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# NOTE ON ROTATING UNIVERSE MODELS

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In some recent papers it has been stated that the Gödel space-time may be generalized to the non-stationary case, simply by allowing the scale-factor to be a function of cosmic time. We here show (for rather general energy-momentum tensors) that this metric allows no solution of Einstein's gravitational field equations.

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## *1. Introduction*

Ever since Lanczos [1] presented the first cosmological solution exhibiting cosmic rotation this theme has captured the interest of relativists. The most famous example of such models is the solution of Gödel [2], which has been studied in detail by many authors.

Both Gamow [3] and Gödel [2] speculated that the observed rotation of galaxies might be due to an overall cosmic rotation. Both because of this Gamow/Gödel hypothesis and because of the non-Machian character of cosmic rotation, it is of much theoretical interest to study expanding cosmological models which also have nonzero cosmic rotation.

Recently Krechet [4] proposed a non-stationary generalization of the Gödel space-time, without explicitly solving the field equations. Later Ivanenko and Krechet [5] applied this metric to the case of a perfect fluid, and deduced a relation between the rotation scalar and the trace of the energy-momentum tensor.

The purpose of the present paper is to point out the mathematical difficulties which arise when one tries to construct expanding rotating universe models. Especially we show that the Gödel metric cannot be generalized to the non-stationary case simply by letting

$R$ be a function of time. This is due to the fact that a dust filled expanding, rotating universe will always also have shear. Hence the energy-momentum tensor must allow for anisotropic pressures.

## 2. *Analysis of the Krechet-Gödel metric*

Krechet proposed the metric

$$ds^2 = R^2(t)\,[dx^2 - \tfrac{1}{2}\,e^{2\sqrt{2}\omega_0 x}dy^2 + dz^2] + 2R(t)e^{\sqrt{2}\omega_0 x}dt\,dy - dt^2 \tag{1}$$

which reduces to the Gödel metric in the case when $R$ is a constant. Now, we introduce a tetrad basis by

$$\theta^1 = R(t)dx, \tag{2a}$$

$$\theta^2 = R(t)\frac{1}{\sqrt{2}}\,e^{\sqrt{2}\omega_0 x}dy, \tag{2b}$$

$$\theta^3 = R(t)dz, \tag{2c}$$

$$\theta^0 = dt - R(t)e^{\sqrt{2}\omega_0 x}dy. \tag{2d}$$

By use of the comoving tetrad formalism, the kinematics of this model can be expressed solely in terms of the structure coefficients of the tetrad basis [6]. We define the structure coefficients by

$$d\theta^\alpha = \tfrac{1}{2}\,c^\alpha{}_{\beta\gamma}\theta^\beta \wedge \theta^\gamma \tag{3}$$

and demand that the metric is Minkowskian in the tetrad frame: $g_{ij} = \mathrm{diag}\,(-1, 1, 1, 1)$, and that $\boldsymbol{u} = \theta^0$. With these assumptions, we find the following general formulae: four-acceleration vector:

$$a_i = c^0{}_{i0}, \tag{4a}$$

vorticity tensor:

$$\omega_{ij} = \tfrac{1}{2}\,c^0{}_{ij}, \tag{4b}$$

expansion (deformation) tensor:

$$\theta_{ij} = \tfrac{1}{2}\,(c_{i0j} + c_{j0i}), \tag{4c}$$

expansion scalar:

$$\theta = \theta^i{}_i = c^1{}_{01} + c^2{}_{02} + c^3{}_{03}, \tag{4d}$$

vorticity vector:

$$\omega^1 = \tfrac{1}{2}\,c^0{}_{23}, \quad \omega^2 = \tfrac{1}{2}\,c^0{}_{31}, \quad \omega^3 = \tfrac{1}{2}\,c^0{}_{12}, \tag{4e}$$

vorticity scalar:

$$(\omega)^2 = \omega_i\omega^i = \tfrac{1}{4}\,[(c^0{}_{23})^2 + (c^0{}_{31})^2 + (c^0{}_{12})^2], \tag{4f}$$

shear tensor:

$$\sigma_{ij} = \theta_{ij} - \tfrac{1}{3}\theta\delta_{ij}, \tag{4g}$$

shear scalar:

$$\begin{aligned}(\sigma)^2 &= \tfrac{1}{2}\sigma_{ij}\sigma^{ij} \\ &= \tfrac{1}{3}\left[(c^1{}_{01})^2+(c^2{}_{02})^2+(c^3{}_{03})^2 \right.\\ &\quad \left. - c^1{}_{01}c^2{}_{02} - c^1{}_{01}c^3{}_{03} - c^2{}_{02}c^3{}_{03}\right] \\ &\quad + \tfrac{1}{4}\left[(c^1{}_{02}+c^2{}_{01})^2+(c^1{}_{03}+c^3{}_{01})^2+(c^2{}_{03}+c^3{}_{02})^2\right].\end{aligned} \tag{4h}$$

The last terms of the shear scalar, rising from cross terms of the expansion tensor (or equivalently, cross terms of the structure coefficients), describe the change of angles between the axes of an imagined coordinate-axis as it moves with the fluid. In all Bianchi models the hypersurface orthogonal frame (and all its Lorentz transforms) has angle preserving motion and hence the shear terms arise only through expansion anisotropy.

Taking the exterior derivatives of the tetrad basis (2), where $\theta^0$ is identified with the four velocity one-form of fundamental observers, we find the following kinematics of Krechet's model

$$\omega^3 = -\omega_0/R, \tag{5a}$$

$$a_2 = \sqrt{2}\,(\dot{R}/R), \tag{5b}$$

$$\sigma_{ij} = 0, \tag{5c}$$

$$\theta_{ij} = (\dot{R}/R)\delta_{ij}. \tag{5d}$$

The fluid has shear-free expansion and nonvanishing vorticity. We also note that this frame has a nonvanishing four-acceleration when $\dot{R} \neq 0$. Since this implies a violation of energy-momentum conservation, a dust cosmology of this type can be ruled out from purely kinematic reasoning.

For completeness we shall, however, write down the Einstein tensor of the above model. The tetrad-basis components of the Einstein-tensor are

$$E_{11} = E_{33} = 2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{\omega_0}{R}\right)^2, \tag{6a}$$

$$E_{22} = -2\frac{\ddot{R}}{R} + 5\left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{\omega_0}{R}\right)^2, \tag{6b}$$

$$E_{10} = \sqrt{2}\,E_{12} = 2\sqrt{2}\,\omega_0\frac{\dot{R}}{R^2}, \tag{6c}$$

$$E_{20} = 2\sqrt{2}\left[\left(\frac{\dot{R}}{R}\right)^2 - \frac{\ddot{R}}{R}\right], \tag{6d}$$

$$E_{00} = -4\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{\omega_0}{R}\right)^2. \tag{6e}$$

If $\dot{R} = 0$, $R = 1$, and $\Lambda = -\omega_0^2 = -4\pi G\varrho$, we recover the Gödel solution. If both $\dot{R}$ and $\omega_0$ are different from zero, the energy-momentum tensor must allow for a non-vanishing anisotropic pressure $\pi_{ij}$, as well as heat conduction terms. In the above space-time, a four--velocity field $\boldsymbol{u}$ proportional to the zeroth tetrad component (this is the velocity field which has vorticity equal to $\omega_0/R$) does not have shear. Hence the usual relation $\pi_{\mu\nu} \sim \sigma_{\mu\nu}$ of a viscous fluid [7] does not lead to the needed anisotropic pressure. We conclude that for a viscous fluid the Krechet-Gödel metric has no non-trivial solution for $R(t)$.

In the case of a dust energy-momentum tensor, the above result is an illustration of a more general theorem of Ellis [8], who showed that generally a dust cosmology can not have both expansion and rotation without also having shear.

We also note that the non-zero four acceleration can be avoided by modifying Krechet's metric, by changing the $\boldsymbol{\theta}^0$ tetrad component to read

$$\boldsymbol{\theta}^0 = \boldsymbol{d}t - e^{\sqrt{2}\omega_0 x} R_0 \boldsymbol{d}y, \tag{7}$$

where $R_0$ is a constant. Then the vorticity decays as $1/R^2$ with expansion. The Einstein tensor in the $\boldsymbol{\theta}$-basis, however, still has nonzero (12), (10), (20) terms, showing that the energy-momentum tensor must allow for both heat flow and anisotropic pressure.

We conclude, that although the modification (7) leads to a geodesic flow of fundamental particles, which a priori is compatible with general relativity, there is no viscous fluid which solves the field equations. Of course, if one puts no restrictions on the energy--momentum tensor, any metric can be regarded as a "solution" of Einstein's field equations.

**Note added in proof.** Recently Korotkii and Krecht [9] have proposed new sources for the Krecht--Gödel metric. These sources are composed of anisotropic or ideal fluids, a minimally coupled complex scalar field, and a radiation field.

## REFERENCES

[1] C. Lanczos, *Z. Phys.* **21**, 73 (1924).
[2] K. Gödel, *Rev. Mod. Phys.* **21**, 447 (1949).
[3] G. Gamow, *Nature* **158**, 549 (1946).
[4] V. G. Krechet, *Sov. J. Phys.* **28**, 957 (1985).
[5] D. D. Ivanenko, V. G. Krechet, *Sov. J. Phys.* **30**, 191 (1987).
[6] H. H. Soleng, *Astrophys. Space Sci.* **137**, 373 (1987); Note that the expressions for the expansion tensor and shear scalar in the above paper is valid only for angle preserving flows.
[7] S. Weinberg, *Astrophys. J.* **171**, 175 (1971).
[8] G. F. R. Ellis, *J. Math. Phys.* **8**, 1171 (1967).
[9] V. G. Korotkii, V. G. Krecht, *Sov. Phys. J.* **31**, 214 (1988).