THERMODYNAMICS OF NONINTERACTING SUPERSTRINGS

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The gas of noninteracting strings is a very crude approximation to the description to a multistring system. This approximation has been used to discuss string models of the big bang and of the last stages of black hole evaporation. We review the thermodynamics of the gas of noninteracting strings, which are an unusual thermodynamic system, because the density of states instead of growing as a power of energy explodes exponentially.

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1. Introduction

String theory is actually the candidate for a theory of everything. There are many versions of string theory, however. The early versions had striking defects. Bosonic strings had ground states with imaginary mass (tachyons). Fermionic strings had not enough internal quantum numbers to describe known particles. The first string theory, which avoids these two difficulties, was the theory of heterotic superstrings [1, 2]. The arguments in the present paper apply to a wide class of string models, all the numerical estimates, however, have been made for heterotic superstrings. Thus, where not stated explicitly otherwise, we mean by string a heterotic superstring.

It is hoped that string theory contains the quantum theory of gravity. Since no one has succeeded in building a satisfactory quantum theory of gravity starting from more standard field theories, this is a very attractive possibility. Simple dimensional arguments indicate that quantum gravitational effects, if existing, can be observable only in very strong gravitational fields. Using simplified versions of string models people have studied the Universe soon after the big bang [3–5] and black holes towards the end of their evaporation [6, 7].

The common approach has been to reduce the dynamics of strings to that of an ideal gas (with suitable quantum statistics) enclosed in a classical vessel of volume V. In this approximation the admittedly unusual features of this gas result only from the fact that the degeneracy of the energy levels in the spectrum of each string grows exponentially

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with the string mass. Therefore, the description becomes very similar to that of the Hagedorn gas [8, 9] and classical papers on the thermodynamics of the Hagedorn gas [10, 11] have been much used. Nevertheless, two important differences should be stressed. First, the masses of excited strings are of the order of the Planck mass, while the masses of Hagedorn's fireballs are in the GeV range. Thus, gravitational collapse of the string gas is a new possibility. Secondly, for the Hagedorn gas only the asymptotic form of the particle spectrum was known, and even there some parameters were controversial. For strings, on the other hand, given a string model the spectrum is known from ground state to arbitrary high masses.

In the present paper we discuss the thermodynamics of noninteracting strings, as well as some limitations of this model. As opposed to earlier papers by other authors on this subject (cf. e.g. [12–14]) we consider the exact discrete string spectrum and not just the asymptotic high mass approximation. This makes it possible to answer quantitatively a number of new questions and to use elementary thermodynamic arguments instead of more complicated estimates of phase space integrals (cf. [15–17]).

2. *Energy spectrum of a heterotic string*

The mass levels of a single heterotic string are given by the formula [2]:

$$M_N = 2\sqrt{\frac{N}{\alpha'}} \qquad N = 0, 1, 2. \dots . \tag{1}$$

In the following we choose the mass unit so that the string constant $\alpha' = 1$, which is believed to correspond to a mass unit of the order of the Planck mass. Then formula (1) simplifies to

$$M_N = 2\sqrt{N} \qquad N = 0, 1, 2, \dots . \tag{2}$$

The degeneracy of the N-th mass level is [2]

$$d(N) = 16 P_F(N) P_{B'}(N), \tag{3}$$

where the factors on the right hand side can be obtained from suitable generating functions:

$$\sum_{N=0}^{\infty} P_F(N) x^N = \prod_{N=1}^{\infty} \left(\frac{1+x^N}{1-x^N}\right)^8, \tag{4}$$

$$\sum_{N=0}^{\infty} P_{B'}(N) x^N = \left[\left(1+480\sum_{m=1}^{\infty} \sigma_7(m) x^m\right) \prod_{n=1}^{\infty} \frac{1}{(1+x^n)^{24}} - 1\right] \frac{1}{x}. \tag{5}$$

In the last formula $\sigma_7(m)$ denotes the sum of the seventh powers of all the divisors of m including m and 1. We have calculated numerically the degeneracies $d(N)$ for the first 100 mass levels of the heterotic string. Their logarithms exact to one part in thousand are given in Table I.

TABLE I

Logarithms of the degeneracies of the first hundred mass levels of the heterotic string

N	$\ln d(N)$	N	$\ln d(N)$	N	$\ln d(N)$	N	$\ln d(N)$	N	$\ln d(N)$
0	8.995	20	76.671	40	112.755	60	141.087	80	165.253
1	16.754	21	78.784	41	114.309	61	142.378	81	166.381
2	22.549	22	80.854	42	115.846	62	143.659	82	167.504
3	27.458	23	82.882	43	117.367	63	144.930	83	168.620
4	31.814	24	84.872	44	118.871	64	146.192	84	169.729
5	35.871	25	86.824	45	120.359	65	147.444	85	170.832
6	39.452	26	88.742	46	121.832	66	148.688	86	171.930
7	42.889	27	90.627	47	123.291	67	149.923	87	173.021
8	46.133	28	92.481	48	124.735	68	151.149	88	174.106
9	49.215	29	94.305	49	126.165	69	152.367	89	175.186
10	52.158	30	96.101	50	127.582	70	153.576	90	176.259
11	54.981	31	97.870	51	128.968	71	154.778	91	177.327
12	57.698	32	99.614	52	130.377	72	155.971	92	178.390
13	60.319	33	101.332	53	131.756	73	157.157	93	179.447
14	62.856	34	103.028	54	133.123	74	158.335	94	180.449
15	65.316	35	104.700	55	134.478	75	159.506	95	181.545
16	67.706	36	106.351	56	125.821	76	160.669	96	182.587
17	70.031	37	107.982	57	137.154	77	161.826	97	183.623
18	72.296	38	109.592	58	138.475	78	162.975	98	184.654
19	74.509	39	111.183	59	139.768	79	164.117	99	185.680

For high masses the degeneracies can be computed from the asymptotic formula

$$d(N) \approx 2^{-13/4} N^{-11/2} e^{2\pi(2+\sqrt{2})\sqrt{N}}. \tag{6}$$

Using relation (2) to eliminate N from this formula one obtains equivalently

$$d(M_N) \approx 2^{31/4} M_N^{-11} e^{\frac{M_N}{T_H}}. \tag{7}$$

This equation defines the Hagedorn temperature

$$T_{\mathrm{H}} = \frac{1}{\pi(2+\sqrt{2})}. \tag{8}$$

From formula (7) one obtains the asymptotic density of levels

$$\varrho(M) = d(M)\frac{dN}{dM} = 2^{27/4} M^{-10} e^{\frac{M}{T_H}}. \tag{9}$$

We shall not use this density, but it is useful for comparison with the work of other authors.

Approximation (6) does not converge very rapidly. For $N = 100$ it gives $\ln(d(N)) = 186.940$, while the value obtained from the exact formula is 186.701. Actually, as pointed out in Ref. [18], only the constant factor in front of the formula depends on the details of the model. For closed strings the constant in the exponent depends only on the central charge of the Virasoro algebra of the model and the power of N is $-\frac{D+1}{2}$, where D is the space-time dimension.

The constant can be calculated using the method introduced in [19]. Using the definition of the θ_4 function

$$\theta_4(0, z) = \prod_{n=1}^{\infty} \frac{1-z^n}{1+z^n} \tag{10}$$

one can write

$$d_F(N) = \frac{1}{2\pi i} \oint \frac{dz}{z^{N+1}\theta_4^8(0, z)}\,. \tag{11}$$

The integration contour is around the point $z = 0$. The integrand has a saddle point for z real and just below one. For large values of N one can use the steepest descent method with the approximation

$$\theta_4(0, z) \approx 2\sqrt{-\frac{\pi}{\ln z}}\, e^{-\frac{\pi^2}{4\ln z}}\,. \tag{12}$$

The standard calculation yields for high N

$$P_F(N) \approx \tfrac{1}{16}(2N)^{-11/4} e^{2\pi\sqrt{2N}}. \tag{13}$$

For the other factor in (3) one finds first, calculating as above [19],

$$P_B(N) = \frac{1}{2\pi i} \oint \prod_{n=1}^{\infty} \frac{dz}{(1-z^n)^{24}} \approx \frac{1}{\sqrt{2}} N^{-27/4} e^{4\pi\sqrt{N}} \tag{14}$$

then

$$P_{B'}(N) \approx P_B(N) + \sum_{m=1}^{N} 480\sigma_7(m) P_B(N-m). \tag{15}$$

Replacing the sum by an integral one sees that it is dominated by the contributions from small m. Expanding the integrand in powers of $\frac{m}{N}$ and keeping only the leading terms one finds

$$P_{B'}(N) \approx \frac{1}{\sqrt{2}} \left[480 \frac{\zeta(8)7!}{(2\pi)^8}\right] N^{-11/4} e^{4\pi\sqrt{N}}, \tag{16}$$

where according to the general definition of the ζ function

$$\zeta(8) = \sum_{n=1}^{\infty} \frac{1}{n^8} . \tag{17}$$

The term in the square bracket exactly equals one, therefore combining (13) and (16) one finds the result (6).

3. *Definition of temperature*

The simplest definition of temperature uses the notion of a heat reservoir. The system has temperature T, when it can be in equilibrium with respect to heat exchange with a large heat reservoir of temperature T. The heat reservoir is at temperature T, when it can be in equilibrium with respect to heat exchange with a thermometer at temperature T. The temperature of the thermometer is by definition possible to read off. E.g. in the gas thermometer the pressure of the gas, or its volume, is a known function of the temperature of the thermometer. This definition of the temperature assumes that it is possible to put the system in equilibrium with respect to heat exchange with some heat reservoir. For strings this is not always the case.

If the system is at temperature T, then the probability of finding it in a quantum state Q of energy E is given by the canonical distribution

$$P(Q) = \frac{1}{Z} e^{-\frac{E}{T}} , \tag{18}$$

where the partition function

$$Z = \sum_{Q} e^{-\frac{E(Q)}{T}} . \tag{19}$$

Here and in the following the temperature scale is chosen so that the Boltzmann constant $k = 1$. Denoting by $g(E)$ the degeneracy of the state with energy E we can rewrite the definition (19) as

$$Z = \sum_{E} g(E) e^{-\frac{E}{T}} . \tag{20}$$

Similarly, the probability that the system has energy E is

$$P(E) = \frac{1}{Z} g(E) e^{-\frac{E}{T}} . \tag{21}$$

Usually the degeneracy $g(E)$ is an increasing function of E, but it increases more slowly than exponentially, so that the exponential dumping factor on the right hand side of equation (21) is sufficient to make the sum convergent for arbitrary temperature T. As seen from formula (7), however, the degeneracy $g(E)$ for a string increases exponentially with E.

The sum converges only for temperatures $T < T_H$. For higher temperatures of the heat bath, the string keeps absorbing energy and never reaches an equilibrium state. Thus T_H is the highest temperature, where the string can be in equilibrium with the heat bath. Consequently temperatures higher than T_H cannot be defined for the string using the heat bath picture.

The statement made sometimes that the Hagedorn temperature T_H is the highest possible temperature for the string refers to the definition of temperature given above. It is possible to give another definition, however, which is also often used in standard thermodynamics. The entropy of a string at mass M is well defined by Boltzmann's formula

$$S(M) = \ln(d(M)). \tag{22}$$

The standard thermodynamic formula is

$$\frac{1}{T} = \frac{\partial S}{\partial M}, \tag{23}$$

where the increases of M and S correspond to a process, in which no external work is done. For strings the mass is quantized. This has two implications. Firstly, the derivative in the definition (23) must be replaced by a ratio of finite differences. Secondly, temperatures are ascribed to transitions between adjacent mass states and not to states themselves. Thus for a state two temperatures can be relevant: one for the transition increasing its mass, the other to the transition decreasing its mass.

Denoting the temperature corresponding to the transition between states N and $N+1$ by $T(N+\frac{1}{2})$, we obtain using formula (2) and the numbers from Table I: $T(\frac{1}{2}) = 0.2578$, $T(1\frac{1}{2}) = 0.1430$, $T(2\frac{1}{2}) = 0.1295$ etc. For large masses, where the mass spectrum is practically continuous, one finds taking the logarithmic derivative of the degeneracy (7) with respect to the string mass

$$T = \frac{T_H}{1 - \frac{11 T_H}{M}}. \tag{24}$$

It is seen that the temperature of the string is always higher than the Hagedorn temperature. With increasing mass it decreases monotonically from $T(\frac{1}{2})$ to the limiting value $T_H \approx 0.0932$. This implies in particular that, if by pumping energy into the string it is possible to reach a phase transition at high mass, the transition temperature must be close to Hagedorn's temperature. A phase transition is expected for two reasons. For the hadron gas, which has been the first dystem known to exhibit a Hagedorn temperature [8], there is the transition of hadrons into the quark-gluon plasma (the deconfinement transition) which has been interpreted as the transition at the Hagedorn temperature [20]. Even if there is no phase transition due to nongravitational interactions, strings are so heavy that gravitational effects, collapse or something more complicated [21] must be included in the analysis.

The definition of temperature (23) and the principle of entropy increase for isolated systems imply that heat flows spontaneously from higher to lower temperature and cannot flow in the opposite direction. Since the string temperature always exceeds T_H, an excited string cannot be in equilibrium with a large heat reservoir at temperature lower than T_H. This is a much simpler derivation of the result deduced before from the discussion of the partition function. A string in its ground state (massless) can be in equilibrium with a heat reservoir at lower temperature, because it has no more energy to loose. As discussed further an excited string can be in equilibrium with a sufficiently small heat reservoir whatever its temperature.

4. The string gas as a two temperature system

For a single string at rest the energy supplied is used to excite the string to higher mass. This changes both the energy and the entropy of the system (string) and the corresponding temperature has been discussed in the preceding section. For a string gas, however, it is also possible to pump energy into the translational motion of the strings. The temperature again can be calculated from formula (23), but the result in general will be different from the string temperature calculated before. Thus the string gas is a two-temperature system. In physics there are many examples of two-temperature systems. In a plasma the temperature of positive ions is usually different from the temperature of the free electrons. In a crystal there can be a temperature of the spins which is different from the temperature of the lattice. In the Universe the temperature of the electromagnetic background radiation is believed to be different from the temperature of the neutrinos. In daily life a system consisting of two cups of coffee, one hot the other cold, is a two-temperature system. In general the system has two temperatures, when there are two ways of supplying energy to the system, with different entropy changes per unit supplied energy. When it will be necessary to distinguish explicitly the two temperatures of the string, we will denote by T_m the temperature corresponding to the mass excitations and by T_k the temperature corresponding to the kinetic energy of the translational motion.

The coupling of the mass excitations to the translational motion is in many ways similar to the coupling of a string to a heat reservoir described in the preceding section. Let us consider one massive string, or more, in a string gas at temperature $T_k < T_H$. Energy will flow from the mass excitations (i.e. one or more of the string masses will decrease) to the translational motion of the gas and the temperature T_k will grow. One of the following two things must happen. Either the massive strings will become massless before the temperature T_k reaches the Hagedorn temperature T_H, or T_k reaches T_H before the strings become massless. In the first case we obtain a gas of light, mostly massless, strings at a temperature below T_H. In the second case there may be an equilibrium at a temperature above T_H. Let us note that according to formula (22) the sum of entropies of two massive strings with masses M_1 and M_2 is lower than the entropy of one string of mass M_1+M_2. Therefore, in equilibrium all the heavy strings will have given their mass excitation energies to just one heavy string. We will refer to the gas of light strings as the L-phase, or low temperature phase, and to the phase where one heavy string is in equilibrium with a gas of light strings, as the H-phase, or high temperature phase. The thermodynamics of the L-phase will be discussed in Section 6 and the thermodynamics of the H-phase in Section 7. First, however, we must discuss the stability condition for the string gas.

5. *Stability condition*

It is a well-known implication of the second principle of thermodynamics that under usual conditions a system in order to be in stable equilibrium must have nonnegative heat capacity

$$C = \frac{\partial E}{\partial T}, \tag{25}$$

i.e. its temperature must increase, when its energy increases. As seen from Table I and from the asymptotic formula (23) the string is an exception in that it has a negative heat capacity. There is nothing unphysical about it. For instance stars, which keep loosing energy by radiation, have in certain stages of their evolution temperatures rising in time, which means according to (25) a negative heat capacity. For a black hole of mass M its temperature, known as the Hawking temperature, is given by the formula [22, 23]

$$T = \frac{1}{8\pi M}. \tag{26}$$

This is a decreasing function of energy (mass) and consequently again the heat capacity is negative. Nevertheless, many of the unusual features of the string gas follow from the observation that, when energy is pumped into the mass of a string, its temperature T_m decreases, so that it becomes easier to put in more energy instead of more difficult as for usual systems.

Let us recall the derivation of the condition for equilibrium with respect to heat transfer. We will see which condition necessary to prove that the heat capacity is positive is not satisfied for strings and how the equilibrium condition is modified when a negative heat capacity occurs. Let us consider a system split into two subsystems A and B. It is not necessary for the two subsystems to be separated in space. It is suficient that energies and entropies are well defined for A and B separately. For instance for strings one may choose as A the mass excitation and as B the translational motion of the strings. Consider a small transfer of energy as heat ΔE from A to B. It is assumed that no work is done on the system in the process. The system is in equilibrium, only if the total entropy of A+B does not increase whatever the sign of ΔE. Using the definitions of temperature and heat capacity one writes the equilibrium condition

$$\Delta S_{\text{tot}} = \left(\frac{1}{T_B} - \frac{1}{T_A}\right)\Delta E + \tfrac{1}{2}\left(\frac{1}{T_A^2 C_A} + \frac{1}{T_B^2 C_B}\right)(\Delta E)^2 \leqslant 0. \tag{27}$$

The assumptions implicit here are that the entropy of A+B is the sum of the entropies of A and B and that the energy of A+B is the sum of the energies of A and B. The second of these assumptions implies because of energy conservation (first principle of thermodynamics) that the energy decrease of subsystem A equals in absolute value the energy increase of subsystem B. The linear term in (27) cannot be positive definite, therefore, it must vanish, which yields $T_A = T_B$. The quadratic term yields the equilibrium condition

$$\frac{1}{C_A} + \frac{1}{C_B} \geqslant 0. \tag{28}$$

This is obviously satisfied, when both heat capacities are positive and obviously not satisfied, when they are both negative. The case, when one is positive and the other negative requires a separate discussion.

The usual argument is that as subsystems A and B one can choose two halves of the total system. Then the two subsystems are identical, which excludes heat capacities differing in sign, and the positivity of the heat capacity follows. This argument does not apply to strings, because the entropy of a string of mass $2M$ is not equal to twice the entropy of a string of mass M. An analogous situation holds for black holes. For a different reason the argument does not apply to stars. There, because of long range forces the energy of the system is not equal to the sum of energies of its halves. Thus, for strings, black holes and stars in certain stages of their evolution negative heat capacities are possible and in fact are realized. Condition (28) means that in the equilibrium between a system with negative heat capacity and a system with positive heat capacity the positive heat capacity must be smaller than the absolute value of the negative heat capacity. This has been pointed out long ago for black holes [23], where it implies that if a black hole is in equilibrium with radiation the amount of radiation, which is the subsystem with positive heat capacity, cannot be too large. For example, if the black hole and the radiation are contained in a box at given temperature, equilibrium is possible only when the volume of the box is smaller than some critical volume V_c. Note the difference with usual systems, where if the systems is in equilibrium with respect to heat transfer (which just means has equal temperature as) a cubic centimeter of water, it will surely be in equilibrium just as well with a cubic mile of water at the same temperature. Also for the H-phase of strings one finds at each temperature a maximal volume V_c (cf. e.g. [6]).

6. *Low temperature phase*

We shall see that the string gas at temperatures below the Hagedorn temperature is dominated by massless strings. Therefore, it is useful to begin by considering the subsystem consisting of all the massless strings in the system. The chemical potential is zero, because strings can be reversibly produced and destroyed. The thermodynamics is similar to that of the photon gas, except that instead of the two polarization states of the photon there are 4032 bosonic states plus 4032 fermionic states and that the space is nine dimensional.

The logarithm of the partition function is given by

$$\ln Z_0 = \frac{V}{(2\pi)^9} 4032 \int d^9p \ln \frac{1+e^{-\frac{p}{T}}}{1-e^{-\frac{p}{T}}}. \tag{29}$$

The integral can be done by changing to hyperspherical coordinates, expanding the logarithm in powers of the exponential and using formulae for ζ and Γ functions. The result is

$$\ln Z_0 = \tfrac{31}{15}\pi^5 V T^9. \tag{30}$$

This yields the free energy F_0, the energy E_0 and the heat capacity at constant volume C_{V0}

$$F_0 = -T \ln Z_0 = -\tfrac{31}{15}\pi^5 V T^{10}, \tag{31}$$

$$E_0 = F_0 - T\frac{\partial F_0}{\partial T} = -9F_0, \tag{32}$$

$$C_{V0} = \frac{\partial E_0}{\partial T} = -\frac{90}{T} F_0. \tag{33}$$

Let us consider the validity limits of this calculation. It has been assumed that the gas is contained in a classical box of volume V and that there is one translational state per each $\hbar^9$ of phase space volume. This is the quasiclassical approximation valid only when the size of the box R is much larger than the inverse temperature (in units $\hbar = c = 1$). For definiteness we will assume that the box is a sphere of radius R. Then it is necessary that

$$R \gg T_H^{-1} \approx 10.7. \tag{34}$$

Without the quasiclassical approximation the calculation could in principle be done, but the results would depend on the exact shape of the box and on the boundary conditions on the walls. Thus they would not be very useful. In nine dimensions the volume of a sphere is related to the radius by the formula

$$V \approx 3.30R^9, \tag{35}$$

Thus condition (34) implies

$$V \gg 6.2\times 10^9. \tag{36}$$

By standards of particle physics, not to mention astrophysics, this condition is not very restrictive, because the unit of volume is $\alpha'^{-9/2}$, which is believed to be of the order of the 9-th power of the Planck length (1.6×10^{-33} cm).

Another limitation is implied by the use of standard thermodynamics, where both the temperature and the energy are simultaneously well defined. Thus, the fluctuations of energy must be much smaller than the energy itself. The mean square fluctuation of energy is T^2C_{V0}, which is small compared to E^2 when

$$V \gg 3.3\times 10^6 \tag{37}$$

i.e. under a condition much weaker than the previous one.

For massive strings one can use the nonrelativistic approximation, because the minimum mass $M_1 = 2$ exceeds by a factor of more than twenty the maximum temperature $T_H = 0.093$. The exact calculation can be done, but it leads to complicated formulae and no more physical insight. The logarithm of the partition function for the subsystem consisting of all the strings of mass M corresponding to the n-th mass level of the string is

$$\ln Z_n = d(M) \frac{V}{(2\pi)^9} e^{-\frac{M}{T}} \int d^9 p e^{-\frac{p^2}{2MT}}. \tag{38}$$

This corresponds to the free energy

$$F_n = -d(M) V (2\pi)^{-\frac{9}{2}} M^{\frac{9}{2}} T^{\frac{11}{2}} e^{-\frac{M}{T}}. \tag{39}$$

For high masses, when the asymptotic approximation to $d(M)$ can be used, this reduces to

$$F_n \approx -2^{-\frac{13}{4}} \pi^{-\frac{9}{2}} V n^{-\frac{13}{4}} T^{\frac{11}{2}} e^{M\left(\frac{1}{T_H} - \frac{1}{T}\right)}. \tag{40}$$

It is seen that the summation over n, necessary to calculate the total free energy, converges rapidly for temperatures not exceeding the Hagedorn temperature T_H. The corrections to the thermodynamic functions of the gas of massless strings are small [17]: about 0.5 per cent for the free energy, about 2 per cent for the energy and about 6 per cent for the heat capacity. The number of strings in the n-th mass state, where $n \geqslant 1$, is about $3 \times 10^{-8}\, V n^{-13/4}$. Thus, it is a very good approximation to interpret the string gas at temperatures below the Hagedorn temperature as a gas of massless strings.

An attempt has been made by Glaser and Taylor [5] to extend a description similar to that for the low temperature phase to all temperatures. These authors point out that the interaction between strings introduces a finite width for each of the massive states. Further, they assume that this width increases with M and at some $M = M_0$ becomes so large, that there is no point in considering higher mass states as bound states. Formally this means that the partition function is obtained by summing over string masses to M_0 instead of to infinity. The assumption about the increasing width is made plausible by analogy with resonances built from light quarks, which indeed tend to become broader, when their mass increases. It is certainly not a general theorem, however, because for bosonic strings it had been demonstrated that the width decreases with increasing mass [24]. Consider the string gas in equilibrium with a heat reservoir at a temperature $T \gg T_H$. Accepting the assumption of Glaser and Taylor, one concludes that since $T \gg T_m$ and since energy flows spontaneously from higher to lower temperature, almost all the strings are in their highest allowed mass states i.e. have $M = M_0$. Glaser and Taylor suggest that

$$M_0 = \frac{1}{\gamma} \ln T, \tag{41}$$

where we have absorbed the slowly varying factor $\ln M_0$ into the constant γ. Thus, at high temperatures M_0 is very large and one can use the crude approximation

$$d(M_0) \approx Ce^{\frac{M_0}{T_H}} \approx CT^{\frac{1}{\gamma T_H}}, \tag{42}$$

where C is a constant. The free energy of an ordinary gas of massless particles in D space dimensions $F \sim VT^{D+1}$, because it must be proportional to the volume (extensive) and have the dimension of energy. The free energy of the gas proposed by Glaser and Taylor has the additional factor (42) and behaves as an ordinary gas in

$$D' = D + \frac{1}{\gamma T_H} \tag{43}$$

spatial dimensions. The same is true for other thermodynamic functions. In particular the entropy grows with temperature as $T^{D'}$ instead of the usual T^D i.e. faster. As stressed in [5] this may be important for cosmology. If the very early universe is filled with the string gas, and if as is usually assumed the product SR^D is constant, than the more rapid decrease of the entropy during the expansion must be compensated by the more rapid increase of R i.e. the Universe expands faster than for models, where the very early Universe is filled with an ordinary gas.

7. *Thermodynamic limit*

In the thermodynamic limit, by definition, the volume of the string gas tends to infinity, while the energy density and the fraction of the energy contained in the heavy string remain constant. As long as the energy density corresponds to temperatures below the Hagedorn temperature, the string gas consists almost entirely of massless strings. Therefore, using the relation (33) between the energy and the temperature of massless strings one finds an estimate of the limiting energy density

$$\varrho_0 = \tfrac{93}{5}\pi^5 T_H^{10} \approx 2.8 \times 10^{-7}. \tag{44}$$

A better estimate of ϱ_0, including the effects of massive string excitations, is higher by about two per cent.

For energy densities exceeding ϱ_0 the gas of light strings becomes unstable with respect to mass excitations and the excess energy density goes over into the mass of one very heavy string, which consequently acquires the mass

$$M_s = (\varrho - \varrho_0)V. \tag{45}$$

In the thermodynamic limit this mass is very high, therefore, the temperature $T_m = T_H = T_k$ and the system is in equilibrium. It is not possible to increase the temperature of the system by pumping into it more energy, because all the surplus energy goes into the mass of the heavy string, which does not increase the temperature. Thus, for the string gas in the thermodynamic limit the Hagedorn temperature is indeed the highest temperature attainable.

As explained in the following section the string gas model used here is likely to break down much before one is any near to the thermodynamic limit. This limit is described here only for completeness.

8. *High temperature phase*

Let us consider now the phase, where $T_m = T_k$. Thus, equilibrium between the mass excitations and translational energy is achieved by equality in temperatures and not, as for the low temperature phase, by pumping all the possible energy out of the mass excitations. Since $T_m > T_H$ this kind of equilibrium is possible only above the Hagedorn temperature. In equilibrium the phase consists of one heavy string of mass M and of a gas of light, mostly massless, strings in equilibrium with it. The equilibrium is stable, when the heat capacity of the massive string exceeds in absolute value (it is negative!) the heat capacity of the gas of light strings. This implies [16]

$$V < 3.25 \times 10^5 \, M^2. \tag{46}$$

In the thermodynamics limit, where both M and V grow to infinity while their ratio remains constant, this inequality is always satisfied. There, however, $T_M \to T_H$, so that the high temperature phase occurs at one temperature only — the highest possible i.e. T_H. For finite systems, on the other hand, stability at given string mass M imposes an upper limit on the volume of the string gas. Since the applicability of the quasiclassical approximation imposes a lower limit for the volume (36), the two limits taken together impose a lower limit on the mass of the heavy string. One finds $N \gg 5000$. This in turn can be translated into an upper limit for the temperature. From formula (24) and the preceding discussion one finds the allowed temperature range

$$T_H < T < 1.007 T_H. \tag{47}$$

Thus, the temperature cannot exceed very significantly T_H.

Serious doubts about the existence of the high temperature phase, as described here, follow from the following argument. The string size should increase with its mass. The relation $R(M)$ for heterotic strings is not known, for bosonic strings, however, Mitchell and Turok [25] find for the root mean square radius the asymptotic high N formula

$$R = \sqrt[4]{\frac{4\pi^2 D^2 N}{6(D-1)}}, \tag{48}$$

where D is the number of space dimensions. Applying this with $D = 9$ as a crude estimate for heterotic strings, one finds [16]

$$M \ll \tfrac{1}{4} R^2. \tag{49}$$

The model of a noninteracting string gas must break down, when the radius of a string becomes bigger than the radius of the box. The importance of this trivial remark has been stressed in the context of the Hagedorn bootstrap by W. Nahm [26]. Combining the estimate from the formula of Mitchell and Turok with the stability condition one finds

$$V \ll 3 \times 10^9 \tag{50}$$

in direct conflict with the condition (36) for the applicability of the quasiclassical approximation. Thus, unless we have grossly overestimated the string radius, the noninteracting string model yields no support for the existence of the high temperature phase of the string gas. A separate problem are gravitational effects, but these will be discussed in a further section.

9. *Phase diagram*

For the gas of noninteracting heterotic strings one can distinguish three phases.

- a) The phase consisting of light, mostly massless, strings.
- b) The phase consisting of one heavy string with little or no light strings. It is assumed here that the energy carried by the light strings is smaller than that necessary to increase the mass of the heavy string to its next mass level.
- c) The phase, where both the heavy string and the light strings carry nonnegligible fractions of energy.

Let us note first that these phases are not phases in the usual thermodynamic sense of the word, because the thermodynamic limit cannot be taken and, therefore, there can be no singularities at the transitions between the phases. One can draw, however, a kind of phase diagram [17] by finding in the energy-volume plane the regions, where the situations (a) or (b) or (c) are respectively the most probable.

The phase consisting of light strings has been described as the low temperature phase in Section 6. It is the most probable phase whenever the temperature calculated from the formula (33) for $E_0 = E$ is less than T_H. The correction due to including higher mass strings in the derivation of the relation between energy and temperature is small and allows somewhat higher energies. Another region, where massless strings dominate, is the region $E < 2$, where there is simply no energy to produce a mass excitation. Note that in this situation the temperature may be arbitrarily high, provided that the volume is sufficiently small to keep the energy within the bound.

The phase (b) dominated by the heavy string is stable if the entropy loss of the string, when its mass drops one level down, exceeds the corresponding entropy gain of the accompanying gas of light strings. Including as light strings massless string only, one finds approximately

$$\Delta S > \tfrac{93}{5}\pi^5 V T^9. \tag{51}$$

Here $\Delta S = S_N - S_{N-1}$ can be calculated from formula (22) and the temperature T of the added gas is fixed by energy conservation

$$\Delta M = \tfrac{62}{3}\pi^5 V T^{10}. \tag{52}$$

The maximal volumes allowed for this phase turn out to be smaller than required by the applicability of the quasiclassical approximation (36), therefore the quantitative applicability of the noninteracting string gas model for this phase is doubtful. Qualitatively, the result must be correct: a massive string contained in a sufficiently small volume will not be able to emit massless strings into this volume.

Finally, the phase with coexisting light and heavy strings, which has been described in the preceding section, occupies the remainder of the M-V plane. In the preceding section we argued that the finite extension of massive strings makes the applicability of our model to this phase unlikely. In the following section we will present further support for these doubts from a discussion of gravitational effects.

10. High energy density limit

When at given volume energy is pumped into the string gas, gravitational collapse must happen at some point. The energy when this happens is of course a function of the volume occupied by the gas. We propose the following (conservative) estimate. Let us replace the gas in the box by a single point of mass $M = E$ located at the centre of the box. The horizon created by this mass is a hypersphere in the 9-dimensional space. The radius of the hypersphere can be calculated from a formula given by Acetta and Gleiser [7]. It is plausible that the box containing the gas must have collapsed, if the assumed radius of the box is smaller than the calculated radius of this horizon. The resulting condition for a non-collapsed box is

$$R < \frac{\pi^{\frac{7}{2}}}{\Gamma(\frac{9}{2})} m_{\rm Pl}^8 R^7 \approx 4.7 m_{\rm Pl}^8 R^7. \qquad (53)$$

When the energy of the string gas consists mainly of the mass of one heavy string, one can combine the present condition with the stability condition (46) to derive a lower bound for the radius of the box

$$R > 0.05 m_{\rm Pl}^{-\frac{16}{5}}, \qquad (54)$$

where the Planck mass should be expressed in units where $\alpha' = 1$. Let us note that according to Ref. [21], even below the Hagedorn temperature the simple model used here will break down because of vacuum polarization.

11. Application to black hole evaporation

There has been a proposal to describe the last stage of black hole evaporation as a transition of the black hole into strings [6, 7]. According to the string paradigm everything, also a black hole, consists of strings. Therefore, some care is necessary when interpreting this assumption. The string system into which the black hole goes over must be sufficiently different from the string system referred to as the black hole. The usual assumption has been that the black hole goes over into a string system, which can be described as a gas of noninteracting strings. Since at total energy M the entropy of the string gas is proportional to M and that of a black hole to M^2, the two string systems are indeed very different from each other.

From the point of view of the present paper the question naturally arises, into which of the three phases of the string gas does a string go over [17]. It seems that, if the black hole ends as a string gas in the sense as specified above, the transition must be directly into ag as of light strings, without the intermediate stage of a gas containing a heavy string alone or containing a heavy string in equilibrium with light strings. Firstly, there are the general doubts, whether the phases containing the heavy strings exist as equilibrium states. Secondly, a heavy string has in most of its states a high angular momentum, while a black hole towards the end of its evaporation is believed to have angular momentum zero, or very nearly zero. Thus, angular momentum conservation would strongly suppress transitions of old black holes into heavy strings.

Another question is, whether this transition takes place at all. The standard argument [6] is that since the entropy of the string gas grows linearly with energy and that of the black hole quadratically, at sufficiently low mass the entropy of the gas is higher than the entropy of a black hole of the same mass. Then, the black hole becomes unstable and the transition takes place. For a gas of massless strings at given total energy the entropy grows with increasing volume as $V^{0.1}$. Therefore, whatever the mass of the black hole it is possible to find a box so large that compared to the black hole, the gas has the same total energy but higher entropy. The very probable existence of black holes in nature [28] means that in spite of that the transitions do not occur. The reason is, of course, that the transition probability of a black hole, which is for its mass a very small object, into a large box of gas is very small and the transition has no time to happen. The same may be true also for smaller black holes. The question, whether black holes go over into a gas of massless strings before something else happens to them, is a dynamical question, which cannot be solved by thermodynamics alone.

12. Conclusions

Our discussion applies to a gas of noninteracting heterotic strings confined in a classical box. The absence of interactions here is understood as in the thermodynamics of the ideal gas: strings can exchange energy, split and join, but in the Hamiltonian the interaction energy is neglected.

Within this model it is possible to describe the string gas using standard thermodynamics. The gas has some unusual features: existence of the Hagedorn temperature, two temperatures which do not have to be equal at equilibrium and a negative heat capacity corresponding to mass excitations. Also for most applications the thermodynamic limit cannot be taken and consequently phases and the phase diagram cannot be defined as usual.

The gas of noninteracting strings in a box can exist in three states, which we call phases.

- A gas of ligh‹, mostly massless strings. This phase occurs at temperatures below the Hagedorn temperature and at any temperature, when there is too little energy for mass excitations. We argue that, if black holes go over into something like a gas of noninteracting strings, then they are the most likely to go over directly into his phase.

- A heavy string with either no light strings accompanying it, or so few that their total energy is too small to produce a mass excitation. At high energy the quantum numbers of this system are essentially the quantum numbers of the heavy string. Thus angular momentum, charge etc. are likely to be high [27, 10]. Since transitions between states differing in angular momentum or charge are impossible, it is not clear that the entropy defined as the logarithm of the total number of states is relevant. In particular a black hole with both angular momentum and charge close to zero can communicate only with a very small subset of states of the heavy string, unless there is some mechanism to remove the surplus charge and angular momentum.
- A gas, where a heavy string is in equilibrium with light strings, which carry a finite fraction of the total energy. The difficulty with this phase is that according to the stability condition the box must be sufficiently small in order to have equilibrium. An estimate of the string size as function of its mass suggests that it may be impossible to squeeze the heavy string into the box.

REFERENCES

[1] D. Gross, J. A. Harvey, E. Martinec, R. Rohm, *Phys. Rev. Lett.* **54**, 502 (1985).
[2] D. Gross, J. A. Harvey, E. Martinec, R. Rohm, *Nucl. Phys.* **B256**, 253 (1985); **B267**, 75 (1986).
[3] E. Alvarez, *Phys. Rev.* **D31**, 418 (1985).
[4] S. H. Tye, *Phys. Lett.* **B158**, 388 (1985).
[5] M. Glaser, J. G. Taylor, *Phys. Lett.* **B164**, 36 (1967).
[6] M. J. Bowick, L. Smolin, L. C. R. Wijewardhana, *Phys. Rev. Lett.* **56**, 424 (1986).
[7] F. S. Accetta, M. Gleiser, *Ann. Phys.* (USA) **176**, 278 (1987).
[8] R. Hagedorn, *Nuovo Cim. Suppl.* **3**, 147 (1965).
[9] R. Hagedorn, *Nuovo Cimento* **52A**, 1336 (1967).
[10] S. Frautschi, *Phys. Rev.* **D3**, 2821 (1971).
[11] R. D. Carlitz, *Phys. Rev.* **D5**, 3231 (1972).
[12] B. Sundborg, *Nucl. Phys.* **B254**, 1665 (1985).
[13] M. J. Bowick, L. C. R. Wijewardhana, *Phys. Rev. Lett.* **54**, 2485 (1985),
[14] D. Pavon, *Gen. Rel. and Grav.* **19**, 375 (1987).
[15] K. Zalewski, Two temperatures in the thermodynamics of strings, CERN preprint TH-4635/87, January 1987.
[16] K. Zalewski, *Phys. Lett.* **B196**, 481 (1987).
[17] K. Zalewski, *Phys. Lett.* **B207**, 278 (1988).
[18] I. Antoniadis, J. Ellis, D. V. Nanopoulos, *Phys. Lett.* **B199**, 402 (1987).
[19] K. Huang, S. Weinberg, *Phys. Rev. Lett.* **25**, 895 (1970).
[20] N. Cabibbo, G. Parisi, *Phys. Lett.* **B59**, 67 (1975).
[21] J. J. Atick, E. Witten, The Hagedorn transition and the number of de greesof freedom of strings, Institute of Advanced Study, Princeton preprint IASSNS-HEP-88/14 (1988).
[22] S. W. Hawking, *Nature* **248**, 30 (1974).
[23] S. W. Hawking, *Phys. Rev.* **D13**, 191 (1975).
[24] M. B. Green, G. Veneziano, *Phys. Lett.* **B36**, 477 (1971).

[25] D. Mitchell, N. Turok, *Nucl. Phys.* **B294**, 1138 (1987).
[26] W. Nahm, *A Critique of Astrophysical Applications of Hagedorn's Boot-Strap* in: *Hadronic Matter at Extreme Energy Densities*, eds. N. Cabibbo and L. Sertorio, Plenum Press, New York 1980, p. 219.
[27] T. L. Curtright, C. B. Thorn, *Nucl. Phys.* **B274**, 520 (1986).
[28] R. D. Blandford, *Astrophysical Black Holes in 300 Years of Gravitation*, eds. S. W. Hawking and W. Israel, Cambridge 1987, p. 277.