L_n\psi = \bar{L}_n\psi = 0, \quad n \geqslant 1,$$

$$(L_0 + \bar{L}_0 - 2)\psi = 0, \tag{3.42}$$

$$(L_0 - \bar{L}_0)\psi = 0. \tag{3.43}$$

The latter constraint has a simple interpretation. From Eqs. (2.45), (2.46), (2.41) and (2.42), we find $L_0 - \bar{L}_0$ in functional form to be

$$L_0 - \bar{L}_0 = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\sigma((P^\mu)^2 - (\bar{P}^\mu)^2)$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} 2x^{\mu\prime}(\sigma) \frac{\delta}{\delta x^\mu(\sigma)}.$$

Consequently $L_0 - \bar{L}_0$ implement the change $\sigma \to \sigma + \varepsilon$ and the constraint $(L_0 - \bar{L}_0)\psi = 0$ tells us that there is no preferred point on the closed string.

There are two types of closed strings called orientated and unorientated. An unorientated string has the symmetry $\sigma \to -\sigma$ and so the waves of the form $f(\tau+\sigma)$ and $g(t-\sigma)$ moving around the closed string have the same amplitude. In terms of oscillators $\sigma \leftrightarrow -\sigma$ implies $\alpha_n^\mu \leftrightarrow \bar{\alpha}_n^\mu$. An unorientated string has no such symmetry and so the amplitudes for left and right moving waves are unrelated. Expanding the most general functional $\psi(x^\mu(\sigma))$ out in terms of Hermite polynomials

$$\psi = (\varphi(x^\mu) + h^\mu(x)\alpha_{-1}^\mu + k^\mu(x)\bar{\alpha}_{-1}^\mu + h_1^{\mu\nu}(x)\alpha_{-1}^\mu \bar{\alpha}_{-1}^\nu$$

$$+ h_2^{\mu\nu}(x)\alpha_{-1}^\mu \alpha_{-1}^\nu + \ldots) \langle x^\mu(\sigma) | 0 \rangle, \tag{3.44}$$

where

$$\alpha_n^\mu \langle x^\mu(\sigma)\,|0\rangle = \bar{\alpha}_n^\mu \langle x^\mu(\sigma)\,|0\rangle = 0, \qquad n \geqslant 1. \tag{3.45}$$

The constraint $(L_0 - \bar{L}_0)\psi = 0$ implies that terms such as the second, third and fifth are not allowed. For an unorientated string we must impose $\alpha_n^\mu \leftrightarrow \tilde{\alpha}_n^\mu$ and hence $h_1^{\mu\nu}$ is a symmetric field.

We find that the lowest physical states for the unorientated string are at the lowest level a tachyon, and at the next level a massless "spin two" and spin zero, By massless "spin two" we mean that it corresponds to the traceless symmetric representation of $SO(D-2)$. It can be shown that the physical states of Eqs (3.41) and (3.42) all have positive norm of $D \leqslant 26$.

3.2. The BRST approach

3.2.1. The BRST action

We begin with the string action of Eq. (2.6) which we reproduce here for convenience

$$A^{\text{orig}} = -\frac{1}{4\pi\alpha'}\int d^2\xi\,\sqrt{-g}\;g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}. \tag{3.46}$$

As explained in Chapter 2, this action is invariant under the two-dimensional diffeomorphisms

$$\delta x^\mu = f^\alpha \partial_\alpha x^\mu,\ \delta g_{\alpha\beta} = f^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha f^\gamma g_{\gamma\beta} + \partial_\beta f^\gamma g_{\alpha\gamma}. \tag{3.47}$$

It will be useful to rewrite, in accord with standard practice, the variation of fields in terms of the covariant derivative. For completeness, we recall that the covariant derivative ∇_α acts on a tensor $T_{\gamma\ldots}^{\beta\ldots}$ according to the rule

$$\nabla_\alpha T_{\gamma\ldots}{}^{\beta\ldots} = \partial_\alpha T_{\gamma\ldots}{}^{\beta\ldots} - \Gamma_{\alpha\gamma}{}^{\delta} T_{\delta\ldots}{}^{\beta\ldots} + \Gamma_{\delta\alpha}{}^{\beta} T_{\gamma\ldots}{}^{\delta\ldots} + \ldots, \tag{3.48}$$

where, in the absence of torsion, the Christoffel symbol is given by

$$\Gamma_{\alpha\beta}{}^{\gamma} = \tfrac{1}{2}\, g^{\gamma\delta}(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}). \tag{3.49}$$

One easily verifies that we may write Eq. (3.47) as

$$\delta x^\mu = f^\alpha \nabla_\alpha x^\mu, \qquad \delta g_{\alpha\beta} = \nabla_\alpha f_\beta + \nabla_\beta f_\alpha. \tag{3.50}$$

The action contains the inverse of $g_{\alpha\beta}$ and we may write its variation as

$$\delta g^{\alpha\beta} = -g^{\alpha\delta}\delta g_{\gamma\delta} g^{\delta\beta} = -(\nabla^\alpha f^\beta + \nabla^\beta f^\alpha), \tag{3.51}$$

where

$$\nabla^\alpha \equiv g^{\alpha\delta}\nabla_\delta. \tag{3.52}$$

We also note that

$$\delta g = g g^{\gamma\delta}\delta g_{\gamma\delta} = 2g\nabla_\alpha f^\alpha. \tag{3.53}$$

Recall that the covariant derivative of the metric vanishes.

We now gauge fix the above action and introduce ghosts and antighosts according to the "standard" BRST method. The final goal in this method is to gauge fix the symmetry so as to yield an invertible propagator and introduce ghosts and antighosts in such a way that the BRST transformations are nilpotent and leave the action invariant. The reader unfamiliar with the procedure will find it useful to carry out the analogous steps for Yang--Mills theory at every step (see the Appendix). The first BRST treatment of the string was in Ref. [26]. We first adopt a gauge-fixing term. According to the theorem given in Chapter 2, we may locally bring the metric to the form $g_{\alpha\beta} = e^{\phi(\xi)}\eta_{\alpha\beta}$, using the two-dimensional general co-ordinate invariance. We could therefore choose this as our gauge-fixing condition, however, to avoid the appearance of the field ϕ, which in fact did not occur in the original action, we choose to gauge fix the ϕ independent (i.e., Weyl invariant) combination $\sqrt{-g}\, g^{\alpha\beta}$. We therefore take this combination to be given by

$$\sqrt{-g}\, g^{\alpha\beta} = \sqrt{-\hat{g}}\, \hat{g}^{\alpha\beta}, \tag{3.54}$$

where $\hat{g}_{\alpha\beta}$ is any two-dimensional metric. One choice is $\sqrt{-\hat{g}}\, \hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$ but we can clearly, according to the above theorem, use any $\hat{g}_{\alpha\beta}$. We therefore introduce the gauge-fixing term

$$A^{\mathrm{GF}} = -\frac{1}{\pi}\int d^2\xi \lambda_{\alpha\beta}(\sqrt{-g}\, g^{\alpha\beta} - \sqrt{-\hat{g}}\, \hat{g}^{\alpha\beta}), \tag{3.55}$$

where $\lambda_{\alpha\beta}$ is the usual Lautrup multiplier field. (The reason for the choice of the factor 2π will become apparent later when we come to define the ghost-antighost anticommutators).

The constraint $\sqrt{-g}\, g^{\alpha\beta} = \sqrt{-\hat{g}}\, \hat{g}^{\alpha\beta}$ involves only two conditions and so only two components of $\lambda_{\alpha\beta}$ contribute in the above action. We may take $\lambda_{\alpha\beta}$ symmetric (i.e., $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$) and traceless with respect to either $g_{\alpha\beta}$ or $\hat{g}_{\alpha\beta}$. Although the latter condition does not correspond precisely with the part of $\lambda_{\alpha\beta}$ which drops from the action, this is immaterial as we have retained in $\lambda_{\alpha\beta}$ the two components that do remain.

As usual the BRST variations of the original fields are given by the substitution

$$f^{\alpha} \to \Lambda c^{\alpha}, \tag{3.56}$$

where Λ is a rigid, i.e., ξ^{α}, independent anticommuting parameter and c^{α} are two-ghost fields into Eq. (3.47). We then have

$$\delta x^{\mu} = \Lambda c^{\alpha}\partial_{\alpha}x^{\mu}, \quad \delta g^{\alpha\beta} = -\Lambda(\nabla^{\alpha}c^{\beta} + \nabla^{\beta}c^{\alpha}). \tag{3.57}$$

Introducing antighosts $b_{\alpha\beta}$ we take them to have the usual variations which are

$$\delta\lambda_{\alpha\beta} = 0, \quad \delta b_{\alpha\beta} = \Lambda\lambda_{\alpha\beta}. \tag{3.58}$$

The tensor character of the antighosts is chosen to match that of $\lambda_{\alpha\beta}$, in view of the latter equation, and so $b_{\alpha\beta}$ is also symmetric and traceless. Finally, we take the variation of the ghosts to be

$$\delta c^{\alpha} = \Lambda c^{\gamma}\nabla_{\gamma}c^{\alpha} = \Lambda c^{\gamma}\partial_{\gamma}c^{\alpha}. \tag{3.59}$$

This is in accord with the Yang-Mills result where that the variation of the ghosts is given by the "structure" constant with two of its indices contracted with those of the ghost fields. In our case, the commutator of two general co-ordinate transformations with parameters ζ_1^{α} and ζ_2^{β} yields a general co-ordinate transformation with parameter

$$\zeta_1^{\alpha}\partial_{\alpha}\zeta_2^{\beta}-(1\leftrightarrow 2) = \zeta_1^{\alpha}\nabla_{\alpha}\zeta_2^{\beta}-(1\leftrightarrow 2). \tag{3.60}$$

The important properties of BRST transformations is that they should be nilpotent, that is $\delta_{\Lambda_1}\,\delta_{\Lambda_2}$ on any field vanishes. In particular, on x^{μ} we have

$$\delta_1\delta_2 x^{\mu} = \Lambda_1 c^{\alpha}\partial_{\alpha}(\Lambda_2 c^{\beta}\partial_{\beta}x^{\mu})+\Lambda_1(\Lambda_2 c^{\delta}\partial_{\delta}c^{\alpha})\partial_{\alpha}x^{\mu} = 0. \tag{3.61}$$

The reader may verify the result for the remaining fields.

The final step is to find an action A^{gh} involving $b_{\alpha\beta}$ and c^{γ} such that

$$A^{\mathrm{T}} = A^{\mathrm{orig}}+A^{\mathrm{gf}}+A^{\mathrm{gh}} \tag{3.62}$$

is BRST invariant. The result is given by

$$A^{\mathrm{gh}} = -\frac{1}{\pi}\int d^2\xi\,\sqrt{g}\,b_{\alpha\beta}(\nabla^{\alpha}c^{\beta}+\nabla^{\beta}c^{\alpha}-g^{\alpha\beta}\nabla_{\delta}c^{\delta}). \tag{3.63}$$

One can find A^{gh} by explicit variation of A^{gf} or by observing that

$$\Lambda(A^{\mathrm{gf}}+A^{\mathrm{gh}}) = -\delta\frac{1}{\pi}\int d^2\xi b_{\alpha\beta}(\sqrt{-g}\,g^{\alpha\beta}-\sqrt{-\hat{g}}\,\hat{g}^{\alpha\beta}) \tag{3.64}$$

and as $\delta^2 = 0$ then $\delta(A^{\mathrm{gf}}+A^{\mathrm{gh}}) = 0$.

It is appropriate to comment on the Hermitian character of the fields. Let us take c^{α} to be Hermitian, then reality of the action implies that $b_{\alpha\beta}$ is antihermitian. We recall that for two anticommuting variables η and ϱ complex conjugation is implemented on the product by $(\eta\varrho)^* = \varrho^*\eta^* = -\eta^*\varrho^*$. Examining the transformations of the fields, we conclude that Λ is antihermitian. Should the above procedure remind the reader of a cook--book recipe, that is because the BRST procedure is rather like one. We refer the reader to the discussion in Chapter 1 and the Appendix. An alternative way to arrive at the above result is to use the well-known insertion of 1 into the functional integral. We would insert

$$1 = \Delta_{\mathrm{FP}}\int_{\mathrm{FP}}\int d^2\zeta\,\delta(\sqrt{-g_{\zeta}}\,g_{\zeta}^{\alpha\beta}-\sqrt{-\hat{g}}\,\hat{g}^{\alpha\beta}), \tag{3.65}$$

where $g_{\zeta}^{\alpha\beta}$ is a diffeomorphism of $g^{\alpha\beta}$ with parameter ζ. Using a standard technique (see, for example, the lectures of B. W. Lee in Les Houches, 1975), we find Δ_{FP} is independent of ζ and can be given in the functional integral by

$$\Delta_{\rm FP} \sim \det \left\{ \frac{\delta}{\delta\zeta^\gamma} (\sqrt{-g_\zeta}\, g_\zeta^{\alpha\beta}) \right\}_{\zeta^\gamma=0}$$

$$= \det \sqrt{-g}\, \frac{\delta}{\delta\zeta^\gamma} (\nabla^\alpha\zeta^\beta + \nabla^\beta\zeta^\alpha - g^{\alpha\beta}\nabla_\delta\zeta^\delta)\Big|_{\zeta^\gamma=0}$$

$$= \det \{ -\sqrt{-g}\, (\nabla^\alpha\delta^\beta_\gamma + \nabla^\beta\delta^\alpha_\gamma - \tfrac{1}{2}\, g^{\alpha\beta}\nabla_\gamma)\delta^2 \}. \tag{3.66}$$

We may write this determination in terms of the integral over Faddeev-Popov ghosts with the action of Eq. (3.63). The integral over ζ may be disregarded and the δ function can be implemented by an integral over $\lambda_{\alpha\beta}$ of the action of Eq. (3.55). We leave the reader to fill in the steps above which are missing and implement the resulting determinants by ghost integrations.

We can integrate over $\lambda_{\alpha\beta}$ in the functional, enforcing $\sqrt{-g}\, g^{\alpha\beta} = \sqrt{-\hat{g}}\, \hat{g}^{\alpha\beta}$, to find the action

$$-\frac{1}{4\pi\alpha'} \int d^2\xi\, \sqrt{-\hat{g}}\, \hat{g}^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$$

$$-\frac{1}{\pi} \int d^2\xi\, \sqrt{-\hat{g}}\, b_{\alpha\beta}(\hat{\nabla}^\alpha c^\beta + \hat{\nabla}^\beta c^\alpha - \hat{g}^{\alpha\beta}\hat{\nabla}_\delta c^\delta), \tag{3.67}$$

where now the $\Gamma^\delta_{\alpha\beta}$ in ∇^α is given by Eq. (3.49) but with $g_{\alpha\beta}$ replaced by $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}e^\phi$ one finds

$$-\frac{1}{4\pi\alpha'} \int d^2\xi \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$$

$$-\frac{1}{\pi} \int d^2\xi b_{\alpha\beta}(\partial^\alpha c^\beta + \partial^\beta c^\alpha - \eta^{\alpha\beta}\partial_\delta c^\delta). \tag{3.68}$$

This result is independent of ϕ as must be the case due to Weyl invariance or, put another way, ϕ does not occur in $A^{\rm orig}$ and $A^{\rm gf}$. The reader may explicitly verify this fact by substituting the Christoffel symbol for the metric $g_{\alpha\beta} = \eta_{\alpha\beta}e^\phi$, which is found to be given by

$$\Gamma_{\alpha\beta}{}^\gamma = \tfrac{1}{2}\, (\partial_\alpha\phi\delta^\gamma_\beta + \partial_\beta\phi\delta^\gamma_\alpha - \partial^\gamma\phi\eta_{\alpha\beta}. \tag{3.69}$$

The above manoeuvres to arrive at the action of Eq. (3.68) like those for the point particle can be criticized from several points of view. The problem centres around whether the gauge fixing of Eq. (3.54) is actually a good one. The invariance of the action places restrictions on the parameter ζ at the end points of the string and it would seem that these missing degrees of freedom in ζ do not allow one to reach the gauge of Eq. (3.54). The good gauge-fixing conditions involve derivatives on $g_{\alpha\beta}$ [27]. The resulting differences are essential for one of two formulations of gauge covariant string theory [27]. Those parts of $g_{\alpha\beta}$ which are subject to derivatives are dynamical and play the role of moduli in the string field theory.

The above problems can also be seen from the fact that the above ghost action has a zero mode problem in that $\nabla_\alpha b^{\alpha\beta} = 0$ has $3(g-1)$ solutions on a Rieman surface of genus g for $g \geqslant 2$ and one solution for a genus one surface. This is of course the number of moduli: however, the identification of the moduli in the extension of the BRST procedure to take into account the zero mode problem has not been completely carried out. For the substantial progress on this problem, we refer the reader to Ref. [28].

The more common method of tackling this problem is to give a prescription for sum over world sheets in the path integral by breaking the integration over $g_{\alpha\beta}$ into diffeomorphisms, a conformal factor and $3(g-1)$ moduli. Since the action is invariant in $D = 26$ under the first two, one is left with the integral over moduli, see Ref. [29] for a review.

Another objection is that the above classical steps are not valid quantum mechanically except in 26 dimensions. Indeed, it is only in 26 dimensions that the conformal anomaly cancels and outside 26 dimensions one finds that the field ϕ occurs explicitly in the action. This action is no longer a free theory and there is no complete method for its quantization, however substantial progress has been made by Bilal, Gervais and Neveu. It would seem that in certain dimensions these Liouville theories are be consistent.

We now quantize the action of Eq. (3.67) or (3.68), the end goal being to describe the on-shell states of the quantized string. We proceed in the same way as in the "old covariant method" given in the previous chapter, but now we have the ghost co-ordinates to include. Before quantizing the action, let us consider its equation of motion. Varying x^μ, c^γ and $b_{\alpha\beta}$ we find the results

$$\partial_\alpha \partial^\alpha x^\mu = 0,$$

$$\partial_\beta b^{\alpha\beta} = 0,$$

$$\partial^\alpha c^\beta + \partial^\beta c^\alpha - \tfrac{1}{2}\eta^{\alpha\beta}\partial_\delta c^\delta = 0. \tag{3.70}$$

We must however apply the discussion given in Chapter 2 on boundary terms. In the closed string case that the boundary term vanishes due to the boundary condition

$$x^\mu(-\pi) = x^\mu(\pi), \quad b_{\alpha\beta}(-\pi) = b_{\alpha\beta}(\pi), \quad c^\beta(-\pi) = c^\beta(\pi), \tag{3.71}$$

while in the open string case we must require as before $x^{\mu'} = 0$ at $\sigma = 0$ and π as well as

$$\delta c^\beta \frac{\delta A}{\delta(\partial_\sigma c^\beta)} = (-2)(\delta c^1 b_{11} + \delta c^0 b_{10}) = 0 \quad \text{at } \sigma = 0 \text{ and } \pi. \tag{3.72}$$

The latter equation will be satisfied if both

$$b_{01} = 0 \quad \text{and} \quad c^1 = 0 \quad \text{at} \quad \sigma = 0 \quad \text{and } \sigma = \pi \tag{3.73}$$

as the variation of a field has the same boundary conditions as the field itself. Other choices are possible, however, one should choose to restrict conjugate variables, identified below, to ensure that conjugate variable have the same number of degrees of freedom. Equation

(3.73) is one choice, but it would make no difference if we chose the other possibility. Equations (3.73) imply, using the equations of motion (3.70) that

$$\partial_1 c^0 = 0 \quad \text{and} \quad \partial_1 b_{00} = 0 \tag{3.74}$$

at $\sigma = 0$ and $\sigma = \pi$.

Writing the ghost action out explicitly, we find it is given by

$$\frac{-2}{\pi}\int d^2\xi\{b_{01}(-\partial_0 c^1+\partial_1 c^0)+b_{00}(-\partial_0 c^0+\partial_1 c^1)\}. \tag{3.75}$$

We recognize that $-2/\pi\ b_{01}$ is conjugate to c^1 and vice versa and $-2/\pi\ b_{00}$ is conjugate to c^0 and vice versa. It is up to us to choose which to take as co-ordinates and which as momentum. We could choose as coordinates b_{01} and c^0 implying that $-2/\pi\ c^1$ and $-2/\pi\ b_{00}$ are their respective momenta. Changing the Poisson brackets (see Chapter 1 for Poisson brackets) for anticommuting variables, to anticommutators according to the Dirac rule we have

$$\{b_{01}(\sigma)c^1(\sigma')\} = -\frac{i\pi}{2}\delta(\sigma-\sigma')$$

$$\{c^0(\sigma), b_{00}(\sigma')\} = -\frac{i\pi}{2}\delta(\sigma-\sigma') \tag{3.76}$$

the rest vanishing. The Schrödinger representation we could take

$$b_{00} = -\frac{i\pi}{2}\frac{\delta}{\delta c^0}, \quad c^1 = -\frac{i\pi}{2}\frac{\delta}{\delta b_{01}}. \tag{3.77}$$

This representation was used in several previous works of the author.

For the *open string*, it is advantageous to extend the range of σ from 0 to π to be from $-\pi$ to π by taking

$$c^0(\sigma) \equiv c^0(-\sigma), \quad c^1(\sigma) = -c^1(-\sigma) \tag{3.78}$$

$$b_{00}(\sigma) = b_{00}(-\sigma), \quad b_{01}(\sigma) = -b_{01}(-\sigma) \tag{3.79}$$

for $0 < \sigma < \pi$. This choice is dictated by requiring that the boundary conditions of Eq. (3.73) and (3.74) at $\sigma = 0$, be automatically encoded. We can then define on the range $-\pi < \sigma < \pi$ the fields

$$c(\sigma) = c^0(\sigma)+c^1(\sigma) \tag{3.80}$$

and

$$b(\sigma) = (-2)\,(b_{00}(\sigma)+b_{01}(\sigma)), \tag{3.81}$$

which have no particular symmetry as $\sigma \to -\sigma$, except for the boundary conditions of Eqs. (3.73) and (3.74) at $\sigma = \pi$. These latter conditions may be expressed as $c(\pi) = -c(\pi)$

and $\partial c/\partial\sigma(\pi) = +\partial c/\partial\sigma(-\pi)$ and similarly for b. We may further extend the range of σ to all σ, since these boundary conditions allow us

$$c(\sigma) = c(\sigma+2\pi), \quad b(\sigma) = b(\sigma+2\pi). \tag{3.82}$$

Conversely, taking the above conditions and demanding that c and b be continuous and have first derivative ensures all the above boundary conditions. In terms of the fields $c(\sigma)$ and $b(\sigma)$, Eq. (3.75) becomes

$$\{b(\sigma), b(\sigma')\} = \{c(\sigma), c(\sigma')\} = 0,$$

$$\{b(\sigma), c(\sigma')\} = 2i\pi\delta(\sigma-\sigma') \quad \text{for} \quad \begin{matrix} -\pi < \sigma < \pi \\ -\pi < \sigma' < \pi \end{matrix} \tag{3.83}$$

from which we recognize $b(\sigma)$ and $c(\sigma)$ as conjugate variables.

The normal mode expansion is given by

$$c(\sigma) = \sum_{n=-\infty}^{\infty} c_n e^{-in\sigma} \tag{3.84}$$

and

$$b(\sigma) = i \sum_{n=-\infty}^{\infty} b_n e^{-in\sigma}, \tag{3.85}$$

where c_n and c_{-n} and b_n and b_{-n} are not related, other than by

$$c_n^\dagger = c_{-n}, \quad b_n^\dagger = b_{-n}. \tag{3.86}$$

We recall that $c(\sigma)$ and $b(\sigma)$ are Hermitian and antihermitian respectively. Using the relation

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\sigma} c(\sigma) d\sigma, \tag{3.87}$$

we find that Eq. (3.83) becomes

$$\{c_n, c_m\} = 0 = \{b_n, b_m\}$$

$$\{c_n, b_m\} = \delta_{n+m,0}. \tag{3.88}$$

Rather than use the representation of Eq. (3.77), it is often more useful to take

$$b(\sigma) = 2i\pi \frac{\delta}{\delta c(\sigma)} \quad \text{or equivalently} \quad b_n = \frac{\delta}{\delta c_{-n}}. \tag{3.89}$$

Note that unlike for $\delta/\delta x^\mu(\sigma)$ we do not require an i for hermiticity since we are dealing with anticommuting quantities and

$$\left(\frac{\delta}{\delta c(\sigma)}\right)^\dagger = \frac{\delta}{\delta c(\sigma)}. \tag{3.90}$$

Important for what follows is the appearance of the zero modes c_0 and b_0 which are Hermitian and obey $\{b_0, c_0\} = 1$.

We now define a vacuum with respect to these oscillators. Clearly we can take

$$c_n|\,\rangle = b_n|\,\rangle = 0 \quad n \geqslant 1 \tag{3.91}$$

as well as the usual condition for the bosonic α_n^μ oscillators. The action of the zero modes on the vacuum, however, requires more care [26]. We can define a vacuum $|+\rangle$ by

$$c_0|+\rangle = 0, \tag{3.92}$$

then under b_0 we find a new vacuum

$$b_0|+\rangle = |-\rangle. \tag{3.93}$$

Since $b_0^2 = 0$, we find that $b_0|-\rangle = 0$. From the relation $\{c_0, b_0\} = 1$, we also find $c_0|-\rangle = |+\rangle$. We note that

$$\langle +|+\rangle = \langle -|c_0 c_0|-\rangle = 0 \tag{3.94}$$

and similarly for $\langle -|-\rangle = 0$. However, we have the relations

$$\langle +|-\rangle = \langle -|c_0 b_0|+\rangle = \langle -|+\rangle \tag{3.95}$$

and we choose $\langle +|-\rangle = 1$. We take $|-\rangle$ vacuum to be Grassman odd and so the $|+\rangle$ vacuum is Grassman even as c_0 is an odd object.

Let us consider the most general function X of $x^\mu(\sigma)$, and the ghosts. In the oscillator basis it may be written as

$$\begin{aligned} |\chi\rangle &= \psi|-\rangle + \varphi|+\rangle \\ &\equiv \sum_{\{n\}\{m\}} c_{n_1}^+ \ldots c_{n_b}^+ b_{m_1}^+ \ldots b_{m_a}^+ \varphi_{n_1 \ldots n_b}{}^{m_1 \ldots m_a}[x^\mu(\sigma)]\,|-\rangle \\ &+ \sum_{\{n\}\{m\}} c_{n_1}^+ \ldots c_{n_{b'}}^+ b_{m_1}^+ \ldots b_{m_{a'}} \varphi_{n_1 \ldots n_{b'}}{}^{m_1 \ldots m_{a'}}[x^\mu(\sigma)]\,|+\rangle. \end{aligned} \tag{3.96}$$

We note that $\psi_{n_1 \ldots n_b}{}^{m_1 \ldots m_a} = \psi_{[n_1 \ldots n_b]}{}^{[m_1 \ldots m_a]} \equiv \psi_b^a$ and if $a+b$ is an odd integer then ψ_b^a is an anticommuting field. We can also expand the functionals of $x^\mu(\sigma)$ in terms of α_n^μ oscillators to obtain

$$\begin{aligned} |\chi\rangle &= (\varphi(x) + A^\mu(x)\alpha_{-1}^\mu + e b_{-1} + \bar{e} c_{-1} \\ &+ e^\mu \alpha_{-1} b_{-1} + f b_{-2} + \ldots)\,|-\rangle \\ &+ (\varphi'(x) + A^{\mu'} \alpha_{-1}^\mu + \ldots)\,|+\rangle. \end{aligned} \tag{3.97}$$

Let us repeat the above for the *closed string*, the difference now being that the fields are periodic with period 2π. Let us choose, as our co-ordinates $c \equiv c^0 + c^1$ and $\bar{c} \equiv c^0 - c^1$ rather than c^0 and b_{01} as above. The corresponding momenta are $\bar{b} = -2(b_{00} + b_{01})1/\pi$

and $\bar{b} = (-2)(b_{00} - b_{01})1/\pi$, the anticommutators being

$$\{c(\sigma),\ b(\sigma')\} = 2i\pi\ \delta(\sigma-\sigma') \tag{3.98}$$

$$\{\bar{c}(\sigma),\ \bar{b}(\sigma')\} = 2i\pi\ \delta(\sigma-\sigma') \tag{3.99}$$

the remaining anticommutators vanishing. The normal mode expansions appropriate to their periodic character are

$$c(\sigma) = \sum_n e^{-in\sigma} c_n, \quad \bar{c}_n = \sum_n e^{in\sigma} \bar{c}_n \tag{3.100}$$

$$b(\sigma) = i\sum_n e^{-in\sigma} b_n, \quad \bar{b}_n = i\sum_n e^{in\sigma} \bar{b}_n \tag{3.101}$$

and the corresponding hermiticity properties are

$$c_n^+ = c_{-n}, \quad \bar{c}_n^+ = \bar{c}_{-n}, \quad b_n^+ = b_{-n}, \quad \bar{b}_n^+ = \bar{b}_{-n}. \tag{3.102}$$

They obey the relation

$$\{c_n, b_m\} = \delta_{n+m,0}; \quad \{\bar{c}_n, \bar{b}_m\} = \delta_{n+m,0} \tag{3.103}$$

the remaining anticommutators vanishing.

For the closed string, we have twice as many zero modes as for the open string, namely b_0, c_0, $\bar{b}_0$, $\bar{c}_0$. We can now write the most general functional of x^μ, $c(\sigma)$, $\bar{c}(\sigma)$ in terms of the oscillators α_n^μ, c_n, b_n, $\bar{c}_n$, $\bar{b}_n$ acting on the vacua. Corresponding to the existence of the zero modes above, we have four types of vacuum

$$|+,+\rangle, \quad |+,-\rangle, \quad |-,+\rangle \quad \text{and} \quad |-,-\rangle \tag{3.104}$$

The first entry refers to the behaviour under β_0, $\bar{\beta}_0$ and the second to that under $\tilde{\beta}_0$, $\tilde{\bar{\beta}}_0$.

3.2.2. The energy-momentum tensor and BRST charge

The action of Eq. (3.68) is Poincaré and BRST invariant and we may compute the corresponding conserved current and charges. Given any action A invariant under a rigid symmetry with parameter Λ (any indices on Λ are not explicitly shown), then the variation of the action, once the parameter Λ is made space-time dependent, must be of the form

$$\delta A = \int d^2\xi(\partial_\alpha \Lambda j^\alpha). \tag{3.105}$$

We identify j^α as the current associated with the rigid symmetry Λ. It is conserved when the equations of motion are used as δA vanishes on-shell for any field variations and in particular those for arbitrary parameter $\Lambda(x)$. The first step in finding a locally Λ invariant theory is to introduce the gauge field h (suppressing any indices) and coupling constant g such that $\delta h = 1/g(\partial_\alpha \Lambda) + O(g_0)$ terms. The locally invariant action to order g_0 or first order in h is then given by

$$A + g\int d^2\xi h_\alpha j^\alpha. \tag{3.106}$$

This strategy also works for determining the self coupling of h_α to itself, although in this case j^α involves h and in general a factor of $\frac{1}{2}$ is required in the above equation.

Hence one can find the current from Eq. (3.105) given the rigid theory or as the coefficient of h_α in Eq. (3.106) given the local theory.

It is conceptually most straightforward to derive the energy-momentum tensor from the former method. Substituting

$$\delta x^\mu = \zeta^\gamma \partial_\gamma x^\mu, \quad \delta c^\alpha = \zeta^\gamma \partial_\gamma c^\alpha, \quad \delta b_{\alpha\beta} = \zeta^\gamma \partial_\gamma b_{\alpha\beta} \tag{3.107}$$

with ζ^γ a function of ξ^γ we find that

$$\begin{aligned} T_{\alpha\beta} &= \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \tfrac{1}{2}\eta_{\alpha\beta}(\partial_\gamma x^\nu \partial^\gamma x^\mu \eta_{\mu\nu}) \\ &\quad -(-2)(2\alpha')(b_{\alpha\delta}\partial_\beta c^\delta - \eta_{\alpha\beta} b_{\gamma\delta}\partial^\gamma c^\delta). \end{aligned} \tag{3.108}$$

We have normalized $T_{\alpha\beta}$ so that the x^μ part agrees with that given previously in Eq. (2.28). The above $T_{\alpha\beta}$ is conserved and traceless upon use of the equations of motion, however it is not symmetric. We may always add to $T_{\alpha\beta}$ a term of the form

$$a\partial^\gamma(b_{\gamma\beta}c_\alpha - b_{\alpha\beta}c_\gamma) \tag{3.109}$$

which is automatically conserved and does not contribute to the total energy and momentum if the fields die off sufficiently fast at infinity. Using the equations of motion (3.70), we may write this term in the form

$$-a(b_{\gamma\beta}\partial_\alpha c^\gamma - \partial^\gamma b_{\alpha\beta}c_\gamma). \tag{3.110}$$

Taking $a = 2(2\alpha^1)$ we find that

$$\begin{aligned} T_{\alpha\beta} &= \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \tfrac{1}{2}\eta_{\alpha\beta}\partial_\gamma x^\mu \partial^\gamma x^\nu \eta_{\mu\nu} \\ &\quad +(2\alpha')(2)\{(b_{\alpha\delta}\partial_\beta c^\delta + b_{\beta\delta}\partial_\alpha c^\delta) + (\partial^\gamma b_{\alpha\beta})c_\gamma\}. \end{aligned} \tag{3.111}$$

The final step is to realize that the equation of motion for $b_{\alpha\beta}$ can be written as

$$\partial_\delta b_{\alpha\beta} - \partial_\alpha b_{\delta\beta} = 0. \tag{3.112}$$

Using this repeatedly on the last term we find the result

$$T_{\alpha\beta} = T^x_{\alpha\beta} + T^{gh}_{\alpha\beta}, \tag{3.113}$$

where

$$\begin{aligned} T^x_{\alpha\beta} &= \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \tfrac{1}{2}\eta_{\alpha\beta}(\partial_\gamma x^\mu \partial^\gamma x^\nu \eta_{\mu\nu}) \\ T^{gh}_{\alpha\beta} &= 4\alpha'\{(b_{\alpha\delta}\partial_\beta c^\delta + b_{\beta\delta}\partial_\alpha c^\delta) \\ &\quad +\tfrac{1}{2}\{(\partial_\alpha b_{\beta\delta})c^\delta + (\partial_\beta b_{\alpha\delta})c^\delta\}\} - \text{trace}. \end{aligned} \tag{3.114}$$

In fact, the trace vanishes on-shell.

We can now define the Virasoro generators L_n for the full system, i.e., x's and ghosts. For the *open string* we find that $(P^\mu)^2$ of Eq. (2.34) is given by

$$(P^\mu(\sigma))^2 = \begin{cases} \frac{1}{\alpha'}(T_{00}+T_{01}) & \text{for} \quad 0 < \sigma < \pi \\ \frac{1}{\alpha'}(T_{00}-T_{01}) & \text{for} \quad -\pi < \sigma < 0 \end{cases} \tag{3.115}$$

in terms of the energy-momentum tensor. Taking this definition over for the full $T_{\alpha\beta}$ of Eq. (3.113) and we define, in analogy with Eq. (2.38).

$$L_n \equiv \frac{1}{4\pi} \int_{-\pi}^{\pi} d\sigma e^{in\sigma} (P^\mu)^2. \tag{3.116}$$

For the closed string we use Eqs. (2.39) and (2.40), but with $T_{\alpha\beta}$ given now by Eq. (3.113) in Eqs. (2.45) and (2.46) to define L_n and $\bar{L}_n$.

In the quantum theory we find the expression for L_n by substituting the anticommuting oscillators given in the previous section. For the open string the result is

$$L_n = L_n^\alpha + L_n^{\text{gh}} \tag{3.117}$$

$$L_n^\alpha = \tfrac{1}{2} : \sum_m \alpha_m^\mu \alpha_{n-m}^\mu : \tag{3.118}$$

$$L_n^{\text{gh}} = : \sum_m b_{n+m} c_{-m}(n-m) : -1. \tag{3.119}$$

The reason for subtracting -1 will become apparent. We note that when normal ordering an anticommuting quantity, we must assign a minus sign for a change of order, i.e.,

$$: c_n b_{-m} : = -b_{-m} c_n \quad \text{for} \quad \begin{matrix} n > 0 \\ m > 0. \end{matrix} \tag{3.120}$$

Note that the $b_0 c_0$ term does not occur in L_n and so we need not worry about how to normal order it. For the other oscillator we normal order with respect to the $|\pm\rangle$ vacuum, i.e., b_{-n} is placed to the left of c_n and c_{-n} to the left of b_n for $n > 0$. In fact, we will never need to normal order b_0 with respect to c_0.

Let us compute the L_n commutator, one finds

$$[L_n, L_m] = (n-m)L_{n+m} + n\frac{(D-26)}{12}(n^2-1)\delta_{n+m,0} \tag{3.121}$$

as a result of the equation

$$[L_n^{\text{gh}}, L_m^{\text{gh}}] = (n-m)L_{n+m}^{\text{gh}} - \tfrac{26}{12}(\delta_{n+m,0})n(n^2-1). \tag{3.122}$$

The normal-ordering constant in Eq. (3.118) was adjusted so as to have no central term in $[L^{gh}_{+1}, L^{gh}_{-1}]$. One may verify the central term in Eq. (3.122) by the same method as used for L_n without ghosts in Eq. (3.20).

Consequently for $D = 26$ there is no central term, or put another way no anomaly in the local conformal symmetry. Thus we see our first advantage of working with the ghosts; the absence of the central term. This is particularly useful when discussing the properties of loop amplitudes.

For the closed string we use Eqs. (2.39) and (2.40), but substitute $T_{\alpha\beta}$ now given by Eq. (3.113) in Eq. (2.45) and (2.46) to define L_n and $\bar{L}_n$. We find

$$L_n = L^{\alpha}_n + L^{gh}_n, \quad \bar{L}_n = \bar{L}^{\alpha}_n + \bar{L}^{gh}_n,$$

where

$$L^{\alpha}_n = :\tfrac{1}{2}\sum_m \alpha^{\mu}_m \alpha^{\mu}_{n-m}:, \quad \bar{L}^{\alpha}_n = \tfrac{1}{2}:\sum_m \bar{\alpha}^{\mu}_m \bar{\alpha}^{\mu}_{n-m}:,$$

$$L^{gh}_n = \sum_m :b_{n+m} c_{-m}(n-m): -1,$$

$$\bar{L}^{gh}_n = \sum_m : \bar{b}_{n+m} \bar{c}_{-m}(n-m): -1.$$

Let us now turn our attention to the BRST current. We may compute it by the same method using the BRST variations of Eqs. (3.57), (3.58) and (3.59). However, as we wish to examine the action with $\lambda_{\alpha\beta}$ eliminated, we must in $\delta b_{\alpha\beta}$ substitute for $\lambda_{\alpha\beta}$ using its equation of motion. The computation of the current is related to the one above for the energy momentum as the BRST variation for the original fields x^{μ}, $g_{\alpha\beta}$ is a translation with parameter which is given by (3.56). The reader may verify that the resulting current is given by

$$J^{BRST}_{\alpha} = c^{\gamma}(T^{x}_{\alpha\gamma} + \tfrac{1}{2}\, T^{gh}_{\alpha\gamma}). \tag{3.123}$$

The first term is easily located by the above argument, while the factor of $\frac{1}{2}$ in the second term is typical of a self-coupling problem. It is conserved as $T^{x}_{\alpha\gamma}$ and $T^{gh}_{\alpha\gamma}$ are separately conserved and $\partial^{\alpha} c^{\gamma} N_{\alpha\gamma} = 0$ by the equation of motion of c^{γ} if $N_{\alpha\gamma}$ is symmetric and traceless.

The BRST charge Q is given by

$$Q = \frac{1}{2\pi\alpha'} \int_a^{\pi} d\sigma J^{BRST}_0, \tag{3.124a}$$

where $a = 0$ for the open string and $-\pi$ for the closed string. We may rewrite Q in the *open string* case as

$$Q = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\sigma c(\sigma)\, (P^{\mu}(\sigma))^2 \tag{3.124b}$$

and substituting the oscillators we find that

$$Q = :\sum_{n=-\infty}^{\infty} c_{-n}(L_n^x + \tfrac{1}{2} L_n^{\text{gh}}). \tag{3.125}$$

Using the above expressions for L_n^x and L_n^{gk} we find that

$$Q = :\sum_{n=-\infty}^{\infty} c_{-n}L_n^x - \tfrac{1}{2}\sum_{n,m,p=-\infty}^{\infty} f_{nm}^p b_p c_{-n} c_{-m} - c_0: \tag{3.126}$$

where f_{nm}^{p} are the structure constants of the conformal group and so are given by

$$[L_n, L_m] \equiv f_{nm}^p L_p = (n-m)L_{n+m}. \tag{3.127}$$

For the *closed string* we find

$$Q = :\sum_{n=-\infty}^{\infty} c_{-n}(L_n^\alpha + \tfrac{1}{2} L_n^{\text{gh}}) + \sum_{n=-\infty}^{\infty} \bar{c}_{-n}(\bar{L}_n^\alpha + \tfrac{1}{2} \bar{L}_n^{\text{gh}}):$$

which admits an obvious interpretation in terms of the structure constants of the L_n's and $\tilde{L}_n$'s.

In this second formulation, the object Q is familiar to physicists. In particular given a general system with first class constraints ϕ_i which generated an algebra with structure constants f_{nm}^p

$$[\varphi_i, \varphi_j] = f_{ij}^k \varphi_k. \tag{3.128}$$

The authors of Ref. [4] introduced for each constraint a ghost c^i and an antighost b^i such that

$$\{c^i, b_j\} = \delta_j^i \tag{3.129}$$

and the BRST charge is given by

$$Q = c^i \varphi_i - \tfrac{1}{2} f_{ij}^k b_k c^i c^j. \tag{3.130}$$

It is straightforward to verify that

$$Q^2 = 0 \tag{3.131}$$

using Eqs (3.128) and (3.129) and the Jacobi identity for f_{ij}^k. Indeed, given the first term of Q and demanding that $Q^2 = 0$, leads one to the second term. The BRST charge for Yang-Mills theories is one example of such an object.

For the string, we must beware of the normal-ordering and one finds that [26]

$$Q^2 = 0 \tag{3.132}$$

only if $D = 26$ and L_0 has the normal-ordering constant discussed above.

The action with the ghosts also has the usual ghost number invariance ($c^\alpha \to e^{i\lambda} c^\alpha$, $b_{\alpha\beta} \to e^{-i\lambda} b_{\alpha\beta}$) and we find that the corresponding current is given by

$$j_\alpha = b_{\alpha\beta} c^\beta. \tag{3.133}$$

The corresponding charge, i.e., ghost number operator is given by

$$N = \frac{1}{\pi}\int_a^\pi d\sigma j_0, \tag{3.134a}$$

where $a = 0$ for the open string and $-\pi$ for the closed string. For the *open string* we may, extending the range of σ, rewrite N as

$$N = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma c(\sigma)b(\sigma), \tag{3.134b}$$

while for the *closed string*

$$N = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma(c(\sigma)b(\sigma)+\tilde{c}(\sigma)\tilde{b}(\sigma)).$$

For the *open string*, we find the suitably normal-ordered oscillator expression is

$$N = \sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n)+\tfrac{1}{2}(c_0b_0-b_0c_0). \tag{3.135}$$

The normal ordering of the last term being so that N is Hermitian. For the closed string N is given by (3.135) plus a similar expression with $\tilde{c}_n$ and b_n. As might be expected, the BRST charge has ghost number $+1$ as is implied by

$$Q = [N, Q]. \tag{3.136}$$

An important property of Q with respect to the Virasoro algebra is

$$\{b_n, Q\} = L_n \equiv L_n^x + L_n^{gh} \tag{3.137}$$

which in turn implies that

$$[L_n, Q] = 0 \tag{3.138}$$

as

$$\begin{aligned}[L_n, Q] &= \{b_n, Q\}Q - Q\{b_n, Q\}\\ &= b_nQ^2 - Q^2b_n = 0.\end{aligned} \tag{3.139}$$

3.2.3. The physical state condition

We now must find what the physical state conditions of Eq. (3.11) look like in the BRST formalism. At first sight it would seem that we have gained very little since we now have a Fock space which is very much larger and even in particular includes anticommuting states. However, as we shall see the physical state condition is particularly simple in the

BRST language. In the usual BRST situation, such as Yang-Mills, the physical state condition is given by

$$Q|\chi\rangle = 0, \tag{3.140}$$

where Q is the BRST charge. As $Q^2 = 0$, a state of the form $|\chi\rangle = Q(|\Lambda\rangle)$ automatically satisfies this equation. Hence if $|\chi\rangle$ is a solution, so is $|\chi\rangle + Q|\Lambda\rangle$. We note that a state $Q|\Lambda\rangle$ has zero norm with all physical states including itself, such states although physical do not contribute to actual processes. To remove this ambiguity we set up the equivalence relation

$$|\chi\rangle_1 \sim |\chi\rangle_2 \quad \text{iff} \quad |\chi\rangle_1 = |\chi\rangle_2 + Q|\Lambda\rangle \quad \text{for some state } |\Lambda\rangle. \tag{3.141}$$

These equivalence classes are often referred to as the cohomology of Q. A physical operator S is one which commutes with Q, i.e., $[Q, S] = 0$ clearly if S is physical so is $S + [Q, U]$ for any U.

In QED we may choose our representatives of the equivalence classes to not depend on ghost oscillators and then the physicall state condition of Eq. (3.140) enforces, in effect, the condition $\partial^\mu(A_\mu^{(+)})|\psi\rangle = 0$ where $+$ denotes the positive frequency part of A_μ, i.e., the Gupta-Bleuler condition. To count the number of degrees of freedom, that is to ascertain that we really have only a spin one irreducible representation of the Poincaré group, we require the above condition and the equation of motion which follows from the action.

In string theory, the situation is somewhat different, although the words and some of the equations are the same. Let us consider for the open string the equation

$$Q|\chi\rangle = 0, \tag{3.142}$$

where Q is given in Eq. (3.124) and $|\chi\rangle$ is the general functional of x^μ, and the ghosts given in Eq. (3.96) or (3.97). Let us first apply Q to the ghost-free state, given by

$$|\chi\rangle = \psi|-\rangle + \phi|+\rangle, \tag{3.143}$$

where ψ and ϕ are functionals of $x^\mu(\sigma)$ alone. We find that Eq. (3.142) implies that

$$Q|\chi\rangle = \sum_{n=0}^{\infty} c_{-n}(L_n^\alpha - \delta_{n,0})\psi|-\rangle + \sum_{n=1}^{\infty} c_{-n}L_n^\alpha\phi|+\rangle \tag{3.144}$$

and so

$$(L_n^\alpha\psi - \delta_{n,0})\psi = 0, \quad n \geqslant 0; \quad L_n^\alpha\phi = 0, \quad n \geqslant 1.$$

We must recover the physical state conditions of Eq. (3.11) which we may identify as those on ψ. We can eliminate the field ϕ if we impose also the condition

$$b_0|\chi\rangle = 0. \tag{3.145}$$

Sine $b_0|+\rangle = |-\rangle \neq 0$, all the remaining states in $|\chi\rangle$ are built on the $|-\rangle$ vacuum. In fact, the constraints

$$b_0|\chi\rangle = 0, \quad Q|\chi\rangle = 0 \tag{3.146}$$

are the correct constraints on $|\chi\rangle$. We have shown above that the physical states which satisfy $L_n^\alpha \psi = 0$, $n \geqslant 1$, $(L_0^\alpha - 1)\psi = 0$ are one solution. However, we must also show that these are the only solutions up to states of the form $Q|\Lambda\rangle$. What one must show in effect is that all states with ghost oscillators acting on $|-\rangle$ are of the form $Q|\Lambda\rangle$. A straightforward proof of this fact can be found in Ref. [30]. In fact, in Ref. [30] they found all the equivalence classes of only $Q|\chi\rangle = 0$ and showed that in fact there were two, the solution above based on the $|-\rangle$ vacuum and another solution based on the $|+\rangle$ vacuum. Our second condition rules out the latter.

We previously stated that all solutions of $L_n^\alpha = 0$ $n \geqslant 1$ $(L_0^\alpha - 1)\psi = 0$ had positive norm and that the states of zero norm were of the form

$$L_{-1}^\alpha |\Omega\rangle, \quad \{L_{-2}^\alpha + \tfrac{3}{2}(L_{-1}^\alpha)^2\} |\Omega'\rangle. \tag{3.147}$$

These latter states may be written as Q of $b_{-2}|\Omega\rangle$ and $(b_{-2} + 3/2 b_{-1} L_{-1}^\alpha |\Omega'\rangle$ respectively. Written in this way, they are obviously of zero norm. Such states, however, do not occur in the count of equivalence classes of Q and so we must conclude that the equivalence classes of Q have positive definite norm. (Here positive norm means after inserting c_0 or ignoring the $|+\rangle$ vacuum as otherwise the scalar product vanishes.)

The reader is now well placed to study the free gauge covariant formulation of string theory [31]. For the open string, the free action is [32, 33]

$$\langle \chi | Q | \chi \rangle, \tag{3.148}$$

where $|\chi\rangle$ is subject to the algebraic constant $N|\chi\rangle = \frac{1}{2}|\chi\rangle$. Clearly $Q|\chi\rangle = 0$ is the equation of motion and examination shows that one can fix the gauge invariance $|\chi\rangle \to |\chi\rangle + Q|\Lambda\rangle$ by $b_0|\chi\rangle = 0$. Since one has not fixed a $|\Lambda\rangle$ of the form $|\Lambda\rangle = Q|\Lambda'\rangle$, one finds in the corresponding second quantized formulation ghost for ghosts. It can be shown, however, that the count of states after gauge fixing is correct [32].

The above brings out the difference between the BRST procedure in, say, Yang-Mills theory and the use which is made of it in string theory. In gauge covariant string theory, we have a gauge *invariant* theory with, as usual, no *ghost fields*. Nonetheless, the best description for this theory is in terms of the *BRST co-ordinates* of the previous section. Another way to see the difference is that in the string $Q|\chi\rangle = 0$ is the equation of motion and $b_0|\chi\rangle = 0$ the gauge fixing condition in the BRST formulation of QED, $Q|\chi\rangle = 0$ is the Gupta-Bleuler condition and $\partial^2 A_\mu - \partial_\mu \partial^\nu A_\nu = 0$ is the equation of motion.

The deep and different use to which the BRST formalism has been used in string theory remains to be understood.

For the closed string the physical state conditions are

$$Q|\chi\rangle = 0, \quad 0 = b_0|\chi\rangle = \bar{b}_0|\chi\rangle. \tag{3.149}$$

This means all states are based on the $|-,-\rangle$ vacuum. We also have

$$\{Q, b_0 - \bar{b}_0\} |\chi\rangle = (L_0 - \bar{L}_0) |\chi\rangle = 0. \tag{3.150}$$

4. *Conformal symmetry*

The reader will have become aware when studying the free quantum string that the dominant structure is the conformal group. For example, the spectrum of the string is determined by the physical state conditions and these are given in terms of the Virasoro operators L_n which are the generators of the conformal group. The purpose of this chapter is to bring out more explicitly the role played by the conformal group. In particular, we shall see that using light-cone co-ordinates, which are naturally suited to the conformal group, greatly simplifies the previous expressions. Conformal concepts also allow us to formulate more generally what can constitute a string theory.

4.1. The conformal group

Given a D dimensional Minkowski space parametrized by x^μ, a conformal transformation is a diffeomorphism $x^\mu \to \bar{x}^\mu(x)$ such that the line element is preserved up to a scale factor, namely

$$d\bar{s}^2 = d\bar{x}^\mu d\bar{x}^\nu \eta_{\mu\nu} = \Omega(x) dx^\mu dx^\nu \eta_{\mu\nu}. \tag{4.1}$$

For all dimensions D, except two, this group is finite dimensional. For $D = 2$, it is an infinite dimensional, meaning it has an infinite number of generators. In the light-cone co-ordinates

$$z = x^0 + x^1, \quad \bar{z} = x^0 - x^1 \tag{4.2}$$

the line element takes the form

$$-ds^2 = (dx^0)^2 - (dx^1)^2 = dz d\bar{z}. \tag{4.3}$$

The derivatives are related by

$$\partial_z \equiv \frac{\partial}{\partial z} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right); \quad \partial_{\bar{z}} \equiv \frac{\partial}{\partial \bar{z}} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}\right).$$

Clearly, it is preserved up to a scale factor [i.e., $\Omega = f'(z)g'(z)$] by a transformation of the form

$$z \to f(z), \quad \bar{z} \to g(\bar{z}). \tag{4.4}$$

for any functions f and g. The only other transformations which preserve the line element up to scale are

$$z \to h(\bar{z}), \quad \bar{z} \to k(z). \tag{4.5}$$

However, these transformations change the orientation defined on the two-dimensional surface and will not be considered further.

The infinitesimal transformations of Eq. (4.4) are of the form

$$z \to z + a_n z^{n+1}; \quad \bar{z} \to \bar{z} + \bar{a}_n (\bar{z})^{n+1} \tag{4.6}$$

and are therefore generated by

$$L_n = z^{n+1}\frac{\partial}{\partial z}, \qquad \bar{L}_n = \bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}. \tag{4.7}$$

The Lie algebra these generators satisfy is given by

$$[L_n, L_m) = (n-m)L_{n+m},$$
$$[L_n, \bar{L}_m] = 0,$$
$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m}. \tag{4.8}$$

Naive substitution of Eq. (4.7) in Eq. (4.8) yields the result above with a minus sign. This is an active viewpoint; to arrive at the sign in (4.8), one must take the passive viewpoint and carry out the manoeuvre on functions.

It is useful to use the Euclidean co-ordinates

$$z = x^0 + ix^1, \qquad \bar{z} = x^0 - ix^1 \tag{4.9}$$

which are obtained by making the substitution $x^1 \to ix^1$, whereupon $\bar{z}$ becomes the complex conjugate of z. In what follows we will often move back and forth between Minkowski and Euclidean space, without pondering the deeper significance of the change of signature. Working in a space with positive definite signature has the advantage of allowing one to make contact with the large mathematical literature on Riemann surfaces. The conformal field theories that are associated with statistical models are in Euclidean space as they arise from the partition function. The conformal field theories associated with string theories are in Minkowski space and can be "rotated" into Euclidean space using the usual Wick rotation $\tau \to -i\tau$ in the path integral.

In general, conformal transformations of the type of Eq. (4.4) are not invertible. The subgroup which is invertible on the Riemann sphere $\hat{C}$, i.e., the complex plane plus the point at infinity is given by the following.

THEOREM

The most general conformal transformation which maps the Riemann sphere $\hat{C}$ onto itself in a one-to-one way is a Möbius transformation which is of the form

$$z \to z' = \frac{az+b}{cz+d}, \tag{4.10}$$

where a, b, c, d are complex parameters. By scaling these parameters without affecting the transformation we may set

$$ad - bc = 1 \tag{4.11}$$

and so the Möbius group is a six real-parameter group. Infinitesimal Möbius transformations are of the form

$$z' = z + \varepsilon_{-1} + \varepsilon_0 z + \varepsilon_1 z^2,$$
$$\bar{z}' = \bar{z} + \bar{\varepsilon}_{-1} + \bar{\varepsilon}_0 \bar{z} + \bar{\varepsilon}_1 \bar{z}^2 \tag{4.12}$$

and so are generated by L_0, $L_{\pm 1}$ and $\bar{L}_0$, $\bar{L}_{\pm 1}$.

We are also interested (for open strings) in the upper half sphere plus the point at infinity, denoted $\hat{H}$. This space is mapped onto itself in a 1 to 1 manner only by conformal transformations of the form

$$z \to z' = \frac{az+b}{cz+d}, \tag{4.13}$$

where now a, b, c, d are *real*. We may scale the parameters, without affecting the transformation, to obtain either $ad-bc = 1$ or $ad-bc = -1$. This three-parameter group is generated by $L_0+\bar{L}_0$, $L_{\pm 1}+\bar{L}_{\pm 1}$.

One easily verified that two Möbius transformations compose to give a third and that the resulting coefficients are the same as one obtains by multiplying the two-by-two matrices. The matrix corresponding to the transformation of Eq. (4.13) being $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The correspondence between two-by-two matrices and Möbius transformations is not one-to-one as the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ lead to the same Möbius transformation. Apart from this ambiguity, they are in one-to-one correspondence. Consequently, the Möbius group is isomorphic to SL(2, C)/Z_2 and the Möbius group with real parameters and $ad-bc = 1$ is isomorphic to SL(2, R)/Z_2.

4.2. The conformal tensor calculus

Consider a two-dimensional manifold on which the metric is conformally equivalent to the metric, i.e.,

$$-ds^2 = e^{\varphi(x)}\{(dx^0)^2-(dx^1)^2\}. \tag{4.14}$$

We now wish to examine transformations that leave the metric in the same form, that is by definition conformal transformations. Since any metric on any oriented Riemann surface can be brought locally into the form of Eq. (4.14), these considerations are quite general. In particular, the above conformal transformations will occur when we relate two co-ordinate patches both of which have had their metrics brought into the form of Eq. (4.14). In the co-ordinates z and $\bar{z}$ the line element takes the form

$$-ds^2 = e^{\varphi(z,\bar{z})}dz d\bar{z}, \tag{4.15}$$

and we conclude, as

$$-ds^2 = g_{zz}(dz)^2+g_{z\bar{z}}dz d\bar{z}+g_{\bar{z}z}d\bar{z}dz+g_{\bar{z}\bar{z}}(d\bar{z})^2 \tag{4.16}$$

that the metric in $\bar{z}$ co-ordinates is given by

$$g_{zz} = 0 = g_{\bar{z}\bar{z}}, \quad g_{z\bar{z}} = g_{\bar{z}z} = \tfrac{1}{2}e^{\varphi}. \tag{4.17}$$

The inverse metric is given by

$$g^{zz} = 0 = g^{\bar{z}\bar{z}}, \quad g^{z\bar{z}} = g^{\bar{z}z} = 2e^{-\varphi}. \tag{4.18}$$

The most general tensor is of the form

$$T^{\overbrace{z\ldots z}^{n}\ \overbrace{\bar z\ldots \bar z}^{\bar n}}_{\underbrace{z\ldots z}_{m}\ \underbrace{\bar z\ldots \bar z}_{\bar m}} \equiv T(z, \bar z) \tag{4.19}$$

and as usual transforms under the co-ordinate change $z \to \omega(z)$, $\bar z \to \bar\omega(\bar z)$ as

$$T(z, \bar z) \to T'(\omega, \bar\omega) = \left(\frac{d\omega}{dz}\right)^{n-m} \left(\frac{d\bar\omega}{d\bar z}\right)^{\bar n-\bar m} T(z, \bar z). \tag{4.20}$$

We call $h = n-m$ and $\bar h = \bar n - \bar m$ the conformal weights or dimensions of the tensor T. A time translation $x^0 \to x^0+a$ is induced by $z \to z+a$, $\bar z \to \bar z + a$ for a real and so is generated by $L_{-1}+\bar L_{-1}$ which is identified to be the Hamiltonian. A space shift $x^1 \to x^1+\lambda$ is induced by $z \to z+\lambda$, $\bar z \to \bar z - \lambda$ and is generated by $L_{-1}-\bar L_{-1}$ which is the total momentum. The remaining generators of the two-dimensional Poincaré group are rotations of the form $\delta x^0 = -\phi x^1$, $\delta x^1 = +\phi x^0$, there are generated by $z \to e^{i\phi} z$, $\bar z \to e^{-i\phi} z$ and so $L_0 - \bar L_0$ is the angular momentum generator. The dilation $z \to \lambda z$, $\bar z \to \lambda \bar z$ for λ real is generated by $L_0+\bar L_0$.

We find that under a dilation $T \to \lambda^{h+\bar h} T$ while under a rotation $T \to \lambda^{i(h-\bar h)\phi} T$. Consequently, we call $h+\bar h$ the dilation weight of T and $h-\bar h$ its spin.

We raise and lower indices with the metric and so in particular

$$T_z = g_{z\bar z} T^{\bar z} = \tfrac{1}{2} e^{\varphi} T^{\bar z}; \qquad T_{\bar z} = \tfrac{1}{2} e^{\varphi} T^{z}. \tag{4.21}$$

The relation between tensors in the $z, \bar z$ co-ordinate system and the original x^0, x^1 co-ordinate system is given by the usual transformation formula, i.e.

$$T_z = \frac{\partial x^\alpha}{\partial z} T_\alpha = \tfrac{1}{2}(T_0 - iT_1), \qquad T_{\bar z} = \tfrac{1}{2}(T_0 + iT_1) \tag{4.22}$$

and for a second rank tensor by

$$\begin{aligned}
T_{zz} &= \tfrac{1}{4}(T_{00} - iT_{01} - iT_{10} - T_{11})\\
T_{z\bar z} &= \tfrac{1}{4}(T_{00} + iT_{01} - iT_{10} + T_{11})\\
T_{\bar z z} &= \tfrac{1}{4}(T_{00} - iT_{01} + iT_{10} + T_{11})\\
T_{\bar z\bar z} &= \tfrac{1}{4}(T_{00} + iT_{01} + iT_{10} - T_{11}).
\end{aligned} \tag{4.23}$$

To find the Minkowski results, we substitute $T_1 \to +iT_1$, $T_{01} \to +iT_{01}$, $T_{10} \to +T_{10}$, $T_{11} \to -T_{11}$. The Christoffel symbol in the $z, \bar z$ co-ordinates is found, from its definition in Eq. (3.48), to be zero except for the components

$$\Gamma_{zz}{}^{z} = \partial_z \varphi; \qquad \Gamma_{\bar z\bar z}{}^{\bar z} = \partial_{\bar z}\varphi. \tag{4.24}$$

4.3. Conformally invariant two-dimensional theories

Before considering two-dimensional theories, let us consider a theory in an arbitrary number of dimensions which is Poincaré invariant. For such a theory, there exists an energy-momentum tensor $T_{\alpha\beta}$ which we can choose to be symmetric. This tensor is the conserved current, i.e., $\partial^\alpha T_{\alpha\beta} = 0$ whose charge generates translations. The current corresponding to Lorentz rotations is a moment of energy-momentum tensor (namely $x_\alpha T_{\beta\delta} - x_\beta T_{\alpha\delta}$, it is conserved on its δ index due to the symmetry of $T_{\alpha\beta}$ and its conservation.

If the theory is dilation invariant, it is invariant under $x_\alpha \to \lambda x_\alpha$ and the associate current is given by a moment of the energy-momentum tensor

$$j_\beta = x^\alpha T_{\alpha\beta}. \tag{4.25}$$

It is conserved provided that $T^\alpha_\alpha = 0$. However, this condition also allows us to construct further conserved moments of $T_{\alpha\beta}$. Any current of the form

$$f^\alpha(x) T_{\alpha\beta} \tag{4.26}$$

provided

$$\partial^\beta f^\alpha + \partial^\alpha f^\beta - \phi\eta^{\alpha\beta} = 0, \tag{4.27}$$

where ϕ is an arbitrary function of x. One example is given by

$$x_\alpha x^\delta T_{\delta\beta} - x^2 T_{\alpha\beta} \tag{4.28}$$

which generates the special translations of the conformal group.

These additional conserved currents define corresponding generators which together with the Poincaré and dilation generator have the conformal group as their algebra. This follows from the fact that the charges generate the transformations $x^\alpha \to x^\alpha + f^\alpha$ and these are precisely the complete set of transformations that leave the metric invariant up to a scale factor provided it satisfies Eq. (4.27).

We have shown therefore that any theory that has $T^\alpha_\alpha = 0$ is not only dilation, but also conformally invariant. In fact, the converse can be shown, a conformally invariant theory has a traceless energy-momentum tensor and its currents are the above momenta of the energy-momentum tensor [34].

Let us now restrict our attention to the two-dimensional case and see how the above general statements are realized. The energy-momentum tensor if it is symmetric and traceless has only two components which we can take to be T_{00} and T_{01}. In the $z, \bar{z}$ system, the traceless condition is

$$T_{z\bar{z}} = 0 \tag{4.29}$$

leaving only T_{zz} and $T_{\bar{z}\bar{z}}$. The conservation condition is given by

$$\partial_{\bar{z}} T_{zz} = 0 = \partial_z T_{\bar{z}\bar{z}} \tag{4.30}$$

which implies that T_{zz} and $T_{\bar{z}\bar{z}}$ are functions of only z and $\bar{z}$ respectively. Given any two functions $f(z)$ and $g(\bar{z})$, we find an infinite set of conserved currents given by

$$f(z) T_{zz} \quad \text{and} \quad g(\bar{z}) T_{\bar{z}\bar{z}} \tag{4.31}$$

as clearly $\partial_{\bar{z}}(f(z)T_{zz}) = 0$. The corresponding generators are

$$L_n = \oint \frac{dz}{z} z^{n+2} T_{zz}, \quad \bar{L}_n = \oint \frac{d\bar{z}}{\bar{z}} (\bar{z})^{n+2} T_{\bar{z}\bar{z}} \tag{4.32}$$

We take the usual $(2\pi i)^{-1}$ to be included in the definition of the contour integral. That is

$$\oint dz \text{ means } \frac{1}{2\pi i} \oint dz.$$

The above holds for any two-dimensional conformal field theory. For a free spin--zero field theory whose action is given in z, $\bar{z}$ co-ordinates by

$$+\frac{1}{4\pi} \int dz d\bar{z} \quad \partial_z \varphi \partial_{\bar{z}} \varphi. \tag{4.33}$$

The corresponding energy-momentum tensor is

$$T_{z\bar{z}} = 0, \quad T_{zz} = \partial_z \varphi \partial_z \varphi, \quad T_{\bar{z}\bar{z}} = \partial_{\bar{z}} \varphi \partial_{\bar{z}} \varphi \tag{4.34}$$

and the reader may verify explicitly that $\partial_{\bar{z}} T_{zz} \doteq 0 = \partial_z T_{\bar{z}\bar{z}}$ using the equation of motion.

Let us now consider quantizing an arbitrary two-dimensional conformal field theory. While the L_n's and $\bar{L}_n$'s classically obey Eq. (4.7), quantum mechanically they become quantum field operators with associated normal ordering. As a result, in the quantum theory they can obey

$$[L_n, L_m] = (n-m)L_{n+m} + c_{n,m} \tag{4.35}$$

as well as a similar equation for $\bar{L}_n$'s. The central term $c_{n,m}$ commutes with the L_n's. Clearly $c_{n,m} = -c_{m,n}$, but it must also obey other conditions in order that the algebra obey the Jacobi identity. It is possible to absorb some of the $c_{n,m}$'s by redefining the L_n's. We now consider what is the most general form of $c_{n,m}$ subject to these considerations. The Jacobi identity

$$0 = [[L_n, L_m], L_p] + [[L_p, L_n], L_m] + [[L_m, L_p], L_n] \tag{4.36}$$

implies the result

$$(n-m)c_{n+m,p} + (p-n)c_{n+p,m} + (m-p)c_{m+p,n} = 0. \tag{4.37}$$

Using the redefinitions

$$L_n \to L_n + \frac{1}{n} c_{n,0},$$

$$L_0 \to L_0 + \tfrac{1}{2} c_{1,-1}, \tag{4.38}$$

we can set $c_{n,0} = 0 = c_{1,-1}$. Taking $p = 0$ in Eq. (4.37), we find

$$(n+m)c_{n,m} = 0 \tag{4.39}$$

and so $c_{n,m} = c_n\delta_{n+m,0}$. Further taking $p = -m-1$, $n = 1$, we find the recursion relation

$$c_{m+1} = \frac{m+2}{m-1}\, c_m, \tag{4.40}$$

which implies that

$$c_m = c\,\frac{m(m^2-1)}{12}; \tag{4.41}$$

where c is a constant. As such we have found that

$$c_{m,n} = c\,\frac{m(m^2-1)}{12}\,\delta_{m+n,0}. \tag{4.42}$$

The coefficient c depends on the conformal theory and is called the central charge. For the theory with one real spin-zero, one finds that $c = 1$ while for a theory with one Majorana-Weyl fermion one finds $c = \frac{1}{2}$. The reader may verify these results by oscillator methods which are almost identical to the evaluation of c for the bosonic string given in Section 3 provided we take only one field say x'. An equivalent method of evaluation to use conformal field theory techniques which are discussed in Section 4.5.

A primary field of weight $(d, \bar{d})$ is a field operator $\phi(z, \bar{z})$ which obeys the equation

$$\varphi(z, z') \to \varphi'(\omega, \bar{\omega}) = \left(\frac{d\omega}{dz}\right)^{d}\left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{\bar{d}} \varphi(\omega, \bar{\omega}). \tag{4.43}$$

This symmetry transformation can be implemented it by the action of the generators L_n, $\bar{L}_n$'s; we define $U = \exp\sum_n (a_n L_n + \bar{a}_n L_n)$ which in the representation of Eq. (4.7) implements $(z, \bar{z}) \to (\omega(z), \bar{\omega}(\bar{z}))$ and then

$$\varphi'(\omega, \bar{\omega}) = u\varphi(z, \bar{z})u^{-1} = \left(\frac{d\omega}{dz}\right)^{d}\left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{\bar{d}} \varphi(\omega, \bar{\omega}). \tag{4.44}$$

The alert reader will realize that between Eq. (4.20) and the above equation we have switched from a passive to an active interpretation in order to agree with the literature.

Taking an infinitesimal transformation $z \to z + \varepsilon z^{n+1}$, $\bar{z} \to \bar{z}$, we find the above equation becomes

$$[L_n, \varphi(z, \bar{z})] = z^n \left\{ z\frac{d}{dz} + (n+1)d \right\} \varphi(z\ \bar{z}) \tag{4.45}$$

and similarly for $\bar{z}$.

In a representation of the Virasoro algebra, we can choose the L_0 to be diagonal. For the representation to have eigenvalues of L_0 which are bounded from below, it must contain a highest weight state $|h\rangle$, such that

$$\begin{aligned} L_n|h\rangle &= 0 \quad n \geq 1 \\ L_0|h\rangle &= h|h\rangle. \end{aligned} \tag{4.46}$$

Clearly the state $L_n|h\rangle$ has an L_0 eigenvalue of $h-n$. If we are dealing with an irreducible representation of the Virasoro algebra alone then the other states must be generated by L_n's and so must be of the form

$$L_{-n_1} \dots L_{-n_p}|h\rangle \tag{4.47}$$

such a state has a L_0 eigenvalue of $\sum_{i=1}^{p} n_i + h$. A representation of the conformal group, whose algebra contains two copies of the Virasoro algebra, is given by the obvious extension of the above to include $\bar{L}_n$'s. An example of a highest weight state is provided by the primary fields since the state

$$|\varphi\rangle = \ln_{z,\bar{z}\to 0} \varphi(z, \bar{z}) \mid \rangle. \tag{4.48}$$

has weight $(d, \bar{d})$ and where the vacuum $|\,\rangle$ is deemed to satisfy $L_n|\rangle = 0, n \geqslant -1$. It is clear that an irreducible representation of the Virasoro algebra is determined entirely by c and h. A useful result is the following.

THEOREM [35]

A *unitary* representation of the Virasoro algebra which has $c < 1$ exists only if

$$c = 1 - \frac{6}{(m+2)(m+3)}, \quad m = 0, 1, 2 \dots \tag{4.49}$$

and the only possible values of h for a fixed c are given by

$$h = \frac{[(m+3)p-(m+2)q]^2-1}{4(m+2)(m+3)}, \tag{4.50}$$

$p = 1, 2, \dots, m+1$, $q = 1, 2, \dots, m$. We refer the reader to the review of Ref. [36] for a more comprehensive discussion.

For $c > 1$ no such systematic understanding of unitary representations of the Virasoro algebra exists. There exist unitary representations for any value of c as can be seen by considering L_n of the form

$$L_n = \oint \frac{dz}{z} z^{n+2} \left\{ : \tfrac{1}{2} \frac{(P(z))^2}{z^2} : + \alpha_0 z \frac{d}{dz} \left\{ \frac{P(z)}{z} \right\} \right\}$$

for a suitable value of α_0. There are, however, interesting reducible representations of the Visasoro group which are irreducible representations of larger groups, such as super--Virasoro groups. Several series of c's each with corresponding allowed values of h's are known for $c > 1$.

In recent years two-dimensional conformal field theories have attracted much attention since it has been found that second order phase transition in two-dimensional systems can be described by such a theory [37]. In particular, it turns out that the so-called critical exponents found in such system must correspond to the values of $(h, \bar{h})$. Indeed for certain systems a precise correspondence has been made between the c and h's in Eqs (4.49) and

(4.50). An important development which inspired much of the current work in the field was the realization in Refs. [38] and [39] that one could solve conformal field theories which contained an irreducible representation of the conformal group with $c = 1-(6(p-q)^2/pq)$ for p and q positive integers. These theories are only unitary if $p = q+1$. By solve, one meant determine "explicitly" all the Green functions. This was possible as a result of the existence of null states for the above c's which in turn lead to differential equations on the Green functions.

This work has led to the hope that one may be able to classify and maybe solve all two-dimensional conformal field theories. This may involve some interesting connections with mathematics involving Yang-Baxter algebras, braid groups, knot theory, etc. [40].

4.4. Conformal symmetry and string theory

Let us begin by reconsidering the bosonic string discussed in Chapters 2 and 3 in the light of the previous sections. We may regard the action of Eq. (2.6) as D free scalar fields x^μ which are coupled to gravity $g_{\alpha\beta}$. The D free scalar fields x^μ in the absence of gravity are invariant under the rigid conformal group discussed above, but when coupled to two--dimensional gravity it is *locally* conformally invariant. Local conformal transformations are the group generated by general co-ordinate transformations and Weyl transformations of Eqs. (2.7) and (2.9). The relation between local conformal transformations and rigid conformal transformations is given by the result. Any locally conformal theory is in the absence of gravity, that is when we set $g_{\alpha\beta} = \eta_{\alpha\beta}$, invariant under the rigid conformal group.

We note that in two dimensions the usual Einstein term for gravity $\sqrt{-g}\,R$ is topological invariant. However, one could consider [41] adding more complicated terms involving gravity, such as those induced by integrating out $D = 2$ fermions, but we will not do this here.

The space-time co-ordinates $x^\mu(\xi)$ are defined on the world-sheet of the string which we parametrized by $\xi^\alpha = (\tau, \sigma)$ and in Chapters 2 and 3 we discussed the string in terms of ξ^α co-ordinates.

Up to now in this chapter, however, we have considered conformal field theories defined on the complex plane or Riemann sphere. We now demonstrate how to conformally map from the string world-sheet to the Riemann sphere. Let us first consider the free *closed string*, its world-sheet is a strip with co-ordinates $-\infty < \tau < \infty$, $-\pi < \sigma < \pi$. Since $x^\mu(\sigma) = x^\mu(\sigma+2\pi)$ we may extend the range of σ from $-\infty$ to ∞ by repeated use of $x^\mu(\sigma) = x^\mu(\sigma+2\pi)$. For light-cone co-ordinates we take

$$\mathfrak{z} \equiv \xi^0+\xi^1 \equiv \tau+\sigma, \quad \bar{\mathfrak{z}} = \xi^0-\xi^1 = \tau-\sigma \tag{4.51}$$

the transition to complex co-ordinates being carried out by $\xi^1 \to i\xi^1$ in which case $\mathfrak{z} = \tau+i\sigma$. The map from the strip to the Riemann sphere $\hat{C}$ is given by

$$\mathfrak{z} = \ln z, \tag{4.52}$$

(see Fig. 4.1).

The free *open string* sweeps out a strip parametrized $-\infty < \tau < \infty$, $0 < \sigma < \pi$.

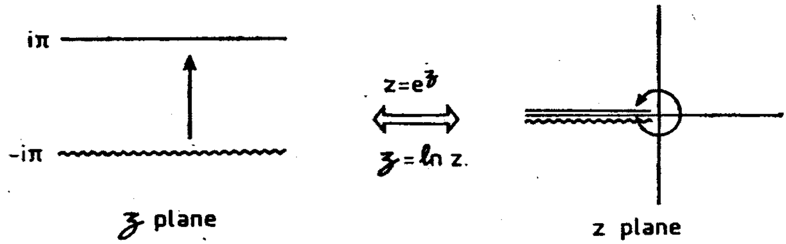

Fig. 4.1. Map from the free closed string world sheet to the Riemann sphere

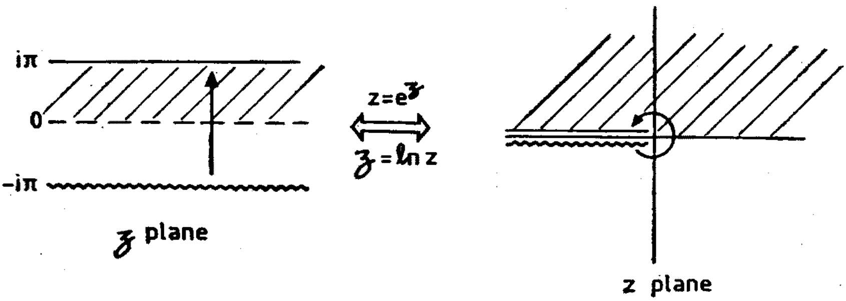

Fig. 4.2. Map from the free open string world sheet to the Riemann sphere

We may extend the range of σ according to Eq. (2.33) from $-\pi < \sigma < \pi$ by the reflection in $\sigma \to -\sigma$ [i.e., $x^\mu(-\sigma) = x^\mu(\sigma)$]. The map to the Riemann sphere is as for the closed string $\zeta = \ln z$ (see Fig. 4.2).

We note that lines of constant τ on the string world-sheet are mapped to circles about $z = 0$ in the complex plane. The incoming string at $t = -\infty$ corresponds to the circle of radius zero around $z = 0$, while the outgoing string at $\tau = +\infty$ corresponds to the circle of infinite radius also around $z = 0$. A τ translation on the strip, i.e., $\tau \to \tau' = \tau+\lambda$, $\sigma \to \sigma' = \sigma$ induces the change $z \to z' = z+\lambda z$, $\bar{z} \to \bar{z}'+\lambda\bar{z}$, λ real is a dilation which is generated by $L_0+\bar{L}_0$ for the closed string and L_0 for the open string. This explains why $L_0+\bar{L}_0$ and L_0 for the open and closed strings respectively play the role of the Hamiltonians.

We now transfer many of the previous expressions in Sections 2 and 3, the ξ^α to the ζ, $\bar{\zeta}$ co-ordinates. We will label tensors in the ζ, $\bar{\zeta}$ co-ordinate system by $+$, $-$, i.e.

$$g_{\zeta\bar{\zeta}} = g_{+-}, \quad T_{\zeta\zeta} = T_{++}, \quad T_{\bar{\zeta}\bar{\zeta}} = T_{--}, \quad \partial_\zeta = \partial_+ \text{ etc.}$$

The gauge choice of Eq. (2.24) is $g_{--} = g_{++} = 0$ and in this gauge the equations of motion are

$$\partial_+\partial_- x^\mu = 0, \quad T_{++} = \partial_+ x^\mu \partial_+ x^\mu = 0, \quad T_{--} = \partial_- x^\mu \partial_- x^\mu = 0. \tag{4.53}$$

For the *closed string* the definitions of (P^μ) and $(\bar{P}^\mu)$ of Eqs (2.39) and (2.40) are recognizable as

$$P^\mu = \frac{2}{\sqrt{2\alpha'}}\partial_+ x^\mu, \quad \bar{P}^\mu = \frac{2}{\sqrt{2\alpha'}}\partial_- x^\mu \tag{4.54}$$

and so

$$(P^\mu)^2 = \frac{1}{2\alpha'} 4T_{++}, \quad (\bar{P}^\mu)^2 = \frac{1}{2\alpha'} 4T_{--}. \tag{4.55}$$

For the *open string*, on the other hand

$$P^\mu = \begin{cases} \frac{2}{\sqrt{2\alpha'}} \partial_+ x^\mu & \text{for} \quad 0 < \sigma < \pi \\ \frac{2}{\sqrt{2\alpha'}} \partial_- x^\mu & \text{for} \quad -\pi < \sigma < 0 \end{cases} \tag{4.56}$$

and hence

$$(P^\mu)^2 = \begin{cases} \frac{1}{2\alpha'} 4T_{++} & \text{for} \quad 0 < \sigma < \pi \\ \frac{1}{2\alpha'} 4T_{+-} & \text{for} \quad -\pi < \sigma < 0. \end{cases} \tag{4.57}$$

Let us now turn our attention to putting the ghost system in $\mathfrak{z}$, $\bar{\mathfrak{z}}$ co-ordinates. The action of Eq. (3.67) becomes

$$A = -\frac{2}{\pi} \int d\mathfrak{z} d\bar{\mathfrak{z}} \left\{ -\frac{1}{4\alpha'} \partial_+ x^\mu \partial_- x^\mu - (b_{++} \partial_- c^+ + b_{--} \partial_+ c^-) \right\}. \tag{4.58}$$

As $b_{\alpha\beta}$ is symmetric and traceless $b_{+-} = 0$ leaving b_{++} and b_{--}. The equations of motion [i.e., Eq. (3.70)] being

$$\partial_+ \partial_- x^\mu = 0, \quad \partial_- b_{++} = 0 = \partial_+ b_{--},$$
$$\partial_- c^+ = 0 = \partial_+ c^-. \tag{4.59}$$

The energy-momentum tensor of Eq. (3.14) is given by

$$T_{++} = \partial_+ x^\mu \partial_+ x^\mu - (2\alpha')(-2)(2b_{++}(\partial_+ c^+) + (\partial_+ b_{++}) c^+)$$
$$T_{--} = \partial_- x^\mu \partial_- x^\mu - (2\alpha')(-2)(2b_{--}(\partial_- c^-) + (\partial_- b_{--}) c^-). \tag{4.60}$$

Considering the open string, Eqs (3.78) and (3.81) have the natural interpretation as

$$c(\sigma) = \begin{cases} c^z & \text{for} \quad 0 < \sigma < \pi \\ c^{\bar{z}} & \text{for} \quad -\pi < \sigma < 0 \end{cases} \tag{4.61}$$

$$b(\sigma) = \begin{cases} -4b_{++} & \text{for} \quad 0 < \sigma < \pi \\ -4b_{--} & \text{for} \quad -\pi < \sigma < 0. \end{cases} \tag{4.62}$$

While for the *closed string* we recognize

$$c = c^+, \quad \bar{c} = c^-$$
$$b = -4b_{++}, \quad \bar{b} = -4b_{--}. \tag{4.63}$$

The reader will be convinced of the simplifying nature of using a conformal co-ordinate system. We could have adopted these co-ordinates at the outset, but when seeing the string for the first time, it is perhaps more palatable when it is first expressed in a more familiar co-ordinate system.

The reader may be puzzled by the off factors of 2 and (-4) which occur in the transition to ξ, $\bar{\xi}$ co-ordinates. The problem is that the conventions used for string theory do not naturally fit into those used in the conformal approach to string theory which inherited the conventions of Ref. [38].

To get between the two we make the change of variables

$$x^\mu \to \tfrac{1}{2} x^\mu, \quad c^+ \to c^+, \quad c^- \to c^-$$

$$b_{++} \to -\tfrac{1}{4} b_{++}, \quad b_{--} \to -\tfrac{1}{4} b_{--}$$

after these changes and setting $2\alpha^1 = 1$, we have the Minkowski space action

$$A = -\tfrac{1}{2}\pi \int d\xi d\bar{\xi}\{\tfrac{1}{2}\partial_+ x^\mu \partial_- x^\mu - (b_{++}\partial_- c^+ + b_{--}\partial_+ c^-)\},$$

where the oscillator expressions for the closed string now are:

$$P^\mu = \partial_+ x^\mu = \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\xi}, \qquad \bar{P}^\mu = \partial_- x^\mu = \sum_{n=-\infty}^{\infty} \bar{\alpha}_n^\mu e^{-in\bar{\xi}},$$

$$b_{++} = \tfrac{1}{2}(b_{00} b_{01} = i \sum_{n=-\infty}^{\infty} b_n e^{-in\xi}, \qquad c^+ = c^0 + c' = \sum_{n=-\infty}^{\infty} c_n e^{-in\xi},$$

$$b_{--} = \tfrac{1}{2}(b_{00} - b_{01}) = i \sum_{n=-\infty}^{\infty} \bar{b}_n e^{-in\bar{\xi}}, \qquad c^- = c^0 - c' = \sum_{n=-8}^{\infty} \bar{c}_n e^{-in\bar{\xi}}$$

where $\xi = \tau + \sigma$, $\bar{\xi} = \tau - \sigma$.

Conformal theories are usually discussed in Euclidean space. The transition from Minkowski space to Euclidean space is carried out by a Wick rotation $\tau \to -i\tau$ *after* which we use the complex co-ordinates $\xi = \tau + i\sigma$, $\bar{\xi} = \tau - i\sigma$, and $z = e^{\xi}$, $\bar{z} = e^{\bar{\xi}}$. Under this change, $\partial_+ \to i\partial \equiv i(\partial/\partial\xi)$, $\partial_- \to i\bar{\partial} = i(\partial/\partial\bar{\xi})$. It is convenient also to make the field redefinition $b_{\pm\pm} \to ib_{\pm\pm}$ with all the other fields unchanged.

After absorbing a factor of $(-i)$ in the definition of the functional integral, namely we use e^{-A} rather than e^{iA}, the action becomes

$$A = -\frac{1}{2\pi}\int d\xi d\bar{\xi}\{-\tfrac{1}{2}\partial x^\mu \bar{\partial} x^\mu - b_{++}\bar{\partial} c^+ - b_{--}\partial c^-\}.$$

The oscillator expressions become

$$P^\mu = i\partial x^\mu = \sum_{n=-\infty}^{\infty} \alpha_n^\mu z^{-n}$$

$$\bar{P}^\mu = i\bar{\partial} x^\mu = \sum_{n=-\infty}^{\infty} \bar{\alpha}_n^\mu \bar{z}^{-n}$$

$$b = b_{++} = \sum_{n=-\infty}^{\infty} b_n z^{-n}, \quad c = c^+ = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

$$\bar{b} = b_{--} = \sum_{n=-\infty}^{\infty} \bar{b}_n \bar{z}^{-n}, \quad \bar{c} = c^- = \sum_{n=-\infty}^{\infty} \bar{c}_n \bar{z}^{-n}.$$

It is a good exercise for the reader to begin in Euclidean space and recover the above equations. The energy-momentum tensor of Eq. (4.60) when scaled by a factor (-2) becomes

$$T_{++} = -\tfrac{1}{2}\,\partial x^\mu \partial x^\mu - 2b(\partial c) - (\partial b)c,$$
$$T_{--} = -\tfrac{1}{2}\,\bar{\partial} x^\mu \bar{\partial} x^\mu - 2\bar{b}(\bar{\partial}\bar{c}) - (\bar{\partial}\bar{b})\bar{c}.$$

We recognize the usual conventions used in the conformal field theory approach to string theory.

Given a tensor on the world strip, we map it with the Riemann sphere using $z = e^{\mathfrak{z}}$. If a tensor of weight $(d, \bar{d})$ has components R in the $\mathfrak{z}, \bar{\mathfrak{z}}$ co-ordinates and R' in the $z, \bar{z}$ co-ordinates they are related by

$$R(\mathfrak{z}, \bar{\mathfrak{z}}) = (z)^d (\bar{z})^{\bar{d}} R'(z, \bar{z}) \tag{4.64}$$

using Eq. (4.43). For example, we have

$$b'(z) = \frac{1}{z^2}\, b(\mathfrak{z}), \quad c'(z) = z c(\mathfrak{z}),$$

$$P'^\mu(z) = \frac{1}{z}\, P^\mu(\mathfrak{z}) \tag{4.65}$$

since b, c and P^μ have weights 2, -1 and 1 respectively. Putting their expansions in terms of oscillators, we have for example

$$b'(z) = \sum_{n=-\infty}^{\infty} b_n z^{-n-2}, \quad c'(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n+1}, \quad P'^\mu(z) = \sum_{n=-\infty}^{\infty} \alpha_n^\mu z^{-n-1}. \tag{4.66}$$

In what follows we will often drop the prime, it being clear from the discussion that the tensors are defined on the Riemann sphere.

When dealing with operators, we may use the representation of the L_n's in terms of α_n^μ's, b_n's and c_n's. In this case, Eq. (4.44) is for a tensor of weight $(d, \bar{d})$ given by

$$U\varphi(z, \bar{z})U^{-1} = \left(\frac{d\omega}{dz}\right)^d \left(\frac{d\bar{\omega}}{d\bar{z}}\right)^{\bar{d}} \varphi(\omega, \omega'), \tag{4.67}$$

where $U = \exp \sum_n (a_n L_n + \bar{a}_n \bar{L}_n)$ and it induces the change $(z, \bar{z}) \to (\omega, \bar{\omega})$ if we were to take $L_n, \bar{L}_n$ to be given as in Eq. (4.7).

The reader may verify that $P^\mu(z)$, $c(z)$ and $b(z)$ do indeed have their expected weights 1, -1 and 2 respectively. It suffices to confirm that they satisfy the infinitesimal form which for a weight $(d, 0)$ is

$$[L_n, \varphi(z)] = z^n \left(z\frac{d}{dz} + d(n+1)\right) \varphi(z) \tag{4.68}$$

One makes use of the relations

$$[L_m, \alpha_n^\mu] = -n\alpha_{n+m}^\mu \tag{4.69}$$

$$[L_m, b_n] = (m-n)b_{n+m}$$

There exist other conformal operators. The object

$$L(z) = \sum_n L_n z^{-n-2} \tag{4.70}$$

is conformal weight (2, 0) when L_n contains ghosts and

$$Q^\mu = q^\mu - ip^\mu \ln z - i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_{-n}^\mu \frac{z^n}{n} \tag{4.71}$$

has weight zero. We note that

$$P^\mu(z) = iz \frac{d}{dz} Q^\mu(z). \tag{4.72}$$

Emission vertices in string theory also have a given conformal weight, for example

$$: e^{ikQ(z)}: \tag{4.73}$$

has weight $k^2/2$. In the above statements it is understood that L_n's are ghost extended. For some purposes, it is useful to work with L_n's in terms of α_n^μ's alone. In this case Q^μ and P^μ have conformal weight zero and one respectively, and $L^\alpha(z) = \sum_n L_n^\alpha z^{-n}$ is almost an operator of weight two, apart from a phase factor which is induced by the presence of the central term in the L_n commutator relations.

In the old dual model and many of the works of Neveu and the author, a conformal field in z, $\bar{z}$ co-ordinates was defined without the $(z)^d(\bar{z})^d$ factors. That is, if ϕ has weight d it would obey Eq. (4.68) with $n+1$ replaced by n.

The above discussion spells out the importance of the conformal group in the formulation of the bosonic string. Examining the discussion we realize that one might just as well consider any two-dimensional (super) conformally invariant theory and by coupling it up to (super)-gravity to yield a locally conformally invariant theory. This latter theory we may consider as a string theory. However, we now have a local symmetry and we must ensure hat it has no quantum anomaly. This can be achieved in the following ways:

(a) calculate the anomaly directly, i.e., evaluate $\langle T_{++}T_{++}\rangle$ at one loop and take ∂_- of it;

(b) verify that the ghost extended L_n's have no central term:

(c) show the BRST charge squares to zero (i.e., $Q^2 = 0$);

(d) put the theory in light-cone gauge and verify that the Lorentz algebra closes. We leave it as an exercise to ponder why these requirements are equivalent!

Although the starting conformal theory may differ considerably from the 26 free scalar fields of the bosonic string, the gauge fixing and and ghost terms are the same as discussed above.

One can apply the same strategy to the superstring. We take a superconformally invariant theory, couple it up to supergravity so that it is locally superconformally invariant. By the same list as above, we must also verify that it has no superconformal anomaly.

Such strings, however, are not in general consistent theories and can develop diseases as interacting string theories. One such disease is anomalies in any space-time gauge or gravitational symmetries of the theory which can occur if the theory has chiral fermions such as occur in superstrings. It has been shown that these space-time anomalies to not occur in closed strings if the theory is modular invariant [42]. We therefore add modular invariance to our list of requirements of a good string theory.

The current attitude is that any (super) conformally invariant theory which has no (super) conformal anomaly, when coupled to (super)-gravity, provides a good string theory if it is modular invariant. This recipe has included a staggering number of four-dimensional string theories [43]. Even constructions which utilize free (super)-conformal theories are enormous in number. In fact, there is no satisfactory method to compute higher loop string corrections if one considers interesting superconformal field theories such as the BPZ series [38].

Unitarity, that is space-time unitarity, is not obviously guaranteed in the above string theories. Nonetheless, it is thought that their physical states have positive norm and that they factorize. The former is shown by going to the light-cone gauge, while the latter is a consequence of applying the known methods of computing in string theory to a free conformal theory.

A final comment concerns BRST anomalies. Space-time gauge and gravitational anomalies are none other than BRST anomalies of the simplest kind. One may wonder however if such anomalies vanish above one loop and if all the other BRST anomalies are automatically absent in any modular and conformally invariant theory. In particular, there are BRST invariances associated with the higher massive modes of the string and these must also be anomaly free. Anomalies of this type are known to arise in the open string. Although one might expect as they arise as boundary terms in moduli space, that modular invariance would ensure their absence in closed string theories. Nevertheless, it is possible that the ambiguities in the superstring are a symptom of an underlying BRST anomalies or equivalently the fact that zero norm physical states do not decouple.

4.2. Some conformal techniques using the operator product expansion

The aim of this section is to familiarize the reader with some of the more elementary technology used in the discussions of conformal field theories [38] and string theory. An essential tool is Wilson's operator product expansion [44] which says that given two quantum operators $A(x^\mu)$, $B(y^\mu)$ of a quantum field theory, then their time-ordered product behaves as x^μ approaches y^μ as

$$A(x^\mu)B(y^\mu) \sim \sum_I C_I(x^\mu - y^\mu)O^I \quad \text{as } x^\mu \to y^\mu, \tag{4.74}$$

where O^I are a set of local operators and C_I are coefficients depending on $(x-y)^\mu$. This equation is to be understood as being valid when the product $A(x)B(y)$ is inserted in a Green function with other operators, that is

$$\lim_{x\to y} \langle|T\{A(x)B(y)-\sum_l C_l(x-y)O^l\}\phi(\omega_1)\phi(\omega_2)\ldots|\rangle = 0, \tag{4.75}$$

where ϕ are operators of the theory. It is in this sense that Eq. (4.74) is automatically time-ordered. Generally, Eq. (4.75) is not valid if some of the ϕ's are composite operators, but it always holds if they are the elementary fields of a Lagrangian quantum field theory. The convergence properties of Eq. (4.75) when valid depend on the fields inserted in the Green function and in general it is an asymptotic series. The coefficients $C_l(x)$ can be shown to be of the form

$$C_l(x) = x^{-d_A-d_B-d_O} \text{ (times a polynomial of } \ln x), \tag{4.76}$$

where d_A, d_B and d_O are the dimensions of the operators A, B and O.

Let us now consider the operator product expansion for two-dimensional conformal field theories. We use $\tau = \ln|z|$ or equivalently $|z|$ as our time-ordering parameter. Given two conformal operators $R(z)$ and $S(z)$ of weights (d_R, O) and (d_S, O) respectively, their operator product expansion is

$$S(z)R(z') \sim \sum_Q \frac{Q(z')}{(z-z')^{d_R+d_S-d_Q}}, \tag{4.77}$$

where $Q(z')$ are conformal operators of dimension d_Q. The Wilson coefficients are determined by conformal invariance to be of the form shown. It is useful to consider the arguments of the operators Q to be z' rather than $(z+z'/2)$, this corresponds only to a rearrangement of the series.

One way to compute the operator product expansion is to normal order the operators. Given the operators $R(z)$ and $S(z')$ we may normal order them to find

$$S(z)R(z') = \sum_l \frac{P_l(z')}{(z-z')^{d_l-d_R-d_S}} + :S(z)R(z'): \tag{4.78}$$

which is valid only if $|z| > |z'|$, where R_l are normal-ordered operators. The singularities arise from taking past each other the infinite number of annihilation and creation operators, in order to achieve the normal-ordered product of the two operators. Any normal-ordered product is finite as $z \to z'$ in the sense that it has finite expectation values between any states with finite occupation number. Hence the singular terms as $z \to z'$ are found in the first term and as $|z| > |z'|$ corresponds to the correct time ordering for the operator product expansion, we may identify the terms in the operator product expansion with these in the normal-ordered expression.

An important application of the above is in the evaluation of the commutator of S_n and R_m where

$$S_n = \oint_{\Gamma_1} \frac{dz}{z} S(z) z^{n+d_S}, \quad R_m = \oint_{\Gamma_2} \frac{dz}{z} R(z) z^{m+d_R}. \tag{4.79}$$

The closed contour Γ_1, say, must enclose the point $z = 0$, but is otherwise arbitrary. The result for S_n being independent of the contour as $S(z)$ is analytic away from $z = 0$.

We may write

$$[S_n, R_m] = S_n R_m - R_m S_n$$
$$= \oint_{\Gamma_1} \frac{dz}{z} z^{n+d_S} \oint_{\Gamma_2} \frac{d\zeta}{\zeta} \zeta^{m+d_R} S(z) R(\zeta)$$
$$- \oint_{\Gamma_3} \frac{dz}{z} z^{n+d_S} \oint_{\Gamma_4} \frac{d\zeta}{\zeta} \zeta^{m+d_R} R(\zeta) S(z). \tag{4.80}$$

We choose the contours such that Γ_1 encloses Γ_2 and Γ_4 encloses Γ_3. This coincides with the time operating used for the operator product expansion. It also coincides with the point where the normal ordering of the two operators converges. We may now use the operator product expansion of Eq. (4.77) but as $T(S(z)R(\zeta)) = S(z)R(\zeta)$ for $|z| > |\zeta|$ and $T(R(\zeta)S(z)) = T(S(z)R(\zeta)) = R(\zeta)S(z)$ for $|\zeta| > |z|$, we use the same expression in the first and second term. Consequently, we have

$$[S_n, R_m] = \left\{ \oint_{\Gamma_1} \frac{dz}{z} z^{n+d_S} \oint_{\Gamma_2} \frac{d\zeta}{\zeta} \zeta^{m+d_R} \right.$$
$$\left. - \oint_{\Gamma_3} \frac{dz}{z} z^{n+d_S} \oint_{\Gamma_4} \frac{d\zeta}{\zeta} \zeta^{m+d_R} \right\} \left\{ \sum_Q \frac{Q(\zeta)}{(z-\zeta)^{d_S - d_R - d_Q}} \right\}. \tag{4.81}$$

Here we have taken S and R to be such that one of them is Grassmann odd. The same conclusion may be achieved by thinking about the normal ordering of the two operators. Let us choose to identify the contours Γ_2 and Γ_4. Then Γ_1 encloses Γ_2 which in term encloses Γ_3 (see Fig. 4.3a) and we consider carrying out the z integration. Since the integrand is analytic everywhere except at $z = \zeta$. We have the result

$$[S_n, R_m] = \oint_{\Gamma_2} \frac{d\zeta}{\zeta} \zeta^{m+d_R} \oint_{\Gamma} \frac{dz}{z} z^{n+d_S} \sum_Q \frac{Q(\zeta)}{(z-\zeta)^{d_S + d_R - d_Q}} \tag{4.82}$$

where Γ is a contour which surrounds the point $z = \zeta$ (see Fig. 4.3b). By standard techniques in complex analysis, in particular Cauchy's theorem, we may then evaluate the integral.

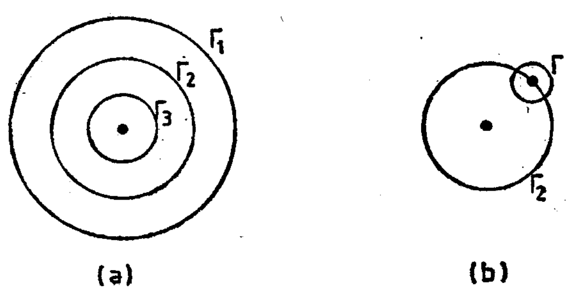

Fig. 4.3. Contours occurring in the evaluation of a commutator

By using the same argument we may show that

$$[S_n, R(\zeta)] = \oint_\Gamma \frac{dz}{z} z^{n+d_S} \sum_Q \frac{\varphi(\zeta)}{(z-\zeta)^{d_S+d_R-d_Q}} . \tag{4.83}$$

What the above demonstrates is the close relation between the commutator of two operators and their operator product expansion. To illustrate this fact, consider $S(z) \equiv T_{zz} = T$ and $R(z)$ to be any conformal operator which must therefore satisfy

$$[L_n, R(z)] = z^n \left\{ z \frac{d}{dz} + (n+1)d \right\} R(z). \tag{4.84}$$

The operator product expansion to recover this result upon using Eq. (4.83) is read off to be

$$T(z)R(\zeta) = \frac{dR(\zeta)}{(z-\zeta)^2} + \frac{\frac{d}{d\zeta}R(\zeta)}{(z-\zeta)} . \tag{4.85}$$

Similarly in order to recover the Virasoro algebra using Eq. (4.81), the operator product of two energy-momentum tensors must be of the form

$$T(z)T(\zeta) = \frac{1}{2}\frac{c}{(z-\zeta)^4} + \frac{2T(\zeta)}{(z-\zeta)^2} + \frac{\frac{d}{d\zeta}T(\zeta)}{(z-\zeta)} . \tag{4.86}$$

It is educational to carry out the above steps for a free scalar theory whose action is

$$\int dz d\bar{z} \left(\frac{-1}{2\pi}\right) \{-\tfrac{1}{2}\, \partial\varphi\bar{\partial}\varphi\}, \tag{4.87}$$

where $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. In order to quantize the theory with respect to the $\tau = \ln |z|$, it is most convenient to map to the string $\tau + i\sigma = \ln z$, quantize the system according to the usual rules and map back to the complex plane. The strategy is identical to that for the free bosonic string with the index μ suppressed. The classical equation of motion $\partial_z \partial_{\bar{z}} \phi = 0$ allows us to write ϕ in the form

$$\varphi(z, \bar{z}) = \varphi_L(z) + \varphi_R(\bar{z}), \tag{4.88}$$

where ϕ_L and ϕ_R depend on z and $\bar{z}$ respectively and be written as

$$\varphi_L(z) = a_L - ib_L \ln z - i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\alpha_{-n}}{n} z^n, \tag{4.89}$$

$$\varphi_R(\dot{z}) = a_R - ib_R \ln \bar{z} - i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\bar{\alpha}_{-n}}{n} \bar{z}^n. \tag{4.90}$$

We define $a = a_L + a_R$ and $b = b_L + b_R$. Clearly ϕ does not depend on $a_L - a_R$. The commutation relations for equal τ can be written as

$$[\varphi(\sigma), \partial_\tau \varphi(\sigma')] = 4i\pi\delta(\sigma - \sigma') \tag{4.91}$$

and these imply the relations

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}; \qquad [a, b] = i \tag{4.92}$$

and similarly for the barred quantities. The propagator for such a theory obeys the equation

$$\partial\bar{\partial}G(z, \bar{z}; \zeta, \bar{\zeta}) = -\delta^2(z-\zeta) \equiv \delta(z-\zeta)\delta(\bar{z}-\bar{\zeta}), \tag{4.93}$$

where

$$\begin{aligned} G(z, \bar{z}; \zeta, \bar{\zeta}) &\equiv \langle\varphi(z, \bar{z})\varphi(\zeta, \bar{\zeta})\rangle \\ &\equiv \langle 0|T\varphi(z, \bar{z})\varphi(\zeta, \bar{\zeta})\rangle. \end{aligned} \tag{4.94}$$

To find the propagator, we require the identity

$$\partial_{\bar{z}} \frac{1}{(z-\zeta)} = \delta^2(z-\zeta), \tag{4.95}$$

which we verify in the usual way for a δ function, by considering

$$\int_D d^2z \left(\partial_{\bar{z}} \frac{1}{(z-\zeta)}\right) f(z) = \int_{\partial D} dz \frac{f(z)}{z-\zeta} = f(\zeta), \tag{4.96}$$

where D is some domain containing the point ζ and ∂D is its boundary. The δ function is thus to be understood as acting only on analytic and antianalytic functions. Equation (4.95) must also be used with care, for example, in the evaluation of $\bar{\partial}\partial \ln z$ which in fact vanishes; a fact which is obvious on the world sheet. As such we recognize that

$$\begin{aligned} G(z, \bar{z}; \zeta, \bar{\zeta}) &= -\{\ln(z-\zeta)+\ln(\bar{z}-\bar{\zeta})\} \\ &= -\ln(|z-\zeta|^2). \end{aligned} \tag{4.97}$$

This expressions for G may be found also by directly evaluating (4.94) using Eqs (4.89), (4.90) and (4.92).

Although ϕ is not single-valued, the following fields are

$$\begin{aligned} P(z) &= i\partial\varphi = \sum_{n=-\infty}^{\infty} \alpha_n z^{-n-1} \\ \bar{P}(\bar{z}) &= i\bar{\partial}\varphi = \sum_{n=-\infty}^{\infty} \bar{\alpha}_n \bar{z}^{-n-1} \end{aligned} \tag{4.98}$$

In what follows, we will concentrate on the operators which are functions of z alone, i.e., left-moving objects and the reader should bear in mind the identical computations for the operators depending on $\bar{z}$ alone, i.e., right-moving objects. Let us find the operator product expansion of two $P(z)$ operators by computing their normal ordering. We have

$$\begin{aligned} P(z)P(\zeta) &= \,:P(z)P(\zeta): + \sum_{n=1}^{\infty} \frac{n}{z\zeta}\left(\frac{\zeta}{z}\right)^n \\ &= \,:P(z)P(\zeta): + \frac{1}{(z-\zeta)^2}, \end{aligned} \tag{4.99}$$

which is valid only if $|z| > |\zeta|$ as only in this region does the infinite sum converge. As such their operator product expansion is

$$P(z)P(\zeta) \sim \frac{1}{(z-\zeta)^2} + \text{analytic terms.} \tag{4.100}$$

We recover the oscillators in terms of $P(z)$ by

$$\alpha_{-n} = \oint \frac{dz}{z} z^{n+1} P(z) \tag{4.101}$$

and we may verify that Eq. (4.82) implies their commutation relation of Eq. (4.92).

The energy-momentum tensor for the free scalar field has the non-zero components

$$T_{zz} \equiv T = -\tfrac{1}{2}\!:\partial\varphi\partial\varphi:$$
$$T_{\bar{z}\bar{z}} \equiv T = -\tfrac{1}{2}\!:\bar{\partial}\varphi\bar{\partial}\varphi: \tag{4.102}$$

In evaluating operator product expansions of composite operators, it is useful to use Wick theorem which expresses the time-ordered product of a set of operators in terms of their normal-ordered product and the two-point Green function. For scalar fields $\phi(x)$, it states that:

$$T\{\phi(x_1)\dots\phi(x_N)\} = :\phi(x_1)\dots\phi(x_2):$$
$$+ \sum_{k<l} :\phi(x_1)\dots\hat{\phi}(x_k)\dots\hat{\phi}(x_l)\dots\phi(x_n): \langle 0|T\phi(x_k)\phi(x_l)\,|0\rangle$$
$$+ \dots + \sum_{k_1<k_2<\dots<k_{2p}} :\phi(x_1)\dots\hat{\phi}(x_{k_1})\dots\hat{\phi}(x_{k_{2p}})\dots\phi(x_n):$$
$$\times \sum_p \langle 0|T\phi(x_{k_{p_1}})\phi(x_{k_{p_2}})\,|0\rangle\dots\langle 0|T\phi(x_{k_{2p-1}})\phi(x_{k_{2p}})\,|0\rangle + \dots, \tag{4.103}$$

where a caret above a term means it is omitted and $\sum_p$ runs over all *distinct* permutations. The simplest example of such a result is:

$$T\phi(x_1)\phi(x_2) = :\phi(x_1)\phi(x_2): + \langle 0|T\phi(x_1)\phi(x_2)\,|0\rangle.$$

One can apply the formula to expressions which themselves contain normal-ordered pieces such as

$$T\{\phi(x_1)\dots:\phi(x_p)\dots\phi(x_q):\dots\phi(x_r)\dots\}. \tag{4.104}$$

The only modification being that one should not contain Green functions of fields which were originally normal ordering. The reader may recover the propagator of Eq. (4.97) by explicitly carrying out the normal ordering in Eq. (4.104) for $|z_1| > |z_2|$. Using Wick theorem we have

$$T\{P(z)P(\zeta)\} = :P(z)P(\zeta): + \langle 0|TP(z)P(\zeta)\,|0\rangle$$

$$= \frac{1}{(z-\zeta)^2} + :P(\zeta)P(\zeta): + \sum_{n=1}^{\infty} \frac{(z-\zeta)^n}{n} :\left[\left(\frac{\partial}{\partial\zeta}\right)^n P(\zeta)\right] P(\zeta):$$

$$= \frac{1}{(z-\zeta)^2} + 2T(\zeta) + \text{terms vanishing as } z \to \zeta. \tag{4.105}$$

Similarly we also find that

$$T\{T(z)P(\zeta)\} = \tfrac{1}{2} T\{: P(z)P(z) : P(\zeta)\}$$

$$= \tfrac{1}{2}: P(z)P(z)P(\zeta): + 2 \cdot \tfrac{1}{2} P(z) \langle 0 | T P(z)P(\zeta) | 0 \rangle$$

$$= \tfrac{1}{2}: P(z)P(z)P(\zeta): + \frac{P(z)}{(z-\zeta)^2}. \tag{4.106}$$

The operator product expansion is therefore

$$T(z)P(\zeta) = P(z) \frac{1}{(z-\zeta)^2} + \text{analytic}$$

$$= \frac{P(\zeta)}{(z-\zeta)^2} + \frac{1}{(z-\zeta)} \frac{\partial}{\partial\zeta}(P(\zeta)) + \text{analytic}. \tag{4.107}$$

We recognize $P(\zeta)$ as an operator of conformal weight one.

Using the same technique, we may evaluate the operator product expansion for two energy-momentum tensors. The time ordered-product, using Wick theorem, is a totally normal-ordered term plus a term with one contraction (i.e., one Green's function) which can be made in four ways and a term with two contractions (i.e., two Green's functions) which can be made in two ways. As a result

$$T(z)T(\zeta) = \tfrac{1}{4}: P(z)P(z): : P(\zeta)P(\zeta):$$

$$\sim \tfrac{4}{4}: P(z)P(\zeta): \frac{1}{(z-\zeta)^2} + \tfrac{2}{4} \frac{1}{(z-\zeta)^4} + \text{analytic}$$

$$= \tfrac{1}{2} \frac{1}{(z-\zeta)^4} + \frac{2T(\zeta)}{(z-\zeta)^2} + \frac{1}{(z-\zeta)} \frac{\partial}{\partial\zeta} T(\zeta) + \text{analytic}. \tag{4.108}$$

We recognize that one real scalar field gives an energy-momentum tensor with $c = 1$.

The reader may like to show that the operator $:e^{ik\phi(z)}:$ obeys the operator product expansion

$$T(z): e^{ik\varphi(\zeta)}: \sim \frac{1}{(z-\zeta)} \frac{\partial}{\partial\zeta} :e^{ik\varphi(\zeta)}:$$

$$+ \frac{k^2}{2} \frac{1}{(z-\zeta)^2} : e^{ik\varphi(\zeta)}: + \text{analytic} \tag{4.109}$$

which implies it has conformal weight $k^2/2$. Another useful formula is given by

$$:e^{ik\varphi(z)}::e^{ik'\varphi(\zeta)}: = \frac{1}{(z-\zeta)^{-k\cdot k'}} :e^{i(k\varphi(z)+k'\varphi(\zeta))}:. \tag{4.110}$$

Finally, let us consider the Grassmann odd fields b and c which have conformal weights $(\lambda, 0)$ and $(1-\lambda, 0)$ whose action [45] is

$$A = -\int dz d\bar{z}\left(-\frac{1}{2\pi}\right)(b\bar{\partial}c). \tag{4.111}$$

Examples of such a system are the ghost of the bosonic string considered earlier for which $\lambda = 2$. The infinitesimal conformal transformations of b and c are given by

$$\begin{aligned}\delta b &= \varepsilon\partial b + \lambda(\partial\varepsilon)b \\ \delta c &= \varepsilon\partial c + (1-\lambda)\partial\varepsilon c.\end{aligned} \tag{4.112}$$

Under these transformations the variation of the action is

$$\begin{aligned}\delta A = &-\int dz d\bar{z}\left(-\frac{1}{2\pi}\right)\{(\varepsilon\partial b + \lambda(\partial\varepsilon)b)\bar{\partial}c + b\bar{\partial}(\varepsilon\partial c + (1-\lambda)(\partial\varepsilon)c)\} \\ &-\int dz d\bar{z}\left(-\frac{1}{2\pi}\right)(\bar{\partial}\varepsilon)\{\lambda b\partial c - (1-\lambda)(\partial b)c\}.\end{aligned} \tag{4.113}$$

Since $\partial_{\bar{z}}\varepsilon$ vanishes, we find that A is conformally invariant. The coefficient of $\partial_{\bar{z}}\varepsilon$ is the energy-momentum tensor:

$$T = -\lambda b\partial c + (1-\lambda)(\partial b)c. \tag{4.114}$$

The equations of motion

$$\bar{\partial}b = 0 = \bar{\partial}c \tag{4.115}$$

imply that we may use the mode expansion

$$c(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n-(1-\lambda)}, \quad b(z) = i\sum_{n=-\infty}^{\infty} b_n z^{-n-\lambda}. \tag{4.116}$$

One can also introduce fields $\bar{c}$ and $\bar{b}$ of weights $(0, 1-\lambda)$ and $(0, \lambda)$ respectively. The analogous results can be thought of as taking the complex conjugate of those above.

To quantize the theory, using $\tau = \ln|z|$ as the time parameter, we impose the anti-commutator

$$\{c(\sigma), b(\sigma')\} = 2i\pi\delta(\sigma-\sigma'), \tag{4.117}$$

which implies that

$$\{c_n, b_m\} = \delta_{n+m,0}, \quad \{c_n, c_m\} = 0 = \{b_n, b_m\}. \tag{4.118}$$

The propagator must satisfy

$$\bar{\partial}\langle c(z)b(\zeta)|\rangle = \delta^2(z-\zeta), \tag{4.119}$$

and hence

$$\langle c(z)b(\zeta)\rangle = \frac{1}{z-\zeta}. \tag{4.120}$$

When normal-ordering anticommuting quantities we must use a minus sign when we interchange this order; for example $Tc(z)b(\zeta) = -b(\zeta)c(z)$ if $|\zeta| > |z|$. Consequently, we find that

$$\begin{aligned}\langle 0|Tb(z)c(\zeta)|0\rangle &= -\langle 0|Tc(\zeta)b(z)|0\rangle \\ &= -\frac{1}{\zeta - z} = \frac{1}{z-\zeta}.\end{aligned} \tag{4.121}$$

Wick theorem asserts that

$$Tc(z)b(\zeta) = :c(z)b(\zeta): + \langle c(z)b(\zeta)\rangle \tag{4.122}$$

and using Eq. (4.18) we find $\langle c(z)b(\zeta)\rangle$ in agreement with the above. In deriving this result we define the vacuum by

$$\begin{aligned} b_n|0\rangle &= 0 \quad n > \lambda - 1, \\ c_n|0\rangle &= 0 \quad n > -\lambda, \end{aligned} \tag{4.123}$$

and normal order with respect to this vacuum. This definition of the vacuum is the one which is SL(2, C) invariant, meaning that it is invariant under $L_{\pm 1}$ and L_0 where L_n is constructed from the energy-momentum tensor of Eq. (4.114) in the usual way

$$\begin{aligned} L_n &= \oint \frac{dz}{z} z^{n+2} T(z) \\ &= :\sum_p b_{-p} c_{n-p}(-p + n\lambda):. \end{aligned} \tag{4.124}$$

From the viewpoint of conformal field theories, as $L_{-1} + \bar{L}_{-1}$ is a translation and so it is natural to insist on such a definition of the vacuum. We recognize from Eq. (4.122) the operator product expansion

$$c(z)b(\zeta) \sim \frac{1}{z-\zeta} + \text{analytic}. \tag{4.125}$$

We may use Eq. (4.125) to arrive at Eq. (4.118).

We now verify that the energy-momentum tensor given above generates the correct transformations on b and c. To demonstrate this, it is sufficient to compute their operator product expansions

$$T(z)b(\zeta) \sim \frac{1}{(z-\zeta)^2}\lambda b(\zeta) + \frac{1}{(z-\zeta)}\frac{\partial}{\partial\zeta} b(\zeta) + \text{analytic}, \tag{4.126}$$

$$T(z)c(\zeta) = \frac{(1-\lambda)c(\zeta)}{(z-\zeta)^2} + \frac{1}{z-\zeta}\frac{\partial}{\partial\zeta} c(\zeta) + \text{analytic}. \tag{4.127}$$

In deriving the above results, we used Wick theorem for the time-ordered product and disregarded the term which has no two-point functions and is totally normal ordered.

It is very useful to verify that the energy-momentum tensor of Eq. (4.114) satisfies the correct operator product expansion, namely

$$T(z)T(\zeta) = \frac{2T(\zeta)}{(z-\zeta)^2} + \frac{1}{(z-\zeta)}\frac{\partial}{\partial\zeta}T(\zeta) + \tfrac{1}{2}\frac{c}{(z-\zeta)^4}. \tag{4.128}$$

We leave the reader to verify the first two terms. The last term comes from contracting all the fields. Although there is only one way to perform these two contractions, there are four terms in all. One of them is given by

$$: -\lambda b(z)\partial_z c(z): \; : -\lambda b(\zeta)\partial_\zeta c(\zeta): \tag{4.129}$$

Carrying out both contractions, we have

$$(-\lambda^2)\partial_\zeta\langle b(z)c(\zeta)\rangle\partial_z\langle c(z)b(\zeta)\rangle = -\frac{(\lambda)^2}{(z-\zeta)^2}. \tag{4.130}$$

A similar analysis on the remaining three terms yields

$$c = 2(-1+6\lambda-6\lambda^2). \tag{4.131}$$

In the above analysis, we took c and b to be Grassmann odd, however, we could just as well have taken them to be Grassman even. The only changes in the above are that now

$$\langle b(z)c(\zeta)\rangle = -\frac{1}{z-\zeta} \tag{4.132}$$

and

$$c = (-2)(-1+6\lambda^2-6\lambda^2) \tag{4.133}$$

all other equations being the same.

The above formula does not apply to real anticommuting fields with $\lambda = 1-\lambda = \frac{1}{2}$, however, the reader may verify that in this case $c = \frac{1}{2}$.

Using the above formula for c, we can test the vanishing of the conformal anomaly for some well known string theories. The BRST structure, i.e., the action and energy--momentum tensor of the ghosts and antighosts is independent of the details of the two--dimensional theory from which one starts and only depends upon the algebra of the local symmetry. Hence if one has a theory with a local conformal invariance we have a $b-c$ system with $\lambda = 2$ which contributes a c of -26. This is cancelled by 26 x^μ's of the bosonic string which have $c = 1$ for each μ.

For a theory with local superconformal invariance, we have a $b-c$ system with $\lambda = 2$ and a commuting $\beta-\gamma$ system with $\lambda = 3/2$. These contribute respectively -26 and $+11$ to c and so the "matter" fields must have a c of $+15$. If we have a system with bosonic fields x^μ and fermionic fields ψ^μ which are both real, but contain left and right movers, then for each μ they contribute to c 1 and $\frac{1}{2}$ respectively. As such we require a space-time dimension of ten, that is the superstring [46].

Finally, let us consider the heterotic string [47] which has the two dimensional content x^μ_L, ψ^μ_L as well as x^μ_R, ψ^i_R, $i = 1, \ldots, 32$. Here L and R denote that the fields depend on z and $\bar{z}$ respectively. For the left movers (L) we have a left supersymmetry and so we have to balance a c of -15 from the ghost sector. Since the μ index takes ten values this works in the same way as the superstring. For the right movers, we have only conformal invariance and so the ghost sector contributes -26 while x^μ_R and ψ^i_R contribute $10 \cdot 1$ and $32 \cdot \frac{1}{2}$, respectively.

The superstring and the bosonic string have a maximal Lorentz symmetry in that all the fields belong to the representation of the Lorentz group of the dimension of the space-time. One can relax this assumption and demand Lorentz symmetry in only a space-time of a lesser dimension which should of course include four dimensions. An example of this is that the heterotic string contains fields ψ^i_R which do not belong to a representation of SO(9, 1). One can go further, and have x^μ_L, ψ^μ_L and x^μ_R, $\mu = 0, 1, 2, 3$ and then add fermions or bosons to balance the c from the ghost sector. Clearly, there are many ways to do this. One has also to ensure modular invariance, but one is still left with many string theories (see previous section).

5. *Conformal mapping and interacting string theory*

So far we have only considered free string theory. Interacting string theory will be discussed later. However, in this chapter we will consider how to represent the surfaces which strings sweep out when they interact. To do this, one does not need to understand the details of how to compute interacting string amplitudes.

Given the central role of conformal invariance for free string theory, it will come as no surprise to the reader to be told that the surface interacting strings sweep out does not have a unique representation, and that any two representations which are related by a conformal mapping will lead to the same string scattering. Conformal mapping, although physically irrelevant, may however introduce or remove technical complications and it turns out that for certain representations of the world surface, it is much easier to compute string scattering than for others. In particular, string computations seem to be most easily carried out by conformally mapping to the Riemann sphere. Such an example was given in the previous chapter where we mapped the free string world sheet when viewed as a strip to the Riemann sphere. We will in this chapter repeat this exercise for the world sheet of the interacting strings. By carrying out this exercise, we will gain a feeling for interacting string theory before being submerged in its complications.

5.1. The world sheet for interacting tree level strings

Two strings can interact by touching at a point to form a third string. Of course, the time reversed process can also occur, namely a string breaks at a point to form two strings. For open strings they join at their end points (Fig. 5.1) while for closed strings two strings may touch at any point (Fig. 5.2). There is also a four-point interaction [48] for the open string which cannot be drawn in the plane, but also has a natural interpretation in terms of touching and joining of strings and there exists an open closed string transition.

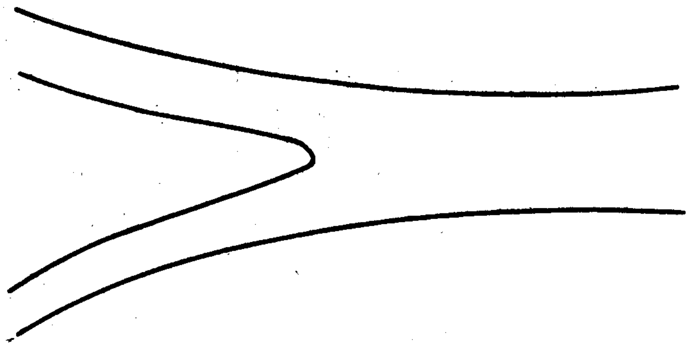

Fig. 5.1. Three open string scattering

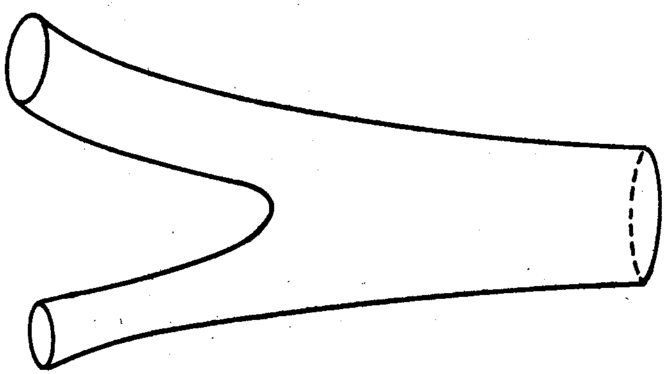

Fig. 5.2. Three closed string scattering

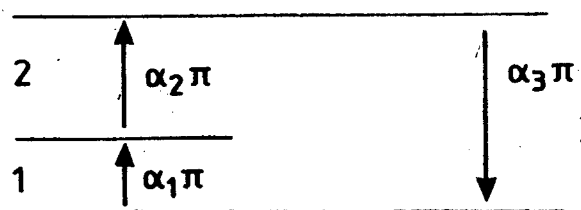

Fig. 5.3. Two strings joining to form a third

Let us consider in detail three open strings scattering which for concreteness we take to be two strings joining to form a third one rather than the reverse. We may represent the world sheet for this process as a single strip (see Fig. 5.3) in the ϱ plane where the strip is

$$0 < \operatorname{Im} \varrho \leqslant \alpha_3 \pi \quad \text{and} \quad -\infty < \operatorname{Re} \varrho < \infty. \tag{5.1}$$

The individual strings have a "length" $|\alpha_r|\pi$ where we take α_r negative for outgoing strings. In the process the "length" of the third string is the sum of the "lengths" of string one and two. We must remember, however, since the string is reparametrization invariant and so the "length" has no physical meaning.

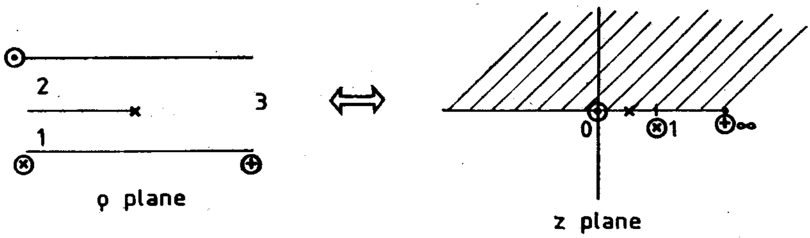

Fig. 5.4. Mapping the world sheet for three string scattering to upper half plane

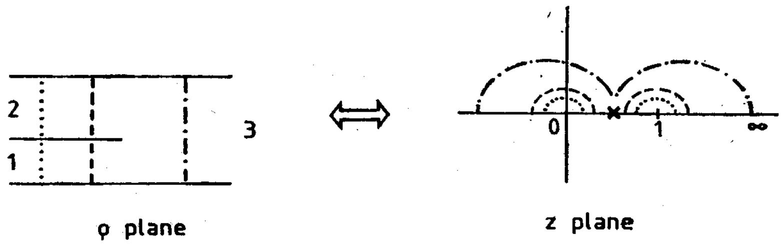

Fig. 5.5. Time evolution on the world sheet and upper half plane

We note that in the interacting light-cone theory and in one of the two gauge covariant string theories, string lengths arise as given above. In the light-cone theory [25], one can choose the gauge such that the string length is none other than the centre-of-mass momentum in the + (i.e., x^0+x^1) direction. In the gauge covariant string theory, the length arises as a remaining component of the metric which cannot be gauged away [27].

We now map the strip onto the upper half plane (see Fig. 5.4) by the transformation

$$\varrho' = \varrho+\tau_b = \alpha_1 \ln(z-1)+\alpha_2 \ln z. \tag{5.2}$$

As the upper half plane is mapped onto itself in a one-to-one way by a real Möbius transformation (see Section 4), the mapping from the ϱ to the z plane is only fixed up to this transformation. To resolve this ambiguity we can select where any three fixed points on the world sheet, such as the ends of the three strings, are mapped onto the upper half plane. In the above map of Eq. (5.2), we have chosen the three end points of the strings three, two and one to be the points ∞, one and zero, respectively in the upper half plane.

One finds in particular that the real axis of the upper half plane is mapped onto the boundary of the string. The three-end points of the strings are mapped onto the three points of the upper half plane as shown. Starting at $z = +\infty$, i.e., at point + and moving along the real axis, we move along the strip boundary to the end of string one (i.e., point × at $z = 1$). Here $(z-1)$ changes argument by π and so we jump to the top of the first string and move towards the interaction point. When we arrive at this point, we then travel along the bottom of string two towards the end of string two, etc. Since the derivatives at the interaction point are discontinuous, the map is not invertible at this point, i.e.,

$$\frac{d\varrho'}{dz} = 0 \rightarrow z_b = -\frac{\alpha_2}{\alpha_3} \tag{5.3}$$

and

$$\tau_b = \alpha_1 \ln\left(-\frac{\alpha_1}{\alpha_3}\right) + \alpha_2 \ln\left(-\frac{\alpha_2}{\alpha_3}\right). \tag{5.4}$$

The time evolution of the string on upper half plane is demonstrated in the diagram 5.5. We will require the inverse of this map, that is the map from the upper half plane to the strip. To this end, let us define the object $y(\gamma, x)$ by

$$\begin{aligned} y &= \gamma \ln(1+xe^y) \\ &= \gamma(xe^y - \tfrac{1}{2}(xe^y)^2 + \dots) \qquad (5.5) \\ &= \gamma x + \frac{\gamma}{2}(2\gamma-1)x^2 + \dots \\ &\equiv \gamma \sum_{n=1}^{\infty} f_n(\gamma) x^n. \qquad (5.6) \end{aligned}$$

It can be shown that [49]

$$f_n(\gamma) = \frac{1}{n\gamma}\binom{n\gamma}{n} = \frac{1}{n!}(n\gamma-1)(n\gamma-2)\dots(n\gamma-n+1). \tag{5.7}$$

The inverse map from the half plane to the strip is given by [50]

$$z = \begin{cases} -\dfrac{1}{\mathfrak{z}_3} e^{-y_3(\mathfrak{z}_3)} & \text{for string 3} \\ e^{-\frac{\alpha_1}{\alpha_2} y_1(\mathfrak{z}_1)} & \text{for string 1} \\ \mathfrak{z}_2 e^{-\frac{\alpha_1}{\alpha_2} y_2(\mathfrak{z}_2)} & \text{for string 2,} \end{cases} \tag{5.8}$$

where

$$\mathfrak{z}_r = e^{\tau_b + \varrho_r}; \qquad y_r \equiv \gamma_r \ln(1+\mathfrak{z}_r e^{y_r})$$

$$\gamma_r = -\frac{\alpha_{r+1}}{\alpha_r}, \qquad (\alpha_{3+1} \equiv \alpha_1). \tag{5.9}$$

One can easily verify by substituting in Eq. (5.2) that this is indeed the inverse map. One can extend the range of σ of the open strings as for the free case. This corresponds in the complex plane to a reflection about $\operatorname{Im} z = 0$.

For the scattering of N open string at tree level, we may proceed similarly. We represent the scattering by a strip in the ϱ plane of the form $0 < \operatorname{Im} \varrho < \pi \sum_{i=1}^{m} \alpha_i$, $-\infty < \operatorname{Re} \varrho < +\infty$ and map it to the upper half plane by the Mandelstam map

$$\varrho = \sum_{r=1}^{N-1} \alpha_r \ln(z - z_r) + \text{constant}. \tag{5.10}$$

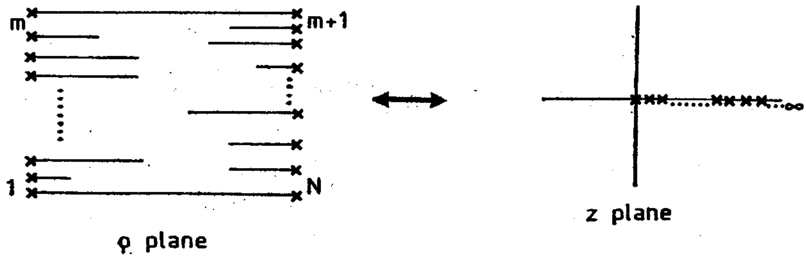

Fig. 5.6. Mapping the world sheet for N string scattering to the upper half plane

(see Fig. 5.6). We choose $z_N = \infty$ and so this term does not appear and we can also set $z_1 = 1$, $z_2 = 0$.

The reader may readily construct for himself the analogous mappings for the closed string.

5.2. The world sheet for quantum string scattering

In this case the world sheet sweep out has holes in it (see Fig. 5.7) and so closed curves are not contractible to a point. The world sheet is therefore a Riemann surface of genus $g \geqslant 1$.

Rather than work with a Riemann surface itself, it is much more convenient to work with a representation of it which is defined on the Riemann sphere. There are two commonly used representations called Shottky and Fuchsian.

Let us consider the Schottky representation of a Riemann surface of genus g. Many mathematics textbooks discuss the Fuchsian representation. The Schottky representation was used in string theory for the introduction of moduli in the classic paper of Alessandrini [51] and has been used in many of the recent calculation of multistring scattering. Consider g elements P_n of SL(2, C) and their action on the Riemann sphere C. Let us denote by G the group generated by P_n. It has elements $P_1, \ldots, P_n, P_1^2, P_1P_2, P_1P_3, \ldots, P_1^3, \ldots$, etc. A fixed point z_0 of a transformation P is one such that $P(z_0) = z_0$. A Schottky group is a group G as above with the restriction that the fixed points of all its elements form a discrete set.

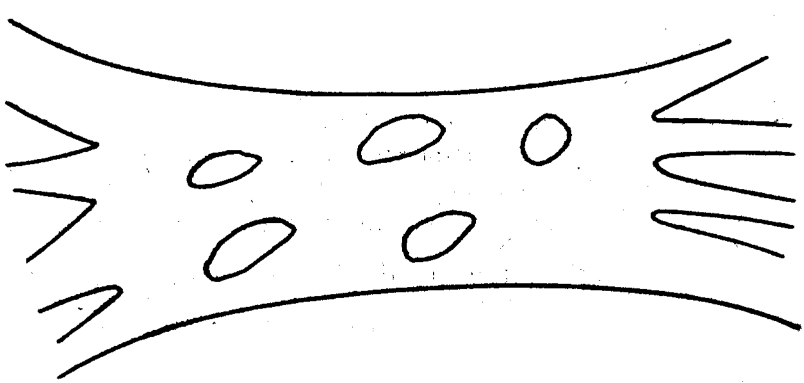

Fig. 5.7. Open string multiloop scattering

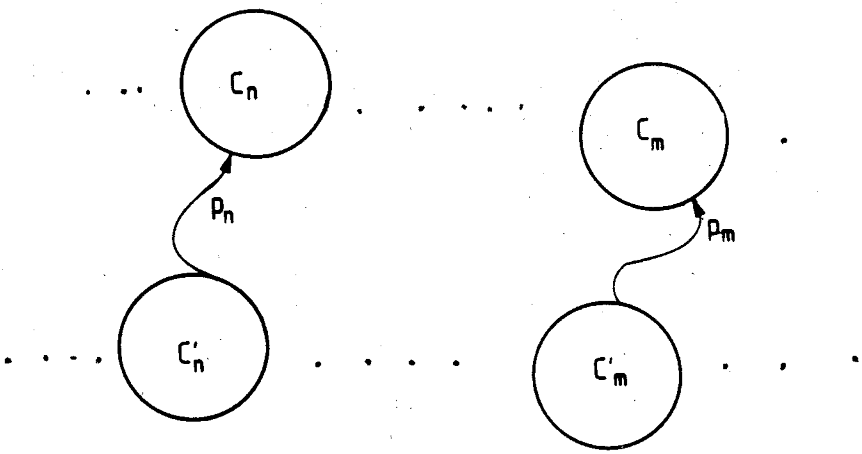

Fig. 5.8. Schottky representation of a Riemann surface

We now can ask what is the fundamental region of G, that is the region F, such that any point of C can be obtained from a point of F by the action of an element of G and no two points of F are related by a group transformation. It has been shown that the fundamental region is, in fact, the region outside the $2n$ isometric circles of P_n and P_n^{-1}, $n = 1, \ldots, g$.. The isometric circle of a transformation $z \to (az+b)/(cz+b)$ is the one such that $|cz+d|^2 = |ad-bc|$. It is straightforward to show that the isometric circle C_n, of P_n^{-1} is mapped into the isometric circle of C_n by the action of P_n as illustrated in Fig. 5.8.

The relation of the above discussion to a Riemann surface is provided by the following:

THEOREM

Any Riemann surface can be represented by the fundamental region of a suitable Schottky group.

Indeed the elements of this Schottky group are unique up to a single Möbius transformation S such that $P_u \to SP_nS^{-1}$ and up to modular transformations.

For example, at one loop the torus is represented by the region outside the two isometric circles corresponding to a transformation P_1 and its inverse P_1^{-1}. By identifying C_1' with C_1, which are mapped into each other by P_1 and are connected by a B cycle we recover the torus (see Fig. 5.9).

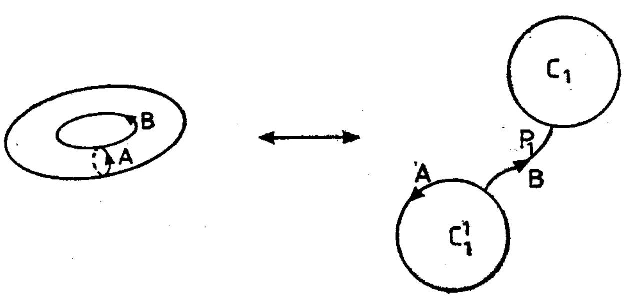

Fig. 5.9. The Schottky representation of a torus

For the closed string, we find that each Riemann surface is characterized by P_n, $n = 1, \ldots, g$ elements of the Möbius group minus one overall Möbius transformation, that is $3g-3$ complex parameters in all. In terms of the $2g$ isometric circles, these correspond to the positions of their centres (i.e., $2g$ complex parameters), their radii (a further g real parameter), as P_n and P_n^{-1} have the same radius namely $\sqrt{ad-bc}/c$ and g real parameters which come from how one identifies a specific point on C_n' with a point on C_n, that is g angles. The -3 is due to the fact that SP_nS^{-1} generate a group which represents the same surface. These $3g-3$ parameters are precisely the encoding of the $3g-3$ of possible complex structures on a Riemann surface of genus g. The case of $g = 1$ is slightly different as here we have only one complex modulus.

For the open string, we first double the surface by the same process as for the $g = 0$ case to construct a closed Riemann surface and then we take the P_n's to be elements of $SL(2, R)/Z_2$. Hence the surface is labelled by $3g-3$ real parameters.

6. Discussion of interacting string theory

In the remaining lectures, I discussed one of the new operator methods for computing string scattering, in particular the group theoretic method of Neveu and myself. Since I have greatly exceeded my allotted pages, I will refer the reader to the review of Ref. [52] on this subject and here only discuss the ill-appreciated relations between the various new oscillator formalisms.

The "traditional" method of computing perturbative string scattering are the old dual model discussed below, the Polyakov method [53] which is reviewed in Ref. [29] and light-cone string field theory [25]. The latter, like gauge covariant string theory, does lead to the correct perturbative results, but they are rather difficult to derive using these methods.

More recently, many calculations of string scattering have been performed in the new oscillation formalism which are the group theoretic method [54–61], the Grassmannian approach [62–66] and the approaches of the two Copenhagen groups [67–73].

All these approaches are in effect an operator approach to string theory, and within the context of past developments can be viewed as most closely related to the old dual model [74], without having its defects. We recall that the old dual model consisted of a three--vertex [75] and a propagator which were sewn together to yield multiloop diagrams. These results contained many of the correct features; however, in general negative norm states propagated uncompensated for in the loops and also zero norm physical external states did not decouple. A strategy used in this formalism was to calculate a multiloop diagram with one external leg [76] and obtain the general diagram by sewing an appropriate tree graph to the one external leg.

Several of the features of the new operator approaches to string theory can be found in a series of papers by Neveu and West [77–81]. In particular in one of these papers [78] the three-vertex for arbitrary bosonic string scattering was extended to include anticommuting (i.e., ghost) oscillators, and shown to be annihilated by the action of the BRST charge. These authors also realized [78–80] that the techniques [77, 78] they had used to construct gauge covariant string field theory could be used in string theory in general, and specifically outside the context of string field theory. In particular, within the context of three-vertices, they wrote down the unintegrated and integrated overlaps (see Eqs. (2) and (7) of Refs. [52]), respectively for arbitrary conformal operators, and extolled the simplicity with which one could compute string vertices using these overlaps. In addition to the three-vertex string with ghosts mentioned above, the three-Neveu-Schwarz vertex [80] and the non--bosonized ghost addition to the fermion emission vertex [81] were derived using overlap relations.

Subsequent to the above developments, the four new approaches were given. The two Copenhagen groups are the most conservative in stategy and so we begin with these. The group [67–71] of P. Di Vecchia, M. Frau, A. Lerda and S. Sciuto adopted essentially the same strategy as the old dual model. They took the ghost extended BRST-invariant three-vertex of Ref. [78] mentioned above, and having found [70] a ghost extension of the old dual model propagator, which also included a ghost insertion, sewed together these objects to compute multiloop graphs [70, 71]. This procedure, due to the BRST properties of the three-vertex and propagator, automatically ensured the correct decoupling properties of negative and zero norm states, thus solving the most important problem of the old dual model. Perhaps the most important achievement in this approach is an explicit expression for the measure in the Schottky representation and vertex [71] for a multiloop amplitude with any number of arbitrary external legs, the latter being in agreement with the result found previously in Ref. [56] using the group theoretic method.

The other Copenhagen group [67–71] of Petersen and Sidenius adopted a strategy that is intermediate between the Polyakov and the old dual models. Namely, they took as their starting point the functional integral for the sum over world surfaces [53], and in particular showed [67] how to recover the three-vertex plus ghosts of Ref. [78] from this viewpoint. The arbitrary external states were treated by taking them to be coherent states when integrating between initial and final states in the sum over world sheets. Of course, having the expression for such an amplitude, one can readily replace it by an oscillator expression using the well-known relation between coherent states and oscillators, namely

$$\alpha_n^\mu |z_n^\mu) = z_n^\mu |\alpha_n^\mu), \qquad n \geqslant 1.$$

To construct multiloop amplitudes, these authors did not carry out the sum over world histories in a direct way as was done in Ref. [82] to find the measure, but rather they chose to carry out the sum by successive sewing with the above three-point vertex in functional form. Antighost insertions in the measure are seen in this approach as arising from integrations over quasi-conformal ghost modes associated with the choice of moduli. This group independently found the multiloop measure in the Schottky representation and the multiloop vertex [72].

The philosophy behind the group theoretic [54–61] and Grassmannian [62–66] approaches was to reformulate string theory itself. In the group theoretic approach, one started, following Refs. [77–81], from an algebraic type of formulation unrelated at first sight to usual starting points for string theory. One assumed that any string vertex (for a review of string vertices in this context the reader may consult Ref. [61]) obeyed overlap relations (Eqs. (2), (6) and (7) of Ref. [52]) for the fundamental conformal fields, and that zero norm physical states decoupled. The overlap relations determined the multiloop vertex for any number of arbitrary external states. Indeed, there is no particular simplicity in taking only one external string. On the other hand, decoupling allowed one to find a set of first-order differential equations for the measure. As there is one such equation for each modulus, they uniquely determine the measure. These equations have been explicitly found [60] when ghosts have been incorporated into the general vertex. In this case, it was shown that it was possible to rescale the vertex so that the measure was one, i.e., the measure has been absorbed into the vertex.

Important in finding these first-order differential equations was an understanding of the conformal properties of string vertices (see the Introduction to this paper for a history of this subject). In particular, one needed to know which conformal transformations annihilated the vertex (the isotropy group) and which changed the moduli. The former were easily found by taking the overlap relations for the conformal operator $L(z) = \sum_n L_{-n}z^n$, while the question of which conformal transformation induced which moduli change could be deduced by their action on the overlap relations defining the vertex [54–61].

Since the group theoretic approach does not start from an action, unitarity and especially factorization are not guaranteed. However, it was simple to show [58] that sewing any two vertices as defined by overlap relations led to a third vertex, which also satisfied its appropriate overlap relation. Thus factorization was almost self-evident. Further, since the three-vertex, when ghost-extended, was constructed by using overlap relations, this result guarantees that the results of the two Copenhagen groups and that of the group theoretic approach will be in agreement.

Some of the explicit results found with the group theoretic method, in addition to those mentioned earlier, were the multiloop vertex for the scattering of any number of arbitrary external strings [56] and the BRST anomaly for all open strings except $SO(2^{13})$ [55] (also found independently using very different methods in Ref. [83]). In Ref. [84] the ghost extension of the three-Neveu-Schwarz vertex was found using overlap relations.

The Grassmannian approach [62–66] defines the state (called a vertex in the other approaches) as an element of a Grassmannian or as a functional integral over the appropriate Riemann surface. It was shown independently [64], at least under certain circumstances, that vertices (i.e., states) obeyed conserved charge equations using one or the other description. Conserved charge equations are *identical* to the integrated overlap equations (i.e., Eq. (7) of Ref. [52]). We note that these had been previously used in Refs. [77–81], and independently for loops in graphs in Refs. [54–61]. Although the derivation of the overlap equations is different in the Grassmannian approach, their use to determine the vertices is the same as in the group theoretic approach. The socalled g vacua are just multiloop vertices with one external leg and follow from the conserved charge relations, taking the special case of only one external leg. As advocated in the old dual model, one can sew a tree vertex to the g vacuum to obtain a multileg graph. This result agrees with finding the graph directly from the overlap relations for an arbitrary number of external legs, due to our previous statements on sewing and overlaps.

The determination of the measure in the Grassmannian approach proceeds by saturating the vertex, when "naturally" normalized, by a suitable vacuum. It is then shown that this state satisfies the Belavin-Knizhnik theorem [84]. Decoupling is shown, as is the fact that this measure is globally defined on moduli space, at least for the bosonic string.

In the Grassmannian approach, the vertices are shown to carry a representation of the conformal group and the energy momentum tensor acts as a connection which defines parallel transport as the moduli change. First found in the Grassmannian approach were the correlation functions for the *b–c* system.

The Grassmannian and group theoretic approaches have many features in common, despite their different starting points, as mentioned above. The vertices are determined in the same way, using overlap relations, and their sewing properties were established independently by both groups in essentially the same way. They identify the same object for the measure; in group theoretic approach it is determined from decoupling, while in the Grassmannian approach it is shown to satisfy the Belavin-Knizhnik theorem which uniquely determines it to be the measure as defined by the Polyakov string. It is also shown to satisfy decoupling [66]. Also similar [60, 66] is the way the actual vertex with the correct decoupling properties is found by multiplying the BRST-invariant vertex, deduced directly from the overlap relations, by ghost factors which are closely related to the conformal properties of the BRST vertex.

All four groups have now tackled the Neveu-Schwarz-Ramond string.

I wish to thank Mike Freeman, André Neveu, Jens Petersen and Raymond Stora for many helpful discussions.

APPENDIX

The BRST formulation of Yang-Mills theory

Let us consider the group G whose antihermitian generators T^a satisfy the Lie algebra

$$[T_a, T_b] = f_{ab}{}^c T_c. \tag{A.1}$$

The gauge fields A^a_μ are in the adjoint representation of the group G and we define the matrix valued field $A_\mu(x) = gA^a_\mu T_a$. It transforms as $A_\mu \to A_\mu + U^{-1}\partial_\mu U$ where $U = \exp(\omega^a(x)T_a)$. The covariant derivative is given by $D_\mu = \partial_\mu + A_\mu$ and the field strength by $F_{\mu\nu} = gF^a_{\mu\nu}T_a = [D_\mu, D_\nu]$. We normalize the T_a generators by $\operatorname{Tr}(T_aT_b) = -C_F\delta_{ab}$.

Starting from the usual Yang-Mills action

$$-\tfrac{1}{4}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu} = \int d^4x \frac{1}{4g^2c_F} \operatorname{Tr} F_{\mu\nu}F_{\mu\nu}, \tag{A.2}$$

we choose to fix the gauge by a condition of the form $G(A_\mu) = 0$; the most popular choice being $\partial_\mu A^\mu = 0$. We fix the gauge to prevent the overcounting in the functional integral. That is, we wish to sum not only over a given gauge field configuration and all those related by gauge transformation, but only over one copy. A practical reason for fixing the gauge is that the usual perturbation theory for small coupling constant requires a propagator which is invertible. We implement this gauge fixing by adding to the Yang-Mills action the gauge fixing term

$$-\int d^4x \,\mathrm{Tr}\,\{BG[A_\mu x]\}, \tag{A.3}$$

where $B = B^a T_a$ is a Lagrange multiplier. The sum of these two terms is no longer gauge invariant, but this symmetry is to be replaced by a rigid BRST symmetry with an anti-commuting parameter Λ which we now construct. The gauge field A_μ transforms as before, except that we must replace the local parameter $\omega^a T_a$ by

$$\delta A_\mu = \Lambda D_\mu c \equiv \Lambda\{\partial_\mu c + [A_\mu, c]\}, \tag{A.4}$$

where c^a are the ghost. The ghosts are to transform as

$$\delta c = -\Lambda c \cdot c = -\frac{\Lambda}{2}\{c, c\}$$

or

$$\delta c^e = -\frac{\Lambda}{2} c^a c^b f_{ab}{}^e, \tag{A.5}$$

where $c \cdot c$ means matrix multiplication. We also introduce antighosts $b = b^a T_a$ which transform as

$$\delta b = \Lambda B, \quad \delta B = 0. \tag{A.6}$$

The most important characteristic of the transformations of A_μ, c, b and B is that

$$\delta_\Lambda \delta_\pi(\text{any field}) = 0, \tag{A.7}$$

i.e.

$$\delta_\Lambda \delta_\pi c = -\delta_\Lambda(\Pi c \cdot c) = -\Pi(\Lambda c \cdot c \cdot c + c \cdot \Lambda c \cdot c) = 0. \tag{A.8}$$

The reader may verify this statement for himself for the other fields. The final task is to extend the action by adding a third term in such a way that the total action is gauge invariant. This is achieved by

$$A^{\mathrm{BRST}} = \int \frac{d^4x}{4g^2 c_k} \mathrm{Tr}\,\{F_{\mu\nu}F_{\mu\nu}\} - \int d^4x \,\mathrm{Tr}\, BG + A^{\mathrm{gh}} \tag{A.9}$$

where

$$A^{\mathrm{gh}} = -\int d^4x \,\mathrm{Tr}\left\{b \frac{\delta G}{\delta A^\mu} D^\mu c\right\}.$$

For the case

$$G = \partial^\mu A_\mu, \quad A^{\mathrm{gh}} = -\int d^4x \,\mathrm{Tr}\,\{b\partial_\mu D^\mu c\} \tag{A.10}$$

owing to its original gauge invariance, the first term is invariant under the BRST transformation. The invariance of the second and third terms may be verified directly or by noting that

$$\Lambda(A^{\mathrm{gh}} + A^{\mathrm{gf}}) = -\int d^4x \delta_\Lambda \,\mathrm{Tr}\,\{b(x)G[A(x)]\} \tag{A.11}$$

and as $\delta^2 = 0$, it follows that $\delta_\Lambda(A^{\mathrm{gf}} + A^{\mathrm{gh}}) = 0$.

Let us now consider the hermiticity properties of the ghost system. Let us choose $c = c^a T_a$ to be antihermitian, that is c^a to be Hermitian. Reality of the action implies that b is Hermitian as

$$\begin{aligned} \mathrm{Tr}\,(b\partial^\mu\partial_\mu c)^\dagger &= \mathrm{Tr}\,(\partial^\mu\partial_\mu c)^\dagger b^\dagger \\ &= -\,\mathrm{Tr}\,(\partial^\mu\partial_\mu c b^\dagger) = \mathrm{Tr}\,(b^\dagger\partial^\mu\partial_\mu c). \end{aligned} \tag{A.12}$$

Similarly as Λc replaces $\omega = \omega^a T$ which is antihermitian, Λ must be antihermitian.

Corresponding to the rigid BRST symmetry of A^{T}, we may construct a BRST current and associated charge Q. In the BRST formalism, we have to impose that physical states should satisfy

$$Q\chi = 0. \tag{A.13}$$

In practice, what this means is that asymptotic states, when they exist, should satisfy this condition. The significance of this condition is further discussed in Chapter 3.

Often, one arrives at the above results in a slightly different manner. One uses the well--known insertion of one into the functional integral and extracts and then disregards the group integration, to arrive at the action of Eq. (A.9). One then notices that this action has the above BRST symmetry (see Les Houches Lectures of B.W. Lee, for example).

The BRST approach has been developed not so much as a general theory but rather as a procedure learnt from a number of examples of which Yang-Mills is the prototype. We now extract the general procedure. Its justification is that the final result namely a nilpotent set of transformations and an invariant which usually define a quantum theory which is unitary and whose physical observables are independent of how the gauge was fixed. Another point in its favour is that it allows one to demonstrate the renormalizability of Yangs-Mills for a general class of gauges.

We replace the original gauge transformations by rigid BRST transformations which have an anticommuting parameter such that:

(*i*) we introduce a ghost c^a and antighost b_a pair for each local invariance ω^a;

(*ii*) the original fields have BRST transformations given by the replacement $\omega^a \to \Lambda c^a$ while those of the ghost are $\delta c = -\Lambda/2\{c, c\}$ where the bracket is evaluated using the relevant action of the generator T_a which can be a differential operator, say, for the group of diffeomorphisms. The variation of δc can also be found by carrying out the commutators in the original gauge invariant theory, making the above substitution and extracting one power of Λ. These transformations are guaranteed to be nilpotent;

(*iii*) in a less geometrical manoeuvre, we introduce the Lagrange multiplier B and the gauge fixing function G. The antighost b and B transform as $\delta b = \Lambda B$, $\delta B = 0$;

(*iv*) finally, we find a BRST invariant action which is given by $\Lambda(A^{\mathrm{gf}} + A^{\mathrm{gh}}) = \delta_\Lambda(\int dx bG)$.

The above works for many systems and in these notes we apply the method to the point particle and string.

Of course, it could happen that having carried out the above procedure, one finds that the ghost action has a local invariance. In this event, one must repeat the procedure adding ghosts for ghosts, etc., until one arrives at a set of nilpotent transformations and an invariant action. Arriving at such an endpoint is, in general, not guaranteed but it almost always works.

Finally, we note that rather than use an anticommuting parameter one can regard the BRST transformations as a derivation in the spirit of $x \to dx$ for the usual space-time manifolds.

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