ONE-PARTICLE IRREDUCIBLE SEPARABLE FEYNMAN DIAGRAMS AND THE $\hat{R}$-OPERATION

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For the case of one-particle irreducible, separable divergent Feynman diagrams the classical definition of the $\hat{R}$-operation is compared with some intuitive approach to the problem of overlapping ultraviolet divergences. The freedom of the generalized $\hat{R}$-operation is analysed.

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1. The $\hat{R}$-operation

The $\hat{R}$-operation is a standard method used in order to remove the ultraviolet divergences from the perturbation calculations of quantum field theory [1, 2, 3].

Notation and definitions: ω_γ — the index of the diagram γ, $\omega_\gamma = 4N-2l$ (only scalar fields and no derivatives), l — nuber of lines of γ (propagators), N — number of independent loops.

The divergent Feynman diagram $\Gamma \equiv$ one-particle irreducible Feynman diagram with nonnegative index or (and) containing some one-particle irreducible divergent subdiagrams. $\{\gamma_1 \dots \gamma_n\}$ is the family of divergent subdiagrams of the diagram Γ (if $\omega_\Gamma \geqslant 0$, then $\Gamma \in \{\gamma_1 \dots \gamma_n\}$). $(\hat{1}-\hat{M}_\gamma)$ is a subtraction of the first $\frac{\omega_\gamma}{2}+1$ terms from the Taylor expansion of the regularized amplitude $A^\varepsilon_\gamma(\boldsymbol{k})$ with respect to the external invariants $\boldsymbol{k}$.

The $\hat{R}$-operation is defined [3]:

$$\hat{R} = \sum_{\text{over forests } F_k} (-\hat{M}_{\gamma_{\alpha_1}}) \dots (-\hat{M}_{\gamma_{\alpha_k}}). \tag{1}$$

This definition may be rewritten in the equivalent factorized form [2]

$$\hat{R} = (\hat{1}-\hat{M}_{\gamma_1})(\hat{1}-\hat{M}_{\gamma_2}) \dots (\hat{1}-\hat{M}_{\gamma_k}) \tag{2}$$

if all product $\hat{M}_{\gamma_i}\hat{M}_{\gamma_k}$ for overlaping γ_i, γ_k are put to be zero operators.

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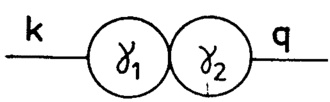
2. The $\hat{R}$-operation for separable diagrams

The diagram is separable if the removal of some vertex divides the diagram into two parts.

Let us consider a separable divergent diagram Γ with two one-particle irreducible divergent subdiagrams γ_1, γ_2 bound by one vertex



$$\left.\begin{array}{l}\omega_{\gamma_1} \geqslant 0\\ \omega_{\gamma_2} \geqslant 0\end{array}\right\} \Rightarrow \omega_\Gamma \geqslant 0$$

$\gamma_1(\gamma_2)$ contains $n_1(n_2)$ vertices, Γ contains n_1+n_2-1 vertices. The "line" $\boldsymbol{k}$ denotes all external lines of γ_1; $\boldsymbol{k}$ itself is a set of external invariants of the diagram γ_1. The same is for γ_2. The sets $\boldsymbol{k}$ and $\boldsymbol{q}$ have one common element. Some internal lines of γ_2 are external lines for γ_1 and vice versa.

For simplicity, let γ_1 and γ_2 be primitively divergent diagrams, i.e. diagrams containing no divergent subdiagrams.

We want to "clean" the amplitude C_Γ^ε from ultraviolet divergences. The most natural way to do it is to come back to the n_1 and n_2 perturbation calculation orders. In these orders the regularized amplitudes for γ_1 and γ_2 were respectively

$$A_{\gamma_1}^\varepsilon(\boldsymbol{k}) = e^{n_1}\left(\frac{a_{-1}(\boldsymbol{k})}{\varepsilon}+a_0(\boldsymbol{k})+O_a(\varepsilon)\right). \tag{3}$$

$$B_{\gamma_2}^\varepsilon(\boldsymbol{q}) = e^{n_2}\left(\frac{b_{-1}(\boldsymbol{q})}{\varepsilon}+b_0(\boldsymbol{q})+O_b(\varepsilon)\right). \tag{4}$$

The residues $a_{-1}(\boldsymbol{k})$ and $b_{-1}(\boldsymbol{q})$ are polynomials of the order $\frac{\omega_{\gamma_1}}{2}$ and $\frac{\omega_{\gamma_2}}{2}$ respectively. The $\hat{R}$-operation applied to $A_{\gamma_1}^\varepsilon$ and $B_{\gamma_2}^\varepsilon$ gives

$$(\hat{1}-\hat{M}_{\gamma_1})A_{\gamma_1}^\varepsilon(\boldsymbol{k}) = e^{n_1}(a_0(\boldsymbol{k})-W_a(\boldsymbol{k})) = A_{\gamma_1}(\boldsymbol{k}), \tag{5}$$

$$(\hat{1}-\hat{M}_{\gamma_2})B_{\gamma_2}^\varepsilon(\boldsymbol{q}) = e^{n_2}(b_0(\boldsymbol{q})-W_b(\boldsymbol{q})) = B_{\gamma_2}(\boldsymbol{q}), \tag{6}$$

where $W_a(\boldsymbol{k})$ is a polynomial, it consists of the first $\frac{\omega_{\gamma_1}}{2}+1$ terms of the Taylor expansion of $a_0(\boldsymbol{k})$. (The same for $W_b(\boldsymbol{q})$.) The terms $O(\varepsilon)$ are dropped out here — they are assigned for liquidation anyway. The operations $(\hat{1}-\hat{M}_{\gamma_1})$ and $(\hat{1}-\hat{M}_{\gamma_2})$ may be represented in the Lagrangian by counterterms Λ_{γ_1} and Λ_{γ_2} [1] proportional (in momentum space) to $e^{n_1}\left(\frac{a_{-1}(\boldsymbol{k})}{\varepsilon}+W_a(\boldsymbol{k})\right)$ and $e^{n_2}\left(\frac{b_{-1}(\boldsymbol{q})}{\varepsilon}+W_b(\boldsymbol{a})\right)$, respectively. The finite amplitude

$C_\Gamma(k \cup q)$ is expected to be equal to

$$C_\Gamma(k \cup q) = \varepsilon^{n_1+n_2-1}(a_0(k) - W_a(k))\,(b_0(q) - W_b(q)). \tag{7}$$

However, the counterterms Λ_{γ_1} and Λ_{γ_2}, if present in the Lagrangian, do not realize the proper $\hat{R}$-operation with respect to Γ. (The reason is the common vertex connecting γ_1 and γ_2.) The $\hat{R}$-operation (1) for the diagram Γ is

$$\hat{R} = \hat{1} - \hat{M}_\Gamma - \hat{M}_{\gamma_1} - \hat{M}_{\gamma_2} + \hat{M}_\Gamma \hat{M}_{\gamma_1} + \hat{M}_\Gamma \hat{M}_{\gamma_2} = (\hat{1} - \hat{M}_\Gamma)\,(\hat{1} - \hat{M}_{\gamma_1} - \hat{M}_{\gamma_2}). \tag{8}$$

In order to realize this $\hat{R}$-operation in the counterterms scheme, a new counterterm (except Λ_{γ_1} and Λ_{γ_2}) would be necessary, namely $\Lambda_\Gamma(k \cup q)$ (see the Appendix).

At this point one can be interested in comparing the intuitive formula for $C_\Gamma(k \cup q)$ (7) with the result given by the $\hat{R}$-operation (8) realized by three counterterms Λ_Γ, Λ_{γ_1}, Λ_{γ_2}. Let us take into account, that

$$(\hat{1} - \hat{M}_\Gamma)\hat{M}_{\gamma_1}\hat{M}_{\gamma_2}\left(\frac{a_{-1}(k)}{\varepsilon} + a_0(k)\right)\left(\frac{b_{-1}(q)}{\varepsilon} + b_0(q)\right)$$

$$= (\hat{1} - \hat{M}_\Gamma)\left(\frac{a_{-1}(k)}{\varepsilon} + W_a(k)\right)\left(\frac{b_{-1}(q)}{\varepsilon} + W_b(q)\right) \equiv 0 \tag{9}$$

because $(\hat{1} - \hat{M}_\Gamma)$ acts here on a polynomial of the order $\frac{\omega_1 + \omega_2}{2}$. (This is a singular case of a general theorem about overlapping divergences, e.g. [4, 5].)

From (8) and (9) we have

$$\hat{R}_\Gamma C_\Gamma^\varepsilon(k \cup q) = \varepsilon^{n_1+n_2-1}(\hat{1} - \hat{M}_\Gamma)\,(\hat{1} - \hat{M}_{\gamma_1})\,(\hat{1} - \hat{M}_{\gamma_2})$$

$$\times \left(\frac{a_{-1}(k)}{\varepsilon} + a_0(k)\right) \times \left(\frac{b_{-1}(q)}{\varepsilon} + b_0(q)\right)$$

$$= \varepsilon^{n_1+n_2-1}(\hat{1} - \hat{M}_\Gamma)\,(a_0(k) - W_a(k))\,(b_0(q) - W_b(q))$$

$$= \varepsilon^{n_1+n_2-1}(a_0(k) - W_a(k))\,(b_0(q) - W_b(q)) \tag{10}$$

(compare with (7)) if

$$\hat{M}_\Gamma(a_0(k) - W_a(k))\,(b_0(q) - W_b(q)) \equiv 0. \tag{10'}$$

Up to now nothing has been assumed about the subtraction points (around these points the Taylor expansions are realized). However, to have (10′) we must take $(k \cup q)_0 = k_0 \cup q_0$.

Our result is the following: in general, the subtraction points cannot be treated as independent variables.

We are now ready to study the case of the generalized $\hat{R}$-operation.

3. *The generalized $\hat{R}$-operation*

Instead of subtractions $(\hat{1}-\hat{M}_\gamma)$ let us introduce generalized "subtractions" [1, 2] $(\hat{1}-\hat{M}_\gamma+\hat{P}_\gamma)$, acting in the following way:

$$(\hat{1}-\hat{M}_\gamma+\hat{P}_\gamma)C^\varepsilon_\gamma(\boldsymbol{k}) = (\hat{1}-\hat{M}_\gamma)C^\varepsilon_\gamma(\boldsymbol{k})+P_\gamma(\boldsymbol{k}). \tag{11}$$

where $P_\gamma(\boldsymbol{k})$ is some polynomial of the order $\frac{\omega_\gamma}{2}$.

The generalized $\hat{R}$-operation is a suitable product of generalized subtraction operators, for which all prescriptions connected with overlapping hold [2].

There are two possible interpretations of this generalization. The first interpretation refers to the choice of the subtraction points. Let $(\hat{1}-\hat{M}_\gamma)$ and $(\hat{1}-\hat{M}'_\gamma)$ be subtractions for some points $\boldsymbol{k}_0$ and $\boldsymbol{k}'_0$ respectively. Of course,

$$\hat{1}-\hat{M}'_\gamma = \hat{1}-\hat{M}_\gamma+\hat{P}. \tag{12}$$

Keeping $\boldsymbol{k}_0$ fixed and manipulating with $\boldsymbol{k}'_0$ we come to some class of polynomials $P(\boldsymbol{k})$. In fact, this "generalization" illustrates only the freedom of choice of the subtraction points. In the previous section we have derived, in which way this freedom should be limited, when a separable (scalar and without derivatives) diagram is concerned.

There is also another, wider as the previous one, possible interpretation of polynomials $\hat{P}$. Let us choose some definite subtraction point. After the subtraction is realized (let the Feynman diagram be primitively divergent for simplicity), the amplitude $(\hat{1}-\hat{M}_\gamma)$ A^ε_γ is not yet renormalized — only the divergent part of it's Taylor series is "amputated". After subtraction there is the time for renormalization: we have to build up the "amputated" part of the Taylor series by some polynomial of the order $\frac{\omega_\gamma}{2}$; this is $\hat{P}_\gamma$.

These two interpretations are not equivalent (cf. [2] page 108). If we want to call $\hat{P}_\gamma$ the finite renormalization operator, we have to take the second interpretation. In the first case the class of possible polynomials P_γ is determined by the class of possible subtraction points. In the second one, P_γ seems to be completely free, not being determined by the theory. Exactly this freedom is studied in what follows.

For our diagram (Fig. 1) the generalized $\hat{R}$-operation is

$$\hat{R}_\Gamma = (\hat{1}-\hat{M}_\Gamma+\hat{P}_\Gamma)(\hat{1}-\hat{M}_{\gamma_1}-\hat{M}_{\gamma_2}+\hat{P}_{\gamma_1}+\hat{P}_{\gamma_2}) \tag{13}$$

$$\hat{R}_\Gamma C^\varepsilon_\Gamma(\boldsymbol{k}\cup\boldsymbol{q}) = \varepsilon^{n_1+n_2-1}\Big\{(\hat{1}-\hat{M}_\Gamma)(\hat{1}-\hat{M}_{\gamma_1}+\hat{P}_{\gamma_1}-\hat{M}_{\gamma_2}+\hat{P}_{\gamma_2})$$

$$\times\left(\frac{a_{-1}(\boldsymbol{k})}{\varepsilon}+a_0(\boldsymbol{k})\right)\left(\frac{b_{-1}(\boldsymbol{q})}{\varepsilon}+b_0(\boldsymbol{q})\right)+P_\Gamma(\boldsymbol{k}\cup\boldsymbol{q})\Big\}. \tag{14}$$

Making use of the formula

$$(\hat{1}-\hat{M}_{\Gamma})(-\hat{M}_{\gamma_1}+\hat{P}_{\gamma_1})(-\hat{M}_{\gamma_2}+\hat{P}_{\gamma_2})\left(\frac{a_{-1}(k)}{\varepsilon}+a_0(k)\right)\times\left(\frac{b_{-1}(q)}{\varepsilon}+b_0(q)\right)$$

$$=(\hat{1}-\hat{M}_{\Gamma})\left[\left(-\frac{a_{-1}(k)}{\varepsilon}-W_a(k)+P_{\gamma_1}(k)\right)\left(-\frac{b_{-1}(q)}{\varepsilon}-W_b(q)+P_{\gamma_2}(q)\right)\right]\equiv 0 \quad (15)$$

$\Bigg($because the contents of the square bracket is a polynomial of the order $\frac{\omega_{\gamma_1}+\omega_{\gamma_2}}{2}\Bigg)$ we have from (14)

$$\hat{R}_{\Gamma}C^{\varepsilon}_{\Gamma}(k\cup q)=e^{n_1+n_2-1}\Bigg\{(\hat{1}-\hat{M}_{\Gamma})(\hat{1}-\hat{M}_{\gamma_1}+\hat{P}_{\gamma_1})(\hat{1}-\check{M}_{\gamma_2}+\hat{P}_{\gamma_2})$$

$$\times\left(\frac{a_{-1}(k)}{\varepsilon}+a_0(k)\right)\left(\frac{b_{-1}(q)}{\varepsilon}+b_0(q)\right)+P_{\Gamma}(k\cup q)\Bigg\}$$

$$=e^{n_1+n_2-1}\{(\hat{1}-\hat{M}_{\Gamma})(a_0(k)-W_a(k)+P_{\gamma_1}(k))(b_0(q)-W_b(q)+P_{\gamma_2}(q))+P_{\Gamma}(k\cup q)\} \quad (16)$$

and to have (16) reduced to

$$\hat{R}_{\Gamma}C^{\varepsilon}_{\Gamma}(k\cup q)=e^{n_1+n_2-1}(a_0(k)-W_a(k)+P_{\gamma_1}(k))(b_0(q)-W_b(q)+P_{\gamma_2}(q)), \quad (17)$$

we must take for $P_{\Gamma}(k\cup q)$ a polynomial, which turns out to be determined by P_{γ_1} and P_{γ_2}:

$$P_{\Gamma}(k\cup q)=\hat{M}_{\Gamma}(a_0(k)-W_a(k)+P_{\gamma_1}(k))(b_0(q)-W_b(q)+P_{\gamma_2}(q)). \quad (18)$$

General conclusion is the following:

If we operate within the framework of pure subtractions (the "first" interpretation) we are — in general — not allowed to treat the subtraction points as independent variables. If we realize the renormalization procedure (the "second" interpretation), similar restrictions refer to the renormalization parameters[1].

APPENDIX

Calculation of $A_{\Gamma}(k\cup q)$

From (8) we have

$$\hat{R}_{\Gamma}C^{\varepsilon}_{\Gamma}(k\cup q)=e^{n_1+n_2-1}(\hat{1}-\hat{M}_{\Gamma})(\hat{1}-\hat{M}_{\gamma_1}-\hat{M}_{\gamma_2})\left(\frac{a_{-1}(k)}{\varepsilon}+a_0(k)\right)\times\left(\frac{b_{-1}(q)}{\varepsilon}+b_0(q)\right)$$

$$\stackrel{\text{df}}{=}e^{n_1+n_2-1}(\hat{1}-\hat{M}_{\Gamma})(\hat{1}-\hat{M}_{\gamma_1}-\hat{M}_{\gamma_2})\tilde{A}^{\varepsilon}_{\gamma_1}(k)\tilde{B}^{\varepsilon}_{\gamma_2}(q), \quad (A1)$$

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[1] Restrictions connected in our example with the separable diagrams in scalar theories are of the same kind as — for example — the Ward identities in gauge theories.

from which

$$\Lambda_\Gamma \sim -e^{n_1+n_2-1}\hat{M}_\Gamma(\hat{1}-\hat{M}_{\gamma_1}-\hat{M}_{\gamma_2})\tilde{A}^\varepsilon_{\gamma_1}(\boldsymbol{k})\tilde{B}^\varepsilon_{\gamma_2}(\boldsymbol{q}). \tag{A2}$$

(The normal product of suitable field operators is absent here.)

From (10′) we have

$$\hat{M}_\Gamma(\hat{1}-\hat{M}_\gamma)(1-\hat{M}_{\gamma_2})\tilde{A}\tilde{B} = 0, \tag{A3}$$

$$\hat{M}_\Gamma(\hat{1}-\hat{M}_{\gamma_1}-\hat{M}_{\gamma_2})\tilde{A}\tilde{B}+\hat{M}_\Gamma\hat{M}_{\gamma_1}\hat{M}_{\gamma_2}\tilde{A}\tilde{B} = 0. \tag{A4}$$

Finally

$$\begin{aligned}\Lambda_\Gamma &\sim e^{n_1+n_2-1}\hat{M}_\Gamma\hat{M}_{\gamma_1}\hat{M}_{\gamma_2}\tilde{A}\tilde{B}\\ &= e^{n_1+n_2-1}\hat{M}_\Gamma\left(\frac{a_{-1}(\boldsymbol{k})}{\varepsilon}+W_a(\boldsymbol{k})\right)\left(\frac{b_{-1}(\boldsymbol{q})}{\varepsilon}+W_b(\boldsymbol{q})\right)\\ &= e^{n_1+n_2-1}\left(\frac{a_{-1}(\boldsymbol{k})}{\varepsilon}+W_a(\boldsymbol{k})\right)\left(\frac{b_{-1}(\boldsymbol{q})}{\varepsilon}+W_b(\boldsymbol{q})\right)\end{aligned} \tag{A5}$$

REFERENCES

[1] N. N. Bogoliubov, D. V. Shirkov, *Vvedenije v teoriyu kvantovannykh polei*, Moskva 1973.
[2] D. Zavyalov, *Perenormirovannye diagrammy Feinmana*, Moskva 1979.
[3] W. Zimmermann, *Commun. Math. Phys.* **11**, 1 (1968).
[4] M. C. Bergere, J. B. Zuber, *Commun. Math. Phys.* **35**, 113 (1974).
[5] P. Węgrzyn, *Acta Phys. Pol.* **B20**, 13 (1989).