EXAMPLES OF NON-ABELIAN CONNECTIONS INDUCED IN ADIABATIC PROCESSES*

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Examples of non-Abelian adiabatic connections derived from 4×4 matrix Hamiltonians are presented. We find a multimonopole type non-Abelian connection.

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1. *Introduction*

Recently there is a revival of interest in adiabatic approximation in quantum mechanics and quantum field theory. It has been initiated by papers [1, 2], in which it has been noticed that a phase factor, which appears when solving Schrödinger equation in the adiabatic approximation cannot be ignored in some cases, in spite of a common practice [3]. It has turned out that the phase factor is essential in many problems, see, e.g. [4] for a review. In particular, one can regard gauge anomalies in field theory as a manifestation of Berry's phase [5].

In paper [1] the phase has been computed for a spin $\vec{s}$ interacting with a slowly changing external magnetic field $\vec{B}$ — the corresponding Hamiltonian is

$$H = \mu \vec{s}\vec{B}, \tag{1}$$

where $\vec{s}$ are the spin operators, $\vec{B} \neq 0$. This Hamiltonian has non-degenerate eigenvalues. In this case Berry's phase can be related to a non-trivial Abelian connection on the space of the adiabatic parameters $\vec{B}$.

In paper [6] it has been noticed that in the case of a Hamiltonian with degenerate eigenvalues a non-trivial, non-Abelian connection might appear. Examples of such non-Abelian connections have been presented in literature [7].

In the present paper we would like to give new examples of non-Abelian adiabatic

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connections. We consider 4×4 matrix Hamiltonians of the form

$$H = B_i^a \sigma^a \otimes \sigma^i, \tag{2a}$$

$$H^{(\pm)} = n^i(\sigma_0 \otimes \sigma^i \pm \sigma^i \otimes \sigma_0), \tag{2b}$$

where σ^a, σ^i, $a = 1, 2, 3$, are the Pauli matrices, σ_0 is the 2×2 unit matrix, and B_i^a, n^i are external, adiabatic parameters. Hamiltonian (2a) can be regarded as the Hamiltonian of a static, SU(2)-gauge quark (in the $A_0^a = 0$ gauge). In the following we shall consider the cases

$$B_i^a = \varepsilon_{aik} n_k, \qquad B_i^a = n_i \delta_{ai},$$

where n_k are external parameters. Hamiltonians (2b) can be regarded as the Hamiltonians of two spin 1/2 particles, with equal or opposite giromagnetic ratios e/m, interacting with an external magnetic field $B^i = \dfrac{mc}{e\hbar} n^i$.

The plan of our paper is the following. In Section 2 we recall the definition of the adiabatic connection, mainly in order to set a framework for subsequent computations and a discussion. In this Section we also generalize to the non-Abelian case some useful formulae for adiabatic curvature. Section 3 contains the examples of non-Abelian connections. We find rather interesting connections of a non-Abelian multimonopole ($m = -2$) type. In Section 4 we present comments and remarks. In this Section we point out that the results of our computations contradict some statements found in literature.

2. *The definition of the adiabatic connection*

Let $\{|a, \lambda\rangle\}$, $a = 1, \ldots, N$, be an orthonormal set of eigenvectors of a Hamiltonian H to an N-fold degenerate eigenvalue E,

$$H(\lambda)|a, \lambda\rangle = E(\lambda)|a, \lambda\rangle. \tag{3}$$

We assume that H depends on n continuous parameters $\lambda = (\lambda^i)$, $i = 1, \ldots, n$. Therefore E and the eigenvectors $|a, \lambda\rangle$ can also depend on λ. The eigenvectors $|a, \lambda\rangle$ span the eigenspace $\mathscr{H}_E$. If we change the parameters λ^i with time, i.e. $\lambda^i = \lambda^i(t)$, then E and $|a, \lambda\rangle$ are time dependent too. $(\lambda^i(t))$ can be regarded as a curve C on a manifold $\Lambda \equiv \{(\lambda^i)\}$ of the parameters λ^i. The manifold Λ by definition consists of all λ^i such that the energy level E is N-fold degenerate with fixed N. This implies that the level E does not cross any other energy levels for any $(\lambda^i) \in \Lambda$.

We would like to solve time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(\lambda(t))\, |\psi(t)\rangle \tag{4}$$

in the case of validity of the adiabatic approximation. Then, if

$$|\psi(t=0)\rangle \in \mathscr{H}_E, \quad \text{i.e.} \quad |\psi(t=0)\rangle = \sum_{a=1}^{N} c_a |a, \lambda(0)\rangle,$$

we can write

$$|\psi(t)\rangle = \sum_{a=1}^{N} c_a \exp\left[-\frac{i}{\hbar}\int_0^t E(t')dt'\right] U_{ab}(t)\,|b, \lambda(t)\rangle, \tag{5}$$

where the matrix $\hat{U} = [U_{ab}]$ is to be determined from the equation

$$(\hat{U}^{-1}\dot{\hat{U}})_{ba} = -\langle a, \lambda(t)|\frac{d}{dt}|b, \lambda(t)\rangle, \tag{6}$$

which follows from (4) and (5) after neglecting transitions to other energy levels. The initial condition for $\hat{U}(t)$ is $U_{ab}(0) = \delta_{ab}$. Equation (6) has the following solution

$$\hat{U}^{-1} = \mathrm{T}\exp\left[\int_0^t dt'\,\frac{d\lambda^i}{dt'}\,\hat{\Gamma}_i(\lambda(t))\right], \tag{7}$$

where T denotes the ordinary time ordering, and $\hat{\Gamma}_i = [\Gamma_i^{ab}]$,

$$\Gamma_i^{ab} \stackrel{\mathrm{df}}{=} \langle b, \lambda|\frac{\partial}{\partial\lambda^i}|a, \lambda\rangle. \tag{8}$$

$\hat{\Gamma}_i$ is the adiabatic connection. From the definition (8) it follows that $\hat{\Gamma}_i$ is anti-Hermitean, $N\times N$ matrix. Therefore $\hat{U}(t)$ is a $U(N)$ matrix.

If we unitarily change the basis eigenvectors $|a, \lambda\rangle$, i.e.

$$|\widetilde{b, \lambda}\rangle = \sum_{a=1}^{N} \Omega_{ba}(\lambda)\,|a, \lambda\rangle, \tag{9}$$

where $\hat{\Omega} = [\Omega_{ab}]$ is a $U(N)$ matrix, then it follows from formula (8) that

$$\tilde{\hat{\Gamma}}_i = \hat{\Omega}\hat{\Gamma}_i\hat{\Omega}^{-1} + \left(\frac{\partial}{\partial\lambda^i}\hat{\Omega}\right)\hat{\Omega}^{-1}, \tag{10}$$

where

$$\tilde{\Gamma}_i^{ab} = \langle\widetilde{b, \lambda}|\frac{\partial}{\partial\lambda^i}|\widetilde{a, \lambda}\rangle.$$

From formula (10) we see that in general $\hat{\Gamma}_i$ transforms like $U(N)$ non-Abelian gauge potential.

In the particular case of the lack of the degeneracy we have $N = 1$, and $\hat{\Gamma}_i$ becomes $U(1)$-type Abelian connection. This case was considered in [1, 2].

It is natural to ask whether the connection $\hat{\Gamma}_i$ can be trivialized (i.e. put to zero) by a unitary redefinition of the orthonormal basis $\{|a, \lambda\rangle\}$, i.e. by transformation (10). In this context it is useful to consider the curvature

$$\hat{F}_{ik}(\hat{\Gamma}) \stackrel{\mathrm{df}}{=} \partial_i\hat{\Gamma}_k - \partial_k\hat{\Gamma}_i - [\hat{\Gamma}_i, \hat{\Gamma}_k], \tag{11}$$

i.e.

$$F_{ik}^{ab} = \left(\frac{\partial}{\partial\lambda^i}\langle b,\lambda|\right)\frac{\partial}{\partial\lambda^k}|a,\lambda\rangle - \sum_c \langle c,\lambda|\frac{\partial}{\partial\lambda^i}|a,\lambda\rangle\langle b\lambda|\frac{\partial}{\partial\lambda^k}|c,\lambda\rangle - (i\leftrightarrow k). \tag{12}$$

$\hat{F}_{ik}(\hat{\Gamma})$ is an anti-Hermitean matrix. From formula (10) it follows that

$$\hat{F}_{ik}(\tilde{\hat{\Gamma}}) = \hat{\Omega}\hat{F}_{ik}(\hat{\Gamma})\hat{\Omega}^{-1}. \tag{13}$$

For $\tilde{\hat{\Gamma}}_i = 0$ we have $\hat{F}_{ik}(\tilde{\hat{\Gamma}}) = 0$. Therefore, the connection can be trivialized only if $\hat{F}_{ik}(\hat{\Gamma}) = 0$. The converse is not necessarily true. It can happen that $\hat{F}_{ik} = 0$, yet the connection cannot be trivialized because of non-trivial topology of the parameter manifold Λ.

Analysis of examples reveals that even for simple Hamiltonians one obtains, as a rule, a topologically non-trivial manifold Λ, and also non-trivial connections, e.g. belonging to non-trivial Chern classes. In this way algebraic topology and mathematical theory of connections find a rather unexpected application in quantum mechanics.

Let us introduce a complete set of orthonormal eigenstates of the Hamiltonian H in Hilbert space $\mathscr{H}$, $\dim\mathscr{H} < \infty$;

$$H|E_\alpha, a^{(\alpha)}\rangle = E_\alpha|E_\alpha, a^{(\alpha)}\rangle, \tag{14}$$

$$I = \sum_{\alpha a^{(\alpha)}} |E_\alpha, a^{(\alpha)}\rangle\langle E_\alpha, a^{(\alpha)}|. \tag{15}$$

Here α enumerates different eigenvalues of H, and $a^{(\alpha)} = 1, \ldots, N$ enumerates eigenstates belonging to the eigenvalue E_α. I is the identity operator in $\mathscr{H}$. It follows from definitions (11), (8) and formula (15) that

$$F_{ik}^{a^{(\alpha)}b^{(\alpha)}} = \left(\frac{\partial}{\partial\lambda^i}\langle E_\alpha, b^{(\alpha)}|\right)\sum_{\substack{\beta, c^{(\beta)}\\ \beta\neq\alpha}} |E_\beta, c^{(\beta)}\rangle\langle E_\beta, c^{(\beta)}|\frac{\partial}{\partial\lambda^k}|E_\alpha, a^{(\alpha)}\rangle - (i\leftrightarrow k). \tag{16}$$

Differentiating both sides of formula (14) with respect to λ^i and using the identity obtained in this manner on the r.h.s. of formula (16) we obtain another useful formula for the adiabatic curvature

$$F_{ik}^{a^{(\alpha)}b^{(\alpha)}} = \sum_{\substack{\beta, c^{(\beta)}\\ \beta\neq\alpha}} (E_\beta - E_\alpha)^{-2}\langle E_\alpha, b^{(\alpha)}|\frac{\partial H}{\partial\lambda^i}|E_\beta, c^{(\beta)}\rangle\langle E_\beta, c^{(\beta)}|\frac{\partial H}{\partial\lambda^k}|E_\alpha, a^{(\alpha)}\rangle - (i\leftrightarrow k). \tag{17}$$

Thus, we see that the well-known formula for $\hat{F}_{ik}$, proposed in [1] for the case without degeneracies, is valid also in the general case after the obvious modifications.

Using formula (16) we can prove that

$$\sum_{\alpha, a^{(\alpha)}} F_{ik}^{a^{(\alpha)}a^{(\alpha)}} = 0, \tag{18}$$

i.e. the sum of diagonal elements of all adiabatic curvatures for given Hamiltonian H vanishes. This formula generalizes the formula given in [8] for the Abelian case. Let us present the proof of formula (18). From (16) it follows that

$$\sum_{\alpha,a^{(\alpha)}} F_{ik}^{a^{(\alpha)}a^{(\alpha)}} = \sum_{\substack{\alpha,\beta \\ a^{(\alpha)},b^{(\beta)}}} \left(\frac{\partial}{\partial\lambda^i}\langle E_\alpha, a^{(\alpha)}|\right)|E_\beta, b^{(\beta)}\rangle\langle E_\beta, b^{(\beta)}|\frac{\partial}{\partial\lambda^k}|E_\alpha, a^{(\alpha)}\rangle-(i\leftrightarrow k)$$

$$= \tfrac{1}{2}\sum_{\substack{\alpha,\beta \\ a^{(\alpha)},b^{(\beta)}}}\left[\left(\frac{\partial}{\partial\lambda^i}\langle E_\alpha, a^{(\alpha)}|\right)|E_\beta, b^{(\beta)}\rangle\langle E_\beta, b^{(\beta)}|\frac{\partial}{\partial\lambda^k}|E_\alpha, a^{(\alpha)}\rangle\right.$$

$$\left.+\left(\frac{\partial}{\partial\lambda^i}\langle E_\beta, b^{(\beta)}|\right)|E_\alpha, a^{(\alpha)}\rangle\langle E_\alpha, a^{(\alpha)}|\frac{\partial}{\partial\lambda^k}|E_\beta, b^{(\beta)}\rangle\right]-(i\leftrightarrow k) = 0.$$

In the last step we have used twice the following identity

$$\left(\frac{\partial}{\partial\lambda^i}\langle E_\alpha, a^{(\alpha)}|\right)|E_\beta, b^{(\beta)}\rangle = -\langle E_\alpha, a^{(\alpha)}|\frac{\partial}{\partial\lambda^i}|E_\beta, b^{(\beta)}\rangle. \tag{19}$$

3. Examples of non-Abelian connections

Let us first consider the simple matrix Hamiltonian (2a). It can be regarded as a 4×4 matrix. The space Λ for Hamiltonian (2a) depends on choice of degree of degeneracy for the eigenvectors. We have many possibilities: no degeneracy; two degenerate levels, the other two non-degenerate levels; two pairs of degenerate levels; triple degenerate levels. The corresponding manifolds Λ are algebraic submanifolds of R^9, defined by algebraic relations between parameters β_i^a (following from the condition that certain eigenvalues are degenerate and the others are not). The situation is too complex to allow for a general analysis. For this reason we shall carry out analysis for Hamiltonian (2a) simplified by restricting B_i^a to some subspaces of R^9.

A. Let us consider as the first example

$$B_i^a = \varepsilon_{aik}n_k, \tag{20}$$

where n_k, $k = 1, 2, 3$, are the adiabatic parameters. Now the Hamiltonian can be written as

$$H = \begin{bmatrix} 0 & \omega^* & -\omega^* & 0 \\ \omega & 0 & 2\varrho & \omega^* \\ -\omega & -2\varrho & 0 & -\omega^* \\ 0 & \omega & -\omega & 0 \end{bmatrix}, \tag{21}$$

where $\omega = n_2 - in_1$, $\varrho = in_3$. It has the following eigenvalues:

$$E_1 = -2|\vec{n}|, \qquad E_2 = 0 \text{ (double degenerate)}, \qquad E_3 = 2|\vec{n}|. \tag{22}$$

We see that degree of degeneracy of the eigenvalues is constant for all $\vec{n}$ except for $\vec{n} = 0$. Thus, $\Lambda = R^3\setminus\{0\}$.

As the corresponding, orthonormal eigenvectors we will choose

$$|E_1\rangle = \frac{1}{2|\vec{n}|}\begin{bmatrix}\omega^* \\ -|\vec{n}|-\varrho \\ |\vec{n}|-\varrho \\ \omega\end{bmatrix}, \qquad (23) \qquad |E_2, 1\rangle = c_0\begin{bmatrix}-2\varrho \\ \omega \\ \omega \\ 0\end{bmatrix}, \qquad (24)$$

$$|E_2, 2\rangle = \frac{c_0}{|\vec{n}|}\begin{bmatrix}-\omega^{*2} \\ \varrho\omega^* \\ \varrho\omega^* \\ \vec{n}^2+n_3^2\end{bmatrix}, \qquad (25) \qquad |E_3\rangle = \frac{1}{2|\vec{n}|}\begin{bmatrix}\omega^* \\ |\vec{n}|-\varrho \\ -|\vec{n}|-\varrho \\ \omega\end{bmatrix}, \qquad (26)$$

where $c_0 = [2(\vec{n}^2+n_3^2)]^{-1/2}$. This choice of the eigenvectors is not a trivial one. The point is that in general explicit formulae for eigenvectors are valid only in some subsets of Λ. In many cases it is not possible to find for mulae forthe eigenvectors which are correct globally, on the whole parameter manifold Λ. Discussion of this problem can be found in [9]. In the spirit of paper [9] existence of the global system of eigenvectors (23)–(26) is due to the fact that

$$\pi_2(\mathrm{U}(4)/\mathrm{U}(2)) = 0.$$

It is easy to check that the adiabatic connections for non-degenerate eigenvalues E_1, E_3 vanish,

$$\Gamma_i^{(1)} = \langle E_1|\frac{\partial}{\partial n_i}|E_1\rangle = 0, \quad \Gamma_i^{(3)} = \langle E_3|\frac{\partial}{\partial n_i}|E_3\rangle = 0.$$

For the degenerate eigenvalue E_2 computation of the connection and curvature is rather tedious. Convenient starting point for computation of the curvature is formula (17), which in the case at hand reads

$$F_{ik}^{ab} = \sum_{\beta=1,3} E_\beta^{-2}\langle E_2 b|\frac{\partial H}{\partial n_i}|E_\beta\rangle\langle E_\beta|\frac{\partial H}{\partial n_k}|E_2, a\rangle - (i\leftrightarrow k). \qquad (27)$$

Because $\partial H/\partial n_k$ is a Hermitean matrix, it is sufficient to find

$$\langle E_2, 1|\frac{\partial H}{\partial n_k}|E_3\rangle = \sqrt{2}\, i(\vec{n}^2+n_3^2)^{-\frac{1}{2}}[n_3(\delta_k^2+i\delta_k^1)-(n_2+in_1)\delta_k^3],$$

$$\langle E_2, 1|\frac{\partial H}{\partial n_k}|E_1\rangle = -\langle E_2, 1|\frac{\partial H}{\partial n_k}|E_3\rangle,$$

$$\langle E_2, 2|\frac{\partial H}{\partial n_k}|E_3\rangle = \sqrt{2}\,|\vec{n}|^{-1}(\vec{n}^2+n_3^2)^{-\frac{1}{2}}[(n_1^2+n_3^2+in_1n_2)\delta_k^2$$

$$-i(n_2^2+n_3^2-in_1n_2)\delta_k^1+n_3(in_1-n_2)\delta_k^3],$$

$$\langle E_2, 2| \frac{\partial H}{\partial n_k} |E_1\rangle = -\langle E_2, 2| \frac{\partial H}{\partial n_k} |E_3\rangle. \tag{28}$$

These formulae follow directly from formulae (23)–(26). Using formulae (28) we obtain from (27) that

$$\hat{b}^s = n_s|\vec{n}|^{-2}(\vec{n}^2+n_3^2)^{-1}\begin{bmatrix} -2in_3 & |\vec{n}|^{-1}(n_2-in_1)^2 \\ -|\vec{n}|^{-1}(n_2+in_1)^2 & 2in_3 \end{bmatrix}, \tag{29}$$

where $\hat{b}^s$, $s = 1, 2, 3$, are defined by

$$(\hat{b}^s)^{ab} = -\tfrac{1}{2}\,\varepsilon_{iks}F_{ik}^{ab}, \tag{30}$$

i.e.

$$F_{ik}^{ab} = -\varepsilon_{iks}(\hat{b}^s)^{ab}.$$

Thus, the curvature is non-zero. From definition (8) we obtain that

$$\hat{\Gamma}_k = i(\vec{n}^2+n_3^2)^{-1}[\sigma^1|\vec{n}|^{-1}(n_1n_3\delta_k^1-n_2n_3\delta_k^2+(n_2^2-n_1^2)\delta_k^3)$$
$$+\sigma^2|\vec{n}|^{-1}(2n_1n_2\delta_k^3-n_2n_3\delta_k^1-n_1n_3\delta_k^2)+\sigma^3(n_1\delta_k^2-n_2\delta_k^1)]. \tag{31}$$

We see that $\hat{\Gamma}_k$ and $\hat{b}^s$ are regular everywhere except for $\vec{n} = 0$. At this point pattern of degeneracies of the Hamiltonian changes. For $|\vec{n}| \to \infty$ $\hat{b}^s$ behaves like $|\vec{n}|^{-2}$. Such behaviour is characteristic for a non-Abelian (SU(2)-type) magnetic monopole. It is well-known that the monopole-type non-Abelian magnetic fields $\hat{b}^s$ are divided into topological classes enumerated by an integer m, [10]. m is called the monopole number. For $m = 1$ ($m = -1$) we have the famous 't Hooft-Polyakov monopole (antimonopole), for $|m| > 1$ one speaks about multimonopoles.

It is interesting to find out to which monopole class belongs our $\hat{b}^s$ given by formula (29). To this end let us introduce a 2×2, Hermitean matrix $\hat{\beta}(\vec{n})$ and a vector $\vec{\beta} = (\beta^a)$, $a = 1, 2, 3$, defined by the formulae

$$\hat{b}^s = -in_s|\vec{n}|^{-3}\hat{\beta}(\vec{n}), \tag{32}$$

$$\hat{\beta} = \beta^a\sigma^a. \tag{33}$$

In the monopole case $\vec{\beta}^2 \to \text{const} \neq 0$ for $|\vec{n}| \to \infty$. Thus, vector $\vec{b}$ can be normalized to 1 (at least for sufficiently large $|\vec{n}|$). Therefore, we can pass to $\vec{e}(\vec{n}) = \vec{\beta}/|\vec{\beta}|$. We shall regard $\vec{e}(\vec{n})$ as a regular mapping from $S_n^2 = \{\vec{n} : |\vec{n}| = \text{const}\}$ into $S_e^2 = \{\vec{e} : |\vec{e}| = 1\}$. Such mappings are divided into homotopy classes enumerated by the integer $m[\vec{e}]$ given by the following formula

$$m[\vec{e}] = \frac{1}{4\pi}\int_{S_n^2} \varepsilon_{iks}e^i\,\frac{\partial e^k}{\partial\vartheta}\,\frac{\partial e^s}{\partial\varphi}\,d\vartheta d\varphi, \tag{34}$$

where (ϑ, φ) are spherical coordinates on the sphere S^2_n. The integer $m[\vec{e}]$ is just the monopole number.

In our case, $\hat{b}^s$ is given by formula (29) — the corresponding $\vec{e}(\vec{n})$ has the form

$$\vec{e}(\vec{n}) = (\vec{n}^2+n_3^2)^{-1}\begin{bmatrix} 2n_1n_2 \\ n_1^2-n_2^2 \\ 2n_3|\vec{n}| \end{bmatrix}. \tag{35}$$

In the spherical coordinates

$$n_1 = |\vec{n}|\sin\vartheta\cos\varphi, \quad n_2 = |\vec{n}|\sin\vartheta\sin\varphi, \quad n_3 = |\vec{n}|\cos\vartheta.$$

Inserting these formulae into the r.h.s. of formula (34) we obtain that in our case

$$m[\vec{e}] = -2. \tag{36}$$

This is a rather interesting result because monopole numbers $|m| \geqslant 2$ are in general hard to come by. For instance, construction of multimonopole solution of Yang-Mills-Higgs system is very complicated and it leads to extremely complex expressions for Yang--Mills and Higgs fields, [1]. In our case $m = -2$ comes out of a relatively simple computation.

Let us remark here that if we take $B_i^a = n_a n_i$ in (2a) then we shall obtain

$$H = \vec{n}\vec{\sigma} \otimes \vec{n}\vec{\sigma}, \tag{37}$$

which looks like a nice candidate for a next example. However, one can check that this Hamiltonian commutes with Hamiltonian (21). Therefore, the both Hamiltonians have common eigenvectors (23)–(26). Hamiltonian (37) has two eigenvalues $\tilde{E}_\pm = \pm\vec{n}^2$, both are doubly degenerate — now the eigenvectors $|E_2, 1\rangle$, $|E_2, 2\rangle$ belong to $\tilde{E}_+ = \vec{n}^2$, while $|E_1\rangle$, $|E_3\rangle$ belong to $\tilde{E}_- = -\vec{n}^2$. From formula (8) we see that adiabatic connection for the level $\tilde{E}_+$ is the same as for the level E_2 of Hamiltonian (21). For the level $\tilde{E}_-$ we now have to compute non-Abelian connection, because of the degeneracy. Thus, we have to compute two more matrix elements, i.e.

$$\tilde{\Gamma}_k^{12} = \langle E_1|\frac{\partial}{\partial n^k}|E_3\rangle, \quad \tilde{\Gamma}_k^{21} = \langle E_3|\frac{\partial}{\partial n^k}|E_1\rangle,$$

which have not been computed in the example A because there the levels E_1, E_3 were not degenerate. After a simple computation we find that

$$\tilde{\Gamma}_k^{12} = \tilde{\Gamma}_k^{21} = 0,$$

i.e. the connection vanishes.

B. As the second example let us take

$$B_i^a = n_a\delta_{ai} \tag{38}$$

(no summation over a), i.e.

$$H = \sum_{a=1}^{3} n_a \sigma^a \otimes \sigma^a. \tag{39}$$

After simple computations we obtain the following energy levels and eigenvectors

$$E_1 = n_1+n_2-n_3, \quad |E_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \quad E_2 = -n_1-n_2-n_3, \quad |E_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix},$$

$$E_3 = n_1+n_3-n_2, \quad |E_3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \quad E_4 = n_2+n_3-n_1, \quad |E_4\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}. \tag{40}$$

Because the eigenvectors do not depend on $\vec{n}$, the corresponding adiabatic connections vanish,

$$\Gamma_i = 0, \tag{41}$$

in spite of the fact that for some values of n_a the energy levels become degenerate.

The eigenvectors (40) do not depend on n_a because matrices $\sigma^1 \otimes \sigma^1$, $\sigma^2 \otimes \sigma^2$, $\sigma^3 \otimes \sigma^3$ commute with each other.

C. As the third example let us consider Hamiltonians (2b). These Hamiltonians do not commute with Hamiltonians (21), (39). It is easy to guess the eigenvectors; they have the form $|\pm\rangle \otimes |\pm\rangle$, where $|\pm\rangle$ are normalized eigenvectors of $\vec{n}\vec{\sigma}$, i.e.

$$\vec{n}\vec{\sigma}|\pm\rangle = \pm|\vec{n}|\,|\pm\rangle. \tag{42}$$

For the sake of completeness let us quote explicit formulae for $|+\rangle$, $|-\rangle$. We have to use two coordinate patches on $\Lambda = R^3\backslash\{0\}$ marked by I and II: $\Lambda^{\mathrm{I}} = \{\vec{n} \in \Lambda : n_3 \neq -|\vec{n}|\}$, $\Lambda^{\mathrm{II}} = \{\vec{n} \in \Lambda : n_3 \neq |\vec{n}|\}$. In Λ^{I} we have

$$|+\rangle^{\mathrm{I}} = [2|\vec{n}|\,(|\vec{n}|+n_3)]^{-\frac{1}{2}}\begin{bmatrix}n_3+|\vec{n}|\\ n_1+in_2\end{bmatrix}, \quad |-\rangle^{\mathrm{I}} = [2|\vec{n}|\,(|\vec{n}|+n_3)]^{-\frac{1}{2}}\begin{bmatrix}in_2-n_1\\ n_3+|\vec{n}|\end{bmatrix}, \tag{43}$$

while in Λ^{II}

$$|+\rangle^{\mathrm{II}} = [2|\vec{n}|\,(|\vec{n}|-n_3)]^{-\frac{1}{2}}\begin{bmatrix}n_1-in_2\\ |\vec{n}|-n_3\end{bmatrix}, \quad |-\rangle^{\mathrm{II}} = [2|\vec{n}|\,(|\vec{n}|-n_3)]^{-\frac{1}{2}}\begin{bmatrix}n_3-|\vec{n}|\\ n_1+in_2\end{bmatrix}. \tag{44}$$

It is easy to check that

$$|\pm\rangle^{\mathrm{II}} = e^{\pm i\lambda}|\pm\rangle^{\mathrm{I}}, \tag{45}$$

where

$$e^{i\chi} = (n_1-in_2)\,(n_1^2+n_2^2)^{-1/2}$$

is a phase factor.

More precisely, Hamiltonians $H^{(+)}$, $H^{(-)}$ have the following eigenvalues and eigenvectors.

$H^{(+)}$:

$$E_1^{(+)} = -2|\vec{n}|, \quad |E_1^{(+)}\rangle = |-\rangle \otimes |-\rangle;$$

$$E_2^{(+)} = 0, \quad |E_2^{(+)}, 1\rangle = |-\rangle \otimes |+\rangle, \quad |E_2^{(+)}, 2\rangle = |+\rangle \otimes |-\rangle;$$

$$E_3^{(+)} = 2|\vec{n}|, \quad |E_3^{(+)}\rangle = |+\rangle \otimes |+\rangle. \tag{46}$$

$H^{(-)}$:

$$E_1^{(-)} = -2|\vec{n}|, \quad |E_1^{(-)}\rangle = |-\rangle \otimes |+\rangle;$$

$$E_2^{(-)} = 0, \quad |E_2^{(-)}, 1\rangle = |+\rangle \otimes |+\rangle, \quad |E_2^{(-)}, 2\rangle = |-\rangle \otimes |-\rangle;$$

$$E_3^{(-)} = 2|\vec{n}|, \quad |E_3^{(-)}\rangle = |+\rangle \otimes |-\rangle. \tag{47}$$

The vectors $|\pm\rangle$ are defined locally, in the patches Λ^{I}, Λ^{II}. In spite of this, it follows from formulae (45) that $|E_2^{(+)}, 1\rangle$, $|E_2^{(+)}, 2\rangle$, $|E_1^{(-)}\rangle$, $|E_3^{(-)}\rangle$ are defined globally on the whole Λ.

After a simple computation based on definition (8) we find that the corresponding adiabatic connections vanish, i.e.

$$\Gamma_i = 0 \quad \text{for the levels} \quad E_{1,3}^{(-)}, E_2^{(+)}. \tag{48}$$

For the eigenvectors $|E_1^{(+)}\rangle$, $|E_3^{(+)}\rangle$ we have to use the patches. This follows from the fact that we only have at our disposal the freedom of choosing a single phase factor standing in front of these vectors — it is too little to cancel ill-defined at $n_3 = \pm|\vec{n}|$ phases occuring in some components of $|E_1^{(+)}\rangle$, $|E_3^{(+)}\rangle$ without introducing them in other components. Therefore, the adiabatic connection has to be computed on the patches. After simple computations we obtain the following results. For the level $E_1^{(+)}$

$$\Gamma_k^{\mathrm{I}} = 2 \quad {}^{\mathrm{I}}\langle -| \frac{\partial}{\partial n_k} |-\rangle^{\mathrm{I}} = -i\, \frac{n_1\delta_k^2 - n_2\delta_k^1}{|\vec{n}|\,(|\vec{n}|+n_3)},$$

$$\Gamma_k^{\mathrm{II}} = -i\, \frac{n_2\delta_k^1 - n_1\delta_k^2}{|\vec{n}|\,(|\vec{n}|-n_3)}, \tag{49}$$

and the corresponding curvature

$$b^s = i\, \frac{n_s}{|\vec{n}|^3} \tag{50}$$

for the both patches.

For the level $E_3^{(+)}$ the connection and curvature differ from the ones given by (49), (50) only by the overall sign, i.e. one should replace i by $-i$. Thus, in these cases the adiabatic curvatures coincide with the Dirac magnetic monopole of the strength ± 2.

Here we choose as the unit Dirac monopole $b^s = -\frac{i}{2}\frac{n_s}{|\vec{n}|^3}$ — such a monopole appears as the adiabatic curvature for Hamiltonian $H = \vec{n}\vec{\sigma}$ considered in [1].

Now let us analyse the level $E_2^{(-)}$. The eigenstates $|E_2^{(-)}, 1\rangle$, $|E_2^{(-)}, 2\rangle$ are defined locally, on the patches. However, it is possible to choose linear combinations of them which are regular everywhere except for $\vec{n} = 0$. Moreover, quite unexpectedly, the new eigenvectors coincide with eigenvectors (24), (25). One can check by an explicit computation that eigenvectors (24), (25) of Hamiltonian (21) are also eigenvectors of $H^{(-)}$, while eigenvectors (23), (26) of Hamiltonian (21) are not eigenvectors of $H^{(-)}$. As an example let us present one of the appropriate linear combinations:

$$|E_2, 1\rangle = -i(|\vec{n}|+n_3)^{-1}[2(\vec{n}^2+n_3^2)]^{-1/2}\{(n_3+|\vec{n}|)^2|+\rangle^{\mathrm{I}} \otimes |+\rangle^{\mathrm{I}}+\omega^2|-\rangle^{\mathrm{I}} \otimes |-\rangle^{\mathrm{I}}\}.$$

From formula (8) it is clear that adiabatic connection is determined by the eigenvectors and the degree of degeneracy — for the levels E_2 and $E_2^{(-)}$ both are identical. Therefore, the adiabatic connection is again the $m = -2$ non-Abelian magnetic monopole found in the example A.

Finally, we would like to remark that $H^{(+)}$ commutes with $H^{(-)}$. As one can see from (46), (47) both operators have all eigenvectors in common. Nevertheless, the adiabatic connections are different for both operators — for $H^{(+)}$ we have the two Dirac monopoles, while for $H^{(-)}$ we have the $m = -2$ non-Abelian multimonopole. This example clearly shows that the adiabatic connection depends not only on eigenstates but also on dimension of eigenspace to which the eigenstates belong.

4. *Conclusions*

The main goal of the present paper is to provide examples of non-Abelian adiabatic connections. We have produced non-trivial monopole-type adiabatic connections.

One may ask about physical relevance of our examples. At the moment we do not know any direct physical application. However, a possibility of a physical application is not excluded. Our 4×4 matrix Hamiltonians can be regarded as particular cases of a general Hamiltonian for a four-level system; it is obvious that any Hamiltonian for such a system can be represented as a 4×4, Hermitean matrix. Four-level physical systems are in abundance, e.g. two static spin 1/2 particles, a static nucleus of spin 3/2, a nucleon with its spin and izospin degrees of freedom. Such a system would provide a physical manifestation of magnetic monopoles in the form of adiabatic connections. In this sense magnetic monopoles do exist in Nature.

The particular case of Hamiltonian (39) with $n_1 = n_2 = n_3 = v$ has been considered in paper [12], with the result that the adiabatic connection is non-zero, in contradiction with the result of our straightforward computation. The method of the computation utilised in [12] is indirect. Our opinion is that the non-zero connection obtained in [12] is a result of a singular (incorrect) choice of the basis of eigenvectors of the Hamiltonian.

Finally, we would like to make a comment on a topological analysis of Berry's phase which is attempted in paper [9]. In that paper the existence of the phase is associated with the impossibility of finding a global, smooth over the parameter manifold Λ, system of eigenvectors of Hamiltonian. Such a situation happens in our example C; in the cases $E_{1,3}^{(+)}$ no global systems of eigenvectors exist. However, in the example A we have the global, smooth system of the eigenvectors, yet there is the non-trivial ($m = -2$) adiabatic connection. Therefore, we think that the topological classification presented in [9] is not complete.

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REFERENCES

[1] M. V. Berry, *Proc. R. Soc. London* **A392**, 45 (1984).

[2] B. Simon, *Phys. Rev. Lett.* **51**, 2167 (1983).

[3] L. I. Schiff, *Quantum Mechanics*, McGraw-Hill, Inc. 1968.

[4] I. J. R. Aitchison, University of Oxford preprint 63/87; R. Jackiw, MIT preprint CTP-1475, May 1987.

[5] A. J. Niemi, G. W. Semenoff, *Phys. Rev. Lett.* **55**, 927 (1985).

[6] F. Wilczek, A. Zee, *Phys. Rev. Lett.* **52**, 2111 (1984).

[7] J. Anandan, L. Stodolsky, *Phys. Rev.* **D35**, 2597 (1987).

[8] G. Giavarini et al., CERN preprint TH 5140/88, August 1988.

[9] E. Kiritsis, *Comm. Math. Phys.* **111**, 417 (1987).

[10] P. Goddard, D. I. Olive, *Rep. Prog. Phys.* **41**, 1357 (1978)

[11] R. S. Ward, *Comm. Math. Phys.* **79**, 317 (1981).

[12] H.-Z. Li, *Phys. Rev.* **D35**, 2615 (1987).