TOPOLOGY AND CHIRAL SYMMETRY BREAKING IN FOUR-FERMION INTERACTION

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The purpose of this paper is to investigate an influence of space-time topology on the breakdown of chiral symmetry in the model with four-fermion interaction due to formation of fermionic condensate.

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1. *Introduction*

The non-linear spinor theory plays considerable role in field theory and in the particle physics. For the first time the non-linear spinor equation has been proposed by Ivanenko fifty years ago [1] and since then it was used by many authors in attempts to construct unified field theory (for the list of references see [2]). Today, the four-fermion interactions are considered in connection with the problems of dynamical symmetry breaking in many field models [3–5]. We consider here only the chiral symmetry breaking in the non-linear spinor field theory.

We believe that the non-perturbative effects such as formation of condensate have great influence on the symmetry breaking. At present there are different methods to estimate the value of fermionic condensate, for example, Monte-Carlo technique [6], the sum rules [7], strong coupling expression of lattice theories [8] etc.

The value of condensate is sensible to external conditions (external fields, temperature etc.). Some time ago several works were published in which the influence of different factors on the condensate value was investigated [9]. In our previous work [10] we considered the dependence of this value on the boundary conditions.

The aim of the present work is to study the behaviour of the vacuum fermionic condensate $\langle \bar{\psi}\psi \rangle$ in four-fermion theory with respect to the non-Euclidean properties of flat space-time. In particular we will try to find a connection of vacuum condensate with the space-time topology and will investigate the evaporation of condensate due to change of topology. Formation of condensate leads to the breakdown of the chiral symmetry. In order to study these effects we will use a mean field method [11] which is a very convenient analytical technique for our case. The philosophy of this approach is analogous to that of the case of mass gap equation. In our earlier papers the method was used in torsion theory [12] and for investigation of the behaviour of effective potential in approximation of mean field [13].

For simplicity we consider in this work the 3-dimensional space-time. The mean field method will be described in the next section for the model of four-fermion interaction with scalar connection. In the third section we will study four different topologies of 3-dimensional Euclidean space and will calculate the value of $\langle\psi\bar{\psi}\rangle$ in these spaces. Fourth section is devoted to discussion of the stable trivial solution of the gap equation.

2. *Mean field method in non-linear fermionic model*

Massless fermionic field with non-linear interaction is described by the Lagrangian

$$\mathscr{L}(x) = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + \frac{\lambda_0}{2}(\bar{\psi}\psi)^2, \tag{1}$$

where $\lambda_0 = g_0^2/\mu_0^2$, μ_0 has a canonical dimension of mass, g_0 is a dimensionless parameter. The Lagrangian (1) is invariant under the chiral transformations

$$\psi \to \gamma_5\psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5. \tag{2}$$

Now, in order to study the dynamical properties of the model we introduce the auxiliary field and rewrite (1) as

$$\mathscr{L}'(x) = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} + g_0\sigma)\psi - \frac{\mu_0^2}{2}\sigma^2. \tag{3}$$

The invariance under the chiral transformations can be maintained at this level if we add to (2) a simultaneous change $\sigma \to -\sigma$.

The generating functional with the Lagrangian (3) can be expressed in the following form

$$Z[\bar{\eta}, \eta, J] = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}J \exp\{i\int d^n x[\mathscr{L}'(x) + \bar{\eta}\psi + \bar{\psi}\eta + J\sigma]\}, \tag{4}$$

where $\bar{\eta}$, η and J are the sources of fermions and auxiliary fields respectively.

Evidently the integration in (4) over σ field leads to (1). The classical field equations for σ and fermionic field can be found by applying the variational procedure to the Lagrangian (3) and have the form

$$(i\gamma^{\mu}\partial_{\mu} + g_0\sigma)\psi = 0, \quad \sigma = g_0/\mu_0^2\bar{\psi}\psi. \tag{5}$$

Thus the field σ presents the bound state of the interacting fermionic field.

The integrating over ψ and J in generating functional (4), therefore gives

$$Z[\bar{\eta}, \eta, J] = \exp\{iW[\bar{\eta}, \eta, J\}$$

$$= \int D\sigma \exp\left\{i\int d^n y d^n x\left[-\bar{\eta}(x)\hat{S}(x, y)\eta(y) + \left(-\frac{\mu_0^2}{2}\sigma^2 + J\sigma\right)\delta^n(x, y)\right]\right\}, \tag{6}$$

where $\hat{S}^{-1}(x, y) = (i\gamma^{\mu}\partial_{\mu} + g_0\sigma)\delta^n(x, y)$ is the Green's function of fermionic field.

A quantization of (6) can be performed under the assumption that

$$\sigma = \sigma_0 + \tilde{\sigma} \tag{7}$$

where σ_0 is the vacuum average with respect to $\langle 0/\sigma/0 \rangle$ and $\tilde{\sigma}$ is the quantum fluctuation of the field σ. One can construct the loops graph's method for the calculation of the Green's functions. These Green's functions will depend on σ_0. Let us find this functional dependence.

The vacuum average of the mean field in the external source J is

$$\sigma_J = \frac{\delta W[J]}{\delta J} = \frac{\langle {}_J0|\sigma|0_J \rangle}{\langle {}_J0|0_J \rangle}. \tag{8}$$

The vacuum average σ_J has the limit $\sigma_0 = \langle \sigma \rangle$ when $J \to 0$. The effective action $\Gamma[G_J]$ may be defind by

$$\Gamma[\sigma_J] = W[J] - (J \cdot \sigma_J). \tag{9}$$

One can write $\Gamma[G_J]$ in the form of series

$$\Gamma[\sigma_J] = \int d^n x \{ -V[\sigma_J] + \tfrac{1}{2} \sigma_J(x)_{,\mu} f(\sigma_J) \sigma_J(x)^{,\mu} + \ldots \} \tag{10}$$

$V[\sigma_J]$ is the effective potential and $f(\sigma_J)$ is a certain function of σ_J.

Combining the results of variation

$$\delta \Gamma[\sigma_J] = \frac{\delta W[J]}{\delta \sigma_J(x)} - J(x) - \int d^n y \, \frac{\delta J(y)}{\delta \sigma_J(x)} \sigma_J(y), \tag{11}$$

with

$$\frac{\delta W[J]}{\delta \sigma_J(x)} = \int d^n y \, \frac{\delta J(y)}{\delta \sigma_J(x)} \sigma_J(y), \tag{12}$$

we obtain

$$\frac{\delta \Gamma}{\delta \sigma_J} = -J. \tag{13}$$

Since we do not expect translational invariance to be dynamically violated, σ_J should be independent of x. Hence it is sufficient to study $\Gamma[\sigma_J]$ for constant σ_J. The value σ_0 is then defined by the equation

$$\left. \frac{\delta \Gamma[\sigma_J]}{\delta \sigma_J} \right|_{\sigma = \sigma_0} = 0. \tag{14}$$

Thus this equation defines the minimum of the effective potential $V[\sigma_J]$.

From (9) and (14) it follows that

$$\mu_0^2 \sigma_0 = i \, \mathrm{Tr} \left. \left(\hat{S} \, \frac{\delta \hat{S}^{-1}}{\delta \sigma} \right) \right|_{\sigma = \sigma_0}. \tag{15}$$

This is the gap equation. It can be written in the momentum representation in the following form

$$\mu_0^2\sigma_0 = -ig_0 \operatorname{Tr} \int \frac{d^n k}{(2\pi)^n} \frac{1}{\gamma^\mu k_\mu + g_0\sigma_0}. \tag{16}$$

Performing the usual Wick rotation $k_0 \to ik_n$, $k_\alpha \to k_\alpha$ we find the gap equation for dynamical fermionic mass $M = g_0\sigma_0$ in Euclidean space

$$M = M\lambda_0 s I(0) \tag{17}$$

where s is the dimension of γ-matrices and

$$I(0) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2+M^2}. \tag{18}$$

3. *Influence of non-Euclidean topology on the fermionic condensate value*

In this section we show that the value of fermionic condensate which is ruled by the equation (17) depends on the space topology. The space-time is assumed to be 3-dimensional flat manifold with boundaries. There are eight 3-dimensional spaces with Euclidean signature which are locally Euclidean, while globally they are non-Euclidean due to boundaries. We examine here only the cases in which the space M_3 is the direct product of one-dimensional Euclidean space by a two-dimensional globally non-Euclidean space. There are only four different situations, namely, when two-dimensional subspaces are the cylinder, the Möbius strip, the torus and the Klein bottle. In all these situations different boundary conditions must be imposed on the spinor field. Let us consider the above mentioned cases in detailes.

3.1. Case $M_3 = R_1 \times R_1 \times S_1$

The fermions satisfy periodic boundary conditions

$$\psi(x_1, x_2, 0) = \psi(x_1, x_2, L).$$

These boundary conditions determine the discrete momentum

$$k_3(n) = \frac{2\pi}{L} n, \quad n \in N.$$

In order to find the influence of the boundary conditions (non-trivial topology) on the gap equation we must calculate the value $I(L)$, which is written for the cylindrical topology as

$$I(L) = \int \frac{d^2 k}{(2\pi)^2} \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{2\pi}{L} n\right)^2 + a^2}. \tag{19}$$

where $a^2 = k_1^2 + k_2^2 + M^2$.

We first perform the summation in (19)

$$\frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\frac{2\pi}{L}n\right)^2+a^2}=\frac{1}{2a}+\frac{1}{a(e^{La}-1)}. \tag{20}$$

Substituting this expression into (19) we have

$$I(L)=I(0)-\frac{1}{2\pi L}\ln(1-e^{-LM}). \tag{21}$$

It can be easily shown that the gap equation (17) for the nontrivial topologies takes the following form:

$$M=M\lambda_0 sI(L). \tag{22}$$

Integrals $I(0)$ and $I(L)$ diverge. Therefore, we have to introduce ultraviolet cutoff

$$\int_{-\Lambda}^{\Lambda}\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+M^2}\sim\Lambda.$$

The ultraviolet divergence of the gap equation can be removed by the renormalization of the λ_0 constant in such a way that

$$\lambda_0\rightarrow\bar{\lambda}=\lambda_0 I(0)s,\qquad \lambda_0/\bar{\lambda}=\Lambda.$$

Then the renormalized gap equation for the cylinder is

$$M=\bar{\lambda}M\left\{1-\frac{1}{\pi L\Lambda}\ln(1-e^{-LM})\right\}. \tag{23}$$

3.2. Case $M_3=R_1\times$ Möbius strip

On the spinor field should be imposed the anti-periodic boundary conditions

$$\psi(x_1,x_2,0)=-\psi(x_1,x_2,L)$$

and

$$k_3(n)=\frac{\pi}{L}(2n+1).$$

The value $I(L)$ for this case can be found by calculation of

$$I(L)=\int\frac{d^2k}{(2\pi)^2}\frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{1}{\left[\frac{\pi}{L}(2n+1)\right]^2+a^2}, \tag{24}$$

where

$$\frac{1}{L}\sum_{n=-\infty}^{\infty}\frac{1}{\left[\frac{\pi}{L}(2n+1)\right]^2+a^2} = \frac{1}{2a}-\frac{1}{a(e^{La}+1)}. \tag{25}$$

Inserting (25) into (24) we obtain

$$I(L) = I(0)+\frac{1}{2\pi L}\ln(1+e^{-LM}). \tag{26}$$

Formula (26) gives us an important result $I(L\to\infty)\to I(0)$. The gap equation for Möbius strip topology is given by

$$M = M\bar{\lambda}\left\{1+\frac{1}{\pi L\lambda}\ln(1+e^{-LM})\right\}. \tag{27}$$

3.3. Case $M_3 = R_1\times S_1\times S_1$

The torus topology has the periodical condition

$$\psi(x_1, 0, 0) = \psi(x_1, L_1, L_2)$$

and the momentum will be

$$k_2(n) = \frac{2\pi n}{L_1}, \quad k_3(m) = \frac{2\pi m}{L_2}, \quad n, m \in N.$$

To find the gap equation we have to calculate

$$I(L_1, L_2) = \frac{1}{L_1L_2}\int\frac{dk_1}{2\pi}\sum_{n,m=-\infty}^{\infty}\left[\left(\frac{2\pi n}{L_1}\right)^2+\left(\frac{2\pi m}{L_2}\right)^2+k_1^2+M^2\right]^{-1}. \tag{28}$$

The sum can be written is s-representation as

$$\sum_{m,n=-\infty}^{\infty}\left[\left(\frac{2\pi n}{L_1}\right)^2+\left(\frac{2\pi m}{L_2}\right)^2+k_1^2+M^2\right]^{-1}$$
$$= \int_0^{\infty} ds e^{-s(k_1{}^2+M^2)}\sum_{m,n=-\infty}^{\infty} e^{-s\left[\left(\frac{2\pi n}{L_1}\right)^2+\left(\frac{2\pi m}{L_2}\right)^2\right]}. \tag{29}$$

Let $L = L_1 = L_2$. Using the formulae

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{x}},$$

we have

$$\sum_{n,m=-\infty}^{\infty} e^{-s\frac{4\pi^2}{L^2}(n^2+m^2)} = \frac{L^2}{4\pi s}\left[1+4\left(\sum_{n=1}^{\infty} e^{-\frac{L^2n^2}{4s}} + \sum_{n,m=1}^{\infty} e^{-\frac{L^2}{4s}(n^2+m^2)}\right)\right]. \quad (30)$$

Combining the results (28), (29) and (30) we find

$$I(L) = \frac{1}{(4\pi)^{3/2}}\int_0^{\infty} dse^{-sM^2}s^{-3/2}\left[1+4\left(\sum_{n=1}^{\infty} e^{-\frac{L^2n^2}{4s}} + \sum_{n,m=1}^{\infty} e^{-\frac{L^2(n^2+m^2)}{4s}}\right)\right]. \quad (31)$$

A final result can be written in terms of the modified Bessel function

$$\int_0^{\infty} x^{\nu-1}e^{-\frac{\alpha}{x}-\gamma x}dx = 2\left(\frac{\alpha}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\alpha\gamma}).$$

Hence, we have

$$M = M\bar{\lambda}\left\{1+\sqrt{\frac{M}{L}}\,\frac{1}{(\pi)^{3/2}\Lambda}\left[\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}K_{-1/2}(nLM)\right.\right.$$

$$\left.\left.+\sum_{n,m=1}^{\infty}\left(\frac{1}{n^2+m^2}\right)^{1/4} K_{-1/2}(LM\sqrt{m^2+n^2})\right]\right\}. \quad (32)$$

This is the gap equation for the torus topology.

3.4. Case $M_3 = R_1 \times$ Klein bottle

In this case the anti-periodical conditions must be imposed on fermions

$$\psi(x_1, 0, 0) = -\psi(x_1, L, L)$$

and

$$k_2(n) = \frac{2\pi n}{L}, \quad k_3(m) = \frac{\pi}{L}(2m+1), \quad n, m \in N.$$

This situation is similar to the previous one, and we have to calculate the following expression

$$I(L) = \frac{1}{L^2}\int\frac{dk_1}{2\pi}\sum_{n,m=-\infty}^{\infty}\left[\left(\frac{2\pi n}{L}\right)^2+\left(\frac{\pi}{L}(2m+1)\right)^2+k_1^2+M^2\right]^{-1}. \quad (33)$$

The formula

$$\sum_{n=-\infty}^{\infty} e^{-\pi(n+\frac{1}{2})^{1/2}x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{\pi n^2}{L}}, \tag{34}$$

helps us to write the sum (33) in the form

$$\sum_{n,m=-\infty}^{\infty} e^{-s\left[\frac{4\pi^2 n^2}{L^2} + \frac{\pi^2}{L^2}(2m+1)^2\right]} = \frac{L^2}{4\pi s}\left[1 + 4\sum_{n,m=1}^{\infty} (-1)^m \right.$$

$$\left. \times e^{-\frac{L^2(n^2+m^2)}{4s}} + 2\sum_{m=1}^{\infty} (-1)^m e^{-\frac{L^2 m^2}{4s}} + 2\sum_{n=1}^{\infty} e^{-\frac{L^2 n^2}{4s}}\right]. \tag{35}$$

Then the gap equation is

$$M = M\bar{\lambda}\left\{1 + \sqrt{\frac{M}{L}} \frac{1}{(\pi)^{3/2}\Lambda}\left[\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}} K_{-1/2}(2LMn)\right.\right.$$

$$\left.\left. + \sum_{n,m=1}^{\infty} (-1)^m K_{-1/2}(LM\sqrt{m^2+n^2}) \left(\frac{1}{n^2+m^2}\right)^{1/4}\right]\right\}. \tag{36}$$

4. *Restoration of chiral symmetry*

We can see that in all the cases mentioned above the gap equations have non-trivial solutions along with trivial solution $M = 0$. The non-trivial solutions lead to the dynamical breakdown of the chiral symmetry. These solutions depend on topological parameter L and on renormalized constant $\bar{\lambda}$. The latter is governed by the Gell-Mann-Low equation, and L can be changed independently. The topological parameter can be regarded as the size of a space region occupied by the fermions.

The most important problem is the investigation of stability conditions of trivial solutions of the gap equations, because the existence of these solutions leads to the restoration of chiral symmetry and to massless fermions. We consider here only the dependence of the stable conditions on the topological parameter and we will show that in some cases the stable trivial solutions can take place at some values of the topological parameter.

The stability conditions for trivial solution of the equation

$$M = f(M, \bar{\lambda}, L) \tag{37}$$

can be found by means of the bifurcation theory [14]. The solution $M = b$ of this equation is stable when

$$f' = (b, \bar{\lambda}, L) < 1 \tag{38}$$

and is unstable when

$$f'(b, \bar{\lambda}, L) \geqslant 1. \tag{39}$$

The function $f(M, \bar{\lambda}, L)$ for the cylindrical topology of the two-dimensional subspace (see 2.1) is

$$f_c(M, \bar{\lambda}, L) = M\bar{\lambda}\left[1 - \frac{1}{\pi \Lambda L} \ln(1 - e^{-LM})\right] \tag{40}$$

and for the Möbius strip topology of the two-dimensional subspace (see 2.3) $f(M, \bar{\lambda}, L)$ is

$$f_M(M, \bar{\lambda}, L) = M\bar{\lambda}\left[1 + \frac{1}{\pi L \Lambda} \ln(1 + e^{-LM})\right]. \tag{41}$$

The first function (40) defines

$$f_c'(0, \bar{\lambda}, L) \to \infty. \tag{42}$$

So, there are no stable trivial solutions if the manifold has the topology of $\mathbf{R}_1 \times \mathbf{R}_1 \times \mathbf{S}_1$ and the fermions are mainly massive

$$M_{f(c)} = -\frac{1}{L} \ln(1 - e^{-\frac{1-\bar{\lambda}}{\bar{\lambda}} \pi L \Lambda}). \tag{43}$$

The second function (41) leads to

$$f_M'(0, \bar{\lambda}, L) = \bar{\lambda}\left[1 + \frac{1}{\pi L \Lambda} \ln 2\right] \tag{44}$$

and in this case there is non-stable point

$$L_c = \frac{\bar{\lambda}}{1 - \bar{\lambda}} \frac{\ln 2}{\pi \Lambda}. \tag{45}$$

According to (38) and (39) the dynamical mass (or fermionic condensate) appears when $L < L_c$, and evaporates when $L > L_c$. Thus, there are two regions in the size scale. The first region is the region in which the fermions are mainly massive ($L < L_c$) and

$$M_{f(M)} = -\frac{1}{L} \ln(e^{\ln 2 \cdot \frac{L}{L_c}} - 1).$$

The other region is that of the massless fermions, when the chiral symmetry is restored.

Unfortunately the gap equation in the cases of torus topology of two-dimensional subspace and Klein bottle topology of two-dimensional subspace are very complicated and cannot be analyzed in any explicit form. But we believe that the correlation of these cases will be analogous to those written above. This is only intuitive proposition and to make the full analysis it is necessary to use a computer. We intend to publish the results in our forthcoming papers.

The above results indicate not only the existence of non-trivial solutions of gap equations in the topologically different spaces but indicate the dependence of the value of fermionic condensate on the size of space-time.

The evaporation of condensate and restoration of chiral symmetry proceed in different ways and are ruled by topology. There are topologies in which these phenomena do not take place. We believe that this will be important in the bag models, because the energy of bag is strongly dependent on non-perturbative effects and boundary conditions.

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