GAUGE TRANSFORMATIONS IN GRAVITATION THEORY

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Different types of gauge transformations in gravitation theory are examined in the framework of its fibre bundle reformulation.

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1. Introduction

The physical specificity of Einstein gravity as the Goldstone-type field was clarified due to the fibre bundle reformulation of gravitation theory [1–3]. By the modified equivalence principle Einstein gravity is responsible for spontaneous breaking of space-time symmetries. From our point of view, a gravitational field could appear as the product of the first phase transition which separated matter (fermions with the Lorentz symmetry group) and the geometric arena (with the symmetry group GL(4, **R**)). Thereby, there are physical reasons for existence of a Higgs gravitation vacuum (or a background metric). Analysis of gauge transformations in gravitation theory also gives us some arguments for existence of such a vacuum.

There are different types of gauge transformations that the Yang-Mills gauge principle fails to discern. This defect appears to be essential in the case of space-time symmetries. Therefore, we base our consideration on the more general principle of the field theory formalization by fibre bundles.

This principle is based on the mathematical definition of a classical matter field as a global section φ of some differential vector bundle $\lambda = \{V, G, X, \Psi\}$ with the typical fibre V, the structure group G, and the base manifold X^4. The bundle atlas $\Psi = \{U_i, \psi_i\}$ (where U_i and ψ_i are patches and morphisms of trivialization of λ) and the coordinate atlas Ψ_X of the base X define the reference frame and the coordinates such that a field $\varphi(x)$ is represented by a family of V-valued functions

$$\{\varphi_i(x) = \psi_i(x)\varphi(x), \quad x \in U_i\}$$

with respect to these atlases. Changes of atlases Ψ and Ψ_X induce gauge transformations and coordinate transformations of field functions $\varphi_i(x)$. Gauge potentials appear as coefficients of a local connection 1-form A on the bundle λ. Thus, gauge theory is the direct issue of the field theory formalization by fibre bundles.

2. Internat symmetry gauge transformations

In gauge theory of internal symmetries one must discern two types of gauge transformations. There are reference frame changes and transformations of fields themselves, with a reference frame fixed. The first-type gauge transformations represent atlas changes

$$\Psi = \{U_i, \psi_i\} \to \Psi' = \{U_i, \psi'_i = g_i(x)\psi_i\} \tag{1}$$

of the matter bundle λ and associated bundles. Here $g_i(x)$ are elements of gauge groups $G(U_i)$ of G-valued functions on U_i, and they play the role of transition functions between charts of atlases Ψ and Ψ'. The corresponding transformation of field functions $\{\varphi_i\}$ reads

$$\varphi_i = \psi_i\varphi \to \varphi'_i = \psi'_i\varphi = g_i\psi_i\varphi = g_i\varphi_i.$$

This transformation does not change a section φ, but alters its representation by field functions $\{\varphi_i\}$.

A local connection 1-form A on the matter bundle λ results from projection of the connection form ω (defined on the total space $\mathrm{tl}\,\lambda_G$ of the principal bundle λ_G) on the base X. Such a projection can take place only relative to some atlas Ψ of λ_G, and $A = \omega\partial(z_i)$ on U_i. Here $\{z_i = \psi_i^{-1}(1_G)\}$ is the family of local sections of λ_G which are defined by a given atlas Ψ, and $\partial(z_i)$ denotes the differential of the mapping z_i. Then, the transformation law of the connection form A under atlas changes (1) reads

$$A'_i = \omega\partial(z_i g_i^{-1})$$

Gauge transformations (1) of atlases over the same cover of the base X compose the group which is the direct product

$$G(\{U_i\}) = \prod_i G(U_i)$$

of groups $G(U_i)$. Let a cover $\{U'_i\}$ contain a cover $\{U_i\}$. Then, there is an embedding of the group $G(\{U_i\})$ into the group $G(\{U'_i\})$. The set of covers of X is provided with the partial order $\{U'_i\} > \{U_i\}$, and there exists the direct limit

$$\begin{array}{ccc} G(X) & \leftarrow & G(\{U'_i\}) \\ & \nwarrow G(\{U_i\}) \nearrow & \end{array}$$

of groups $G(\{U_i\})$ with respect to this order. If groups $G(\{U_i\})$ are non-abelian, the limit $G(X)$ fails to be a group. This is a pseudo-group. Any atlas $\Psi = \{U_i, \varphi_i\}$ of the bundle λ corresponds to a certain coordinate atlas

$$\Psi_{\mathrm{tl}\lambda} = \{U_i, \psi_i : \mathrm{tl}\,\lambda \ni p \to (x, v) \in U_i \times V\}$$

of the total space $\mathrm{tl}\,\lambda$ of λ. Therefore, atlas transformations (1) represent elements of the pseudo-group of coordinate transformations of $\mathrm{tl}\,\lambda$.

Since atlas changes are equivalence transformations of fibre bundles, the requirement of the invariance of a field action functional under the first-type transformations seems to be quite natural.

The second-type gauge transformations are generated by equivariant mappings F of the total space $\mathrm{tl}\,\lambda_G$ of the principal bundle λ_G [4], i.e.

$$F(pg) = F(p)g, \quad p \in \mathrm{tl}\,\lambda_G, \quad \pi(F(p)) = \pi(p), \quad \pi: \mathrm{tl}\,\lambda_G \to X.$$

This mapping can be written as $F(p) = p\gamma(p)$ where γ is a G-valued function on $\mathrm{tl}\,\lambda_G$ such that $\gamma(pg) = g^{-1}\gamma(p)g$.

Mappings F induce transformations of sections φ of the matter bundle λ. Sections φ can be defined by V-valued functions f on $\mathrm{tl}\,\lambda_G$ such that

$$\varphi(\pi(p)) = [p]f(p), \quad f(pg) = g^{-1}f(p),$$

where $[p], p \in \mathrm{tl}\,\lambda_G$ denotes the mapping

$$[p]: V \to V_{\pi(p)} \in \mathrm{tl}\,\lambda$$

by the law $(p, V) \to (p, V)/G$ (when (pg, v) is identified with (p, gv)). Then, gauge transformations F of sections φ read

$$f(p) \to f'(p) = f(p\gamma(p)) = \gamma^{-1}(p)f(p),$$

$$\varphi = [p]f(p) \to \varphi' = [p]f'(p) = [p]\gamma^{-1}(p)f(p). \tag{2}$$

Let $\Psi = \{U_i, \psi_i = [z_i]^{-1}\}$ be an atlas of the bundle λ. Transformation (2) yields the following transformation of field functions

$$\varphi_i(x) = [z_i(x)]^{-1}\varphi(x) = f(z_i(x)) \to \varphi_i'(x) = f(z_i(x)\gamma(z_i)) = \gamma^{-1}(z_i)\varphi_i(x) \tag{3}$$

with respect to the atlas Ψ. This transformation looks like a gauge transformation of the first type between atlases Ψ and

$$\Psi' = \{U_i, \psi_i' = \gamma^{-1}(\psi_i^{-1}(1_G))\psi_i\}.$$

According to the property of γ, atlases Ψ and Ψ' possess the same transition functions $\varrho_{ij}' = \varrho_{ij}$.

Thus, for any second-type gauge transformations (2) of matter fields $\varphi \to \varphi'$ there exists the gauge transformation (3) of the first type $\Psi \to \Psi'$ such that fields φ_i look relative to Ψ' just as φ_i' look relative to Ψ, i.e. $\psi_i\varphi' = \psi_i'\varphi$. This rule is also true for gauge fields

$$\omega \to \omega' = \omega(\partial\gamma),$$

$$A_i = \omega(\partial\psi_i^{-1}) \to A_i' = \omega'(\partial\psi_i^{-1}) = \omega(\partial(\gamma^{-1}\psi_i)^{-1}) = \omega(\partial(\psi_i')^{-1}).$$

Therefore, a field action functional, being invariant under gauge transformations of the first type, turns out to be also invariant under the second-type gauge transformations.

There is a one-to-one correspondence between the functions γ, generating the second-type gauge transformations, and global sections of the associated bundle $\tilde{\lambda}_G$, possessing the typical fibre G. However, as distinguished from the principal bundle λ_G, the structure group of $\tilde{\lambda}_G$ acts on G by the adjoint representation $g: G \to gGg^{-1}$. Thereby, the group of the second-type gauge transformations is the group $\tilde{G}(X)$ of global sections of the bundle $\tilde{\lambda}_G$. This group differs from the pseudo-group $G(X)$ of the first-type gauge transformations.

Gradation of gauge transformations is important for quantum theory. By the Noether theorems the invariance of an action functional S under the first-type gauge transformations causes constraint appearance in the system of field equations. Such a system can be described as a generalized Hamilton system. Its solutions of the same physical coset are related by gauge transformations whose generators, represented by certain linear superpositions of the first-class constraints, compose the Lie algebra L_G of the group G. These gauge transformations are transformations of the second type, although the coincidence of their family with the group $\tilde{G}(X)$ is open to question. The group $\tilde{G}(X)$ is usually applied in quantum gauge theory to construct the measure support in generating functionals [5]. At the same time, one can choose another Hamilton system, equivalent to the first one in the physical sector, but whose algebra of gauge transformation generators differs from the algebra L_G.

3. *Gauge gravitation theory*

In fibre bundle terms the Einstein gravitational field on an orientable paracompact manifold X^4 is defined to be a global section g of the fibre bundle B of pseudo-Euclidean bilinear forms on tangent spaces T_x over X^4. This bundle B is associated with the tangent bundle TX, possessing the structure group $GL^+(4, \mathbf{R})$, and B is isomorphic with the fibre bundle Q in quotient spaces $GL^+(4, \mathbf{R})/SO(3, 1)$. A global section h of Q describes the Einstein gravitational field in the tetrad form. According to the well-known theorems, this section exists if and only if the structure group $GL^+(4, \mathbf{R})$ of TX contracts to the Lorentz group, i.e. there is an atlas $\Psi^{(g)} = \{U_i, \psi_i^g\}$ of TX such that all transition functions of $\Psi^{(g)}$ are reduced to the elements of Lorentz gauge groups $SO(3, 1)\,(U_i \cap U_j)$. All local metric functions $g_i = \psi_i^g g$ coincide with the Minkowski metric η, and tetrad functions $h_i = \psi_i^g h$ take on their values in the centre of the quotient space $GL^+(4, \mathbf{R})/SO(3, 1)$ relative to $\Psi^{(g)}$.

The tetrad field h is usually written as a family of local sections $\{h_i(x), x \in U_i\}$ of the principal $GL(4, R)$-bundle up to right multiplying h_i by elements of gauge Lorentz groups $SO(3, 1)\,(U_i)$ i.e. $h_i = h_i SO(3, 1)\,(U_i)$. This freedom reflects the non-uniqueness of the atlas $\Psi^{(g)}$. With respect to any atlas Ψ of TX these tetrad functions take the form of matrix functions $h_i = \psi_i(\psi_i^g)^{-1}$, acting in the typical fibre of TX and describing gauge transformations from some atlas $\Psi^{(g)}$ to a given atlas $\Psi = \{U_i, \psi_i = h_i\psi_i^g\}$.

Choice of a certain atlas Ψ of the tangent bundle TX defines a reference frame in gravitation theory. Hence, all reference frame changes compose the pseudo-group $GL^+(4, \mathbf{R})\,(X)$ of the first type gauge transformations in gravitation theory.

This definition of reference frames is close to the one used in the tetrad formulation of gravitation theory. If the atlas $\Psi = \{U_i, \psi_i\}$ of TX is fixed, the vierbein $\{t_i(x)\} = \psi_i^{-1}(x)\,\{t\}$ (where $\{t\}$ is the basis of the typical fibre $\mathbf{R}^4$ of TX) can be erected in every point of the space-time manifold X^4. Functions $t_i(x)$ represent local sections z_i of the associated principal bundle LX in linear frames, which are defined by the atlas Ψ. Inversely, if a family of such sections is fixed, they define some atlas Ψ of TX.

The traditional generally covariant form of gravitation theory corresponds to the special case of holonomic atlases when a bundle atlas $\Psi = \{U_i, \psi_i = \partial\chi_i\}$ is correlated with a coordinate atlas $\Psi_X = \{U_i, \chi_i\}$ of the manifold X^4, i.e. $t_\mu(x) = \partial_\mu$ are oriented along coordinate lines. Respectively, the gauge pseudo-group GL(4, **R**) (X) of reference frame changes contains the pseudo-group of holonomic gauge transformations

$$(\psi_i')^{-1}\psi_i = g_i(x) \colon t_\mu(x) \to t'_\mu(x') = t_\delta(x)\partial x^\delta/\partial x'^\mu$$

accompanied by coordinate transformations $\chi_i'\chi_i^{-1} : x^\mu \to x'^\mu$.

The relativity principle in fibre bundle terms proves to be identical to the gauge principle, and gravitation theory can be built directly as the gauge theory of space-time symmetries. However, in contrast with the internal symmetry case, in gauge models of space--time symmetries there are two classes of the first-type gauge transformations (see Section 4). These are familiar atlas transformations of a matter bundle λ and atlas changes of the tangent bundle TX, because space-time transformations act on both fields and operators of partial derivatives $\partial\mu$ as vectors of tangent spaces. The specificity of the gauge principle in the space-time symmetry case lies in the fact that the invariance of the matter action functional under space-time gauge transformations of the second class makes it necessary to introduce the metric field besides gauge potentials.

However, the relativity principle fails to fix the Minkowski signature of this metric, and the correspondence of these gauge fields to the Lorentz group as it takes place in gravitation theory is not indicated. Therefore, the equivalence principle is called into play in the gauge gravitation theory.

The equivalence principle in gravitation theory must guarantee transition to special relativity in a certain reference frame. Since special relativity can be characterized in the geometric terms as the geometry of Lorentz invariants, the equivalence principle must postulate the existence of a reference frame where Lorentz invariants can be defined everywhere on X^4, and these have to be conserved under parallel transport.

This postulate admits the adequate mathematical formulation. The connection A on the tangent bundle and associated bundles must be reduced to the Lorentz connection, i.e. there are atlases Ψ^L of these bundles such that A_i take values in the Lie algebra L of the Lorentz group. Transition functions of these atlases are elements of Lorentz gauge groups SO(3, 1) $(U_i \cap U_j)$. Therefore, the structure group of TX and associated bundles contracts to the Lorentz group, and consequently the corresponding pseudo-Riemannian metric g (or tetrad field h) exists everywhere on X^4 (i.e. $\Psi^L = \Psi^{(g)}$).

Thus, the equivalence principle establishes that the metric g and the connection A, introduced by the relativity principle, are the pseudo-Riemannian metric and the Lorentz connection. These quantities are independent, and the gauge gravitation theory is the affine--metric theory. Note that if Ψ is an arbitrary atlas the Lorentz connection form A takes values in the Lie algebra of the group GL(4, **R**), but the identity $(d-A)g = 0$ holds.

The equivalence principle singles out the Lorentz group as the exact symmetry subgroup of the gauge group GL(4, $\mathbb{R}$) (X) of space-time symmetries, broken spontaneously due to the Einstein gravitational field as the Goldstone field [1–3].

Note that, in contrast with Goldstone fields of internal symmetries, the Einstein gravitational field (identified in holonomic atlases with deviations of g_i from η) cannot be removed by any gauge. The reason lies in the fact that, as it was mentioned above, space-time transformations act also on operators of partial derivatives ∂_μ. But these operators include tetrad functions $\partial_a = h^\mu_a \partial_\mu$ relative to a nonholonomic atlas $\Psi^{(g)}$ where metric functions g_i are reduced to the Minkowski metric η. In other words, the Einstein gravitational field does not vanish, but its metric form is transformed into the tetrad form under the gauge $g_i = \eta$.

4. Gauge transformations in gravitation theory

As it was mentioned above, space-time symmetries (in comparison with internal symmetries) transform both fields φ^A (where A denotes an intrinsic spin index) and non-field quantities such as differential forms dx^μ and derivatives ∂_μ. Therefore, one must discern two kinds of the first-type space-time transformations. There are atlas changes of a matter bundle λ (type (1.1)) and atlas changes of the tangent bundle (type (1.2)). Though there are equivalent atlases of these bundles, their atlases are not always the same, because spinor bundles λ admit only atlases $\Psi^{(g)}$ with Lorentz transition functions.

Moreover, spinor bundles $\lambda = (V, \mathrm{SO}(3,1), X, \Psi^{(g)})$ and $\lambda' = (V, \mathrm{SO}(3, 1), X, \Psi^{(g')})$, associated with TX, are not equivalent if atlases $\Psi^{(g)}$ and $\Psi^{(g')}$ correspond to different gravitational fields g and g'. These atlases cannot be transformed into each other by Lorentz gauge transformations, and they, being equivalent as atlases of TX, are nonequivalent as atlases of spinor bundles.

Transformations of type (1.2) do not alter spinor field functions φ_i. Transformations of type (1.1) are reduced to Lorentz gauge transformations of atlases λ. They act on the gauge gravitation field A_i, but do not change the Einstein gravitational field. These transformations alter tetrad functions

$$h_i \to h_i g_i, \qquad g_i \in \mathrm{SO}\,(3, 1)\,(U_i). \tag{4}$$

However, as it was mentioned above, the Einstein gravitational field h is defined by tetrad functions h_i up to transformations (4).

As in the case of internal symmetries, the second-type partners (type (2.1)) of space-time gauge transformations of the type (1.1) represent fibre-to-fibre morphisms of matter spinor bundles, and they compose the Lorentz gauge group $\widetilde{\mathrm{SO}}(3, 1)\,(X)$. These transformations as well as gauge transformations of the type (1.1) act on the gauge gravitation field A, but do not alter the Einstein gravitational field.

In the case of gauge transformations of the type (1.2) one should take into account that these transformations act on derivatives ∂_μ which contain no field quantities with respect to holonomic atlases. This property must be conserved under the second-type gauge transformations, namely, if γ is such a transformation, $\gamma(\partial_\mu)$ must be reduced to $\partial_{\mu'}$, with respect to some holonomic atlas. As a consequence, only holonomic transformations possess the second-type partners (type (2.2)). These represent fibre-to-fibre morphisms of the tangent bundle TX and associated tensor bundles which are induced by diffeomorphisms of the manifold X^4. Being written with respect to some coordinate atlas Ψ_X and some holonomic atlas Ψ, such transformations read

$$\gamma: X \ni x \to x'(x) \in X,$$

$$\partial_\gamma: T_x \ni t^\nu \frac{\partial}{\partial x^\nu} \to t'^\nu \frac{\partial}{\partial x'^\nu} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right) t^\mu \frac{\partial}{\partial x'^\nu} \in T_{x'(x)}. \tag{5}$$

Transformations (5) yield familiar covariant transformations

$$\partial_\gamma: \tau^{\mu\cdots}_{\alpha\cdots}(x) \to \tau'^{\mu\cdots}_{\alpha\cdots}(x') = \frac{\partial x'^\mu}{\partial x^\nu} \cdots \frac{\partial x^\beta}{\partial x'^\alpha} \cdots \tau^{\nu\cdots}_{\beta\cdots} \tag{6}$$

of tensor fields. For any morphism (5) one can find the holonomic transformation of Ψ (i.e. the first-type transformation) such that the corresponding transformation of tensor fields looks like transformation (6). Therefore, the invariance of an action functional under holonomic transformations entails its invariance under the second-type transformations (5).

In comparison with fibre-to-fibre morphisms of bundles in the case of internal symmetries, morphism (5) is not the identity mapping of the base X. However, one can consider mappings, induced by morphisms (5) on spaces of global sections τ of tensor bundles:

$$L_\gamma: \tau(x) \to \tau'(x) = (\partial_\gamma)\tau(\gamma^{-1}(x)) \tag{7}$$

which transform tensor fields $\tau(x) \to \tau'(x)$ in the same point (type (2.2')). If diffeomorphism γ can be represented as the flow $\gamma(s)$ along integral curves $\dot{x}(s) = \varepsilon(x(s))$ of some vector field $\varepsilon(x)$, the generator of morphism (7) takes the familiar form of the Lie derivative

$$L_\varepsilon: \tau^{\mu\cdots}_{\alpha\cdots}(x) \to -\partial_\alpha \varepsilon^\beta \tau^{\mu\cdots}_{\beta\cdots} - \ldots - \varepsilon^\sigma \partial_\sigma \tau^{\mu\cdots}_{\alpha\cdots} + \partial_\nu \varepsilon^\mu \tau^{\nu\cdots}_{\alpha\cdots} + \ldots$$

Let us point out two different actions of morphisms (5) and (7) on integrals $\int_U \omega$ where $U \subset X$ and ω is some 4-form on X^4. In the first case

$$\int_U \omega \to \int_{\gamma(U)} \omega' = \int_U \omega,$$

this quantity is invariant, whereas in the second case

$$\int_U \omega \to \int_U \omega' = \int_{\gamma^{-1}(U)} \omega,$$

this is invariant only if $\gamma(U) = U$. For instance,

$$L_\varepsilon: \int_U L d^4x \to \int_U \partial_\mu(\varepsilon^\mu L) d^4x = \int_{\partial U} \varepsilon^\mu L ds_\mu = 0$$

because ε^μ is equal to zero on ∂U or it is tangent to ∂U if $\gamma(s)U = U$.

Note that there are no first-type gauge transformations such that the corresponding transformation of tensor fields imitates transformation (7). As a consequence, the invariance of an action functional $S = \int Ld^4x$ under transformations (7) results from its invariance under holonomic transformations only if the supplementary condition $\gamma(U) = U$ holds.

Gauge transformations of types (2.2) and (2.2′) act not only on tensor fields, but they also concern spinor matter fields. These transformations alter a tetrad gravitational field $h \to h'$ and consequently altases $\Psi^{(g)} = \{U_i, \psi_i^g = h_i^{-1}\psi_i\}$ (where $\Psi = \{U_i, \psi_i\}$ is the fixed holonomic atlas):

$$\psi^{(g)} \to \psi^{(g')} = \{U_i', \quad \psi_i^{g'} = (h_i')^{-1}\psi_i\}.$$

Herewith, atlases $\Psi^{(g)}$ and $\Psi^{(g')}$ fail to be transformed into each other by Lorentz gauge transformations. Therefore, as it was mentioned above, a spinor field φ in different gravitational fields g and g' is represented by sections of nonequivalent bundles λ and λ', even if φ is expressed by the same field functions $\varphi^A(x)$ with respect to atlases $\Psi^{(g)}$ and $\Psi^{(g')}$.

The invariance of a field action functional under the (1.1) and (2.2)-type space-time gauge transformations entails the Lagrangian degeneracy, and so leads to the Hamilton formalism with constraints. Herewith, the secondary first-class constraints H_α ($\alpha = 1, 2, 3$), $H_\perp$, H_{ij} play the role of generators of the second-type gauge space-time transformations. These constraints (except $H_\perp$) are purely kinematic, i.e. they have the same form for any action functional S, invariant under the first-type gauge transformation. As a consequence (though $H_\perp$ depends on the choice of S), the first-class constraint algebra is the same for different gravitation Lagrangians R, $R+T^2$, $R+R^2+T^2$ (where T denotes torsion) [6, 7]. For example, the R case differs from the $R+R^2+T^2$ case only by so-called if-constraints (second-class) which do not influence essentially gauge symmetries of these models, although the second type models are usually built on the basis of Poincaré gauge symmetries.

Since spinor fields in different Einstein gravitational fields are represented by sections of nonequivalent bundles, the problem of the fermion functional space in a generating functional of quantum gravity arises. Note that this difficulty does not concern gauge gravitation fields, and, in our opinion, this is related with the Higgs-Goldstone nature of the Einstein gravitational field. This problem can be solved if one assumes existence of a classical background Einstein gravity.

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