# AN APPROXIMATE ELECTROSTATIC POTENTIALS BETWEEN MASSIVE SPIN-0, DUFFIN-KEMMER AND SPIN-$\frac{1}{2}$ DIRAC PARTICLES. II*

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*(Received January 28, 1988; revised version received February 8, 1989)*

Two-body relativistic electrostatic potentials between charged, massive, spin-0 and massive, spin-$\frac{1}{2}$ Dirac particles are derived using Barut's method. Only terms proportional to products of the electric charges are considered.

PACS numbers: 11.10.Qr

In this paper we shall address ourselves to the same as in [1] problem of finding an approximate electrostatic potential between charged scalar and spinor particles. The difference, however, consists using the Duffin–Kemmer description of a scalar particle [2, 3] instead of the usual Klein–Gordon description. This has the advantage as the resulting equations for the bound-state wave functions become linear in the energy variable, and become one-time equations.

We start with the Lagrangian

$$L = -A^{\dagger}(id+m)A-\tilde{A}^{\dagger}(id+\tilde{m})\tilde{A}$$

$$+\bar{\psi}(i\partial\!\!\!/-M)\psi+\bar{\tilde{\psi}}(i\partial\!\!\!/-\tilde{M})\tilde{\psi}-e\tilde{e}\int dy\bar{D}(x-y)j^{\mu}(x)\tilde{j}_{\mu}(y). \tag{1}$$

where

$$j^{\mu} = A^{\dagger}\beta^{\mu}A-\bar{\psi}\gamma^{\mu}\psi \tag{2}$$

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* The work supported by the CPBP 01.03 Program of the Ministry of National Education.

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and similarly for $\tilde{j}^{\mu}$ with fields $\tilde{A}$ and $\tilde{\psi}$ replacing $A$ and $\psi$,

$$\partial\!\!\!/ = \gamma^{\mu}\partial_{\mu} \quad \text{and} \quad d = \beta^{\mu}\partial_{\mu}. \tag{3}$$

The Duffin–Kemmer matrices $\beta^{\mu}$ are taken in the form

$$\beta^{0} = \begin{bmatrix} 0 & i & 0 & 0 & 0 \\ -i & & & & \\ 0 & & & & \\ 0 & & & 0 & \\ 0 & & & & \end{bmatrix}, \quad \beta^{1} = \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & & & & \\ i & & & & \\ 0 & & & 0 & \\ 0 & & & & \end{bmatrix},$$

$$\beta^{2} = \begin{bmatrix} 0 & 0 & 0 & i & 0 \\ 0 & & & & \\ 0 & & & & \\ i & & 0 & & \\ 0 & & & & \end{bmatrix}, \quad \beta^{3} = \begin{bmatrix} 0 & 0 & 0 & 0 & i \\ 0 & & & & \\ 0 & & & & \\ 0 & & & 0 & \\ i & & & & \end{bmatrix}. \tag{4}$$

and satisfy the relations (cf. e.g. [4])

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda}+\beta^{\lambda}\beta^{\nu}\beta^{\mu} = \beta^{\mu}g^{\lambda\nu}+\beta^{\lambda}g^{\mu\nu}, \qquad \mu, \nu, \lambda = 0, 1, 2, 3 \tag{5}$$

and

$$\|g^{\mu\nu}\| = \mathrm{diag}\,(1, -1, -1, -1).$$

The fields $A$ and $\tilde{A}$ are given by the five-dimensional column in terms of the scalar fields $\phi$ and $\tilde{\phi}$,

$$A = \begin{bmatrix} m^{1/2} \\ -m^{-1/2}D_0 \\ m^{-1/2}D_1 \\ m^{-1/2}D_2 \\ m^{-1/2}D_3 \end{bmatrix}\phi, \quad \tilde{A} = \begin{bmatrix} \tilde{m}^{1/2} \\ -\tilde{m}^{-1/2}D_0 \\ \tilde{m}^{-1/2}D_1 \\ \tilde{m}^{-1/2}D_2 \\ \tilde{m}^{-1/2}D_3 \end{bmatrix}\tilde{\phi}, \tag{6}$$

where $D_{\mu}$ is a covariant derivative

$$D_{\mu}\phi = \partial_{\mu}\phi+ieA_{\mu}\phi, \quad D_{\mu}\tilde{\phi} = \partial_{\mu}\tilde{\phi}+i\tilde{e}A_{\mu}\tilde{\phi}.$$

Conjugated fields are formed by the Hermitian conjugation and multiplication by the matrix of conjugation $\Gamma$:

$$\Gamma = \mathrm{diag}\,(1, 1, -1, -1, -1), \tag{7}$$

$$A^{\dagger} = [m^{1/2}, \quad -m^{-1/2}D_0^{*}, \quad -m^{-1/2}D_1^{*}, \quad -m^{-1/2}D_2^{*}, \quad -m^{-1/2}D_3^{*}]\phi^{*}, \tag{8}$$

$$\tilde{A}^{\dagger} = [\tilde{m}^{1/2}, \quad -\tilde{m}^{-1/2}D_0^{*}, \quad -\tilde{m}^{-1/2}D_1^{*}, \quad -\tilde{m}^{-1/2}D_2^{*}, \quad -\tilde{m}^{-1/2}D_3^{*}]\tilde{\phi}^{*}. \tag{9}$$

The electromagnetic field $A^{\mu}$ has been eliminated by means of the formula (5) of [1], and the scalar fields $\phi$, $\tilde{\phi}$ are replaced now by the fields $A$, $\tilde{A}$.

We introduce now the bilocal fields as follows

$$\Phi(x, y) \equiv A(x) \otimes \tilde{A}(y), \quad 25 \text{ components}, \tag{10}$$

$$\Psi(x, y) \equiv A(x) \otimes \tilde{\psi}(y), \quad 20, \tag{11}$$

$$\tilde{\Psi}(x, y) \equiv \psi(x) \otimes \tilde{A}(y), \quad 20, \tag{12}$$

$$\Theta(x, y) \equiv \psi(x) \otimes \tilde{\psi}(y), \quad 16. \tag{13}$$

Due to the global U(1) invariances of the Lagrangian the zero-components of the currents $j^0$ and $\tilde{j}^0$ lead to the following time-independent expressions

$$\int d^3y(-A^{\dagger}\beta^0 A + \bar{\psi}\gamma^0\psi)(y) = 1, \tag{14}$$

and

$$\int d^3y(-\tilde{A}^{\dagger}\beta^0 \tilde{A} + \bar{\tilde{\psi}}\gamma^0\tilde{\psi})(y) = 1. \tag{15}$$

The action can be rewritten in terms of the bilocal fields and yields

$$S = S_{\Phi} + S_{\Psi} + S_{\tilde{\Psi}} + S_{\Theta}, \tag{16}$$

where we have denoted the subsequent pieces as:

$$S_{\Phi} = \int dxdy\Delta(x, y) \cdot \left\{\Phi\left[\beta^0 \otimes (id + \tilde{m}) + (id + m) \otimes \beta^0 - \frac{e\tilde{e}}{4\pi r}\beta^{\mu} \otimes \beta_{\mu}\right]\Phi\right\}, \tag{17}$$

$$S_{\Psi} = \int dxdy\Delta(x, y) \cdot \left\{\Psi\left[-\beta^0 \otimes (i\partial - \tilde{M}) - (id + m) \otimes \gamma^0 + \frac{e\tilde{e}}{4\pi r}\beta^{\mu} \otimes \gamma_{\mu}\right]\Psi\right\}, \tag{18}$$

$$S_{\tilde{\Psi}} = \int dxdy\Delta(x, y) \cdot \left\{\tilde{\Psi}\left[-\gamma^0 \otimes (id + \tilde{m}) - (i\partial - M) \otimes \beta^0 + \frac{e\tilde{e}}{4\pi r}\gamma^{\mu} \otimes \beta_{\mu}\right]\tilde{\Psi}\right\}, \tag{19}$$

$$S_{\Theta} = \int dxdy\Delta(x, y) \cdot \left\{\Theta\left[\gamma^0 \otimes (i\partial - \tilde{M}) + (i\partial - M) \otimes \gamma^0 - \frac{e\tilde{e}}{4\pi r}\gamma^{\mu} \otimes \gamma_{\mu}\right]\Theta\right\}, \tag{20}$$

and the function $\Delta(x, y)$ is given by the formula

$$\Delta(x, y) \equiv \tfrac{1}{2}\left[\delta(x^0 - y^0 - r) + \delta(x^0 - y^0 + r)\right], \qquad r \equiv |\vec{x} - \vec{y}|. \tag{21}$$

The field equations for $\Phi$, $\Psi$, $\tilde{\Psi}$ and $\Theta$ follow then by variations with the respect to the conjugated fields. They are of the first-order in the derivatives and read:

$$\left[\beta^0 \otimes (id + \tilde{m}) + (id + m) \otimes \beta^0 - \frac{e\tilde{e}}{4\pi r}\beta^{\mu} \otimes \beta_{\mu}\right]\Phi = 0, \tag{22}$$

$$\left[\beta^0 \otimes (i\partial - \tilde{M}) + (id + m) \otimes \gamma^0 - \frac{e\tilde{e}}{4\pi r}\beta^{\mu} \otimes \gamma_{\mu}\right]\Psi = 0, \tag{23}$$

$$\left[\gamma^0 \otimes (id+\tilde{m})+(i\tilde{\partial}-M) \otimes \beta^0 - \frac{e\tilde{e}}{4\pi r}\gamma^\mu \otimes \beta_\mu\right]\tilde{\Psi} = 0, \tag{24}$$

$$\left[\gamma^0 \otimes (i\tilde{\partial}-\tilde{M})+(i\tilde{\partial}-M) \otimes \gamma^0 - \frac{e\tilde{e}}{4\pi r}\gamma^\mu \otimes \gamma_\mu\right]\Theta = 0. \tag{25}$$

This system of equations we want to study more closely. First we pass from the coordinates $(x, y)$ to the relative coordinates and coordinates of the centre-of-mass:

$$r_\mu = x_\mu - y_\mu, \quad R_\mu = ax_\mu+(1-a)y_\mu, \tag{26}$$

thus

$$x_\mu = R_\mu+(1-a)r_\mu, \quad y_\mu = R_\mu - ar_\mu. \tag{27}$$

For the momenta we have

$$\varrho_1^\mu = \alpha P^\mu + p^\mu, \quad p_2^\mu = (1-\alpha)P^\mu - p^\mu. \tag{28}$$

Transforming the derivatives we obtain the following set of equations for the bilocal fields

$$\left[\tilde{m}\beta^0 \otimes \mathbf{1}+m\mathbf{1} \otimes \beta^0 + P_\mu(-\beta^0 \otimes \beta^\mu + \alpha\Gamma^\mu)+p_k\Gamma^k - \frac{e\tilde{e}}{4\pi r}\beta^\mu \otimes \beta_\mu\right]\Phi = 0, \tag{29}$$

$$\left[-\tilde{M}\beta^0 \otimes \mathbf{1}+m\mathbf{1} \otimes \gamma^0 + P_\mu(-\beta^0 \otimes \gamma^\mu + aT^\mu)+p_kT^k - \frac{e\tilde{e}}{4\pi r}\beta^\mu \otimes \gamma_\mu\right]\Psi = 0, \tag{30}$$

$$\left[\tilde{m}\gamma^0 \otimes \mathbf{1}-M\mathbf{1} \otimes \beta^0 + P_\mu(-\gamma^0 \otimes \beta^\mu + a\hat{T}^\mu)+p_k\tilde{T}^k - \frac{e\tilde{e}}{4\pi r}\gamma^\mu \otimes \beta_\mu\right]\tilde{\Psi} = 0, \tag{31}$$

$$\left[-\tilde{M}\gamma^0 \otimes \mathbf{1}-M\mathbf{1} \otimes \gamma^0 + P_\mu(-\gamma^0 \otimes \gamma^\mu + aS^\mu)+p_kS^k - \frac{e\tilde{e}}{4\pi r}\gamma^\mu \otimes \gamma_\mu\right]\Theta = 0, \tag{32}$$

where the matrices $\Gamma^\mu$, $T^\mu$, $\tilde{T}^\mu$ and $S^\mu$ are defined as follows:

$$\Gamma^\mu \equiv \beta^0 \otimes \beta^\mu - \beta^\mu \otimes \beta^0, \quad T^\mu \equiv \beta^0 \otimes \gamma^\mu - \beta^\mu \otimes \gamma^0,$$

and

$$S^\mu \equiv \gamma^0 \otimes \gamma^\mu - \gamma^\mu \otimes \gamma^0, \quad \tilde{T}^\mu \equiv \gamma^0 \otimes \beta^\mu - \gamma^\mu \otimes \beta^0,$$

$$\mu = 0, 1, 2, 3, \quad k = 1, 2, 3. \tag{33}$$

We see that the time-derivative $p_0$ does not appear in the equations (29)–(32) what permits to consider the functions $\Phi$, $\Psi$, $\tilde{\Psi}$ and $\Theta$ as not depending on $r^0$

$$\Phi(x^\mu, y^\mu) \equiv \Phi(R^\mu, r^\mu) = \Phi(R^\mu, r^k), \tag{34}$$

and similarly for other wave functions.

For the stationary case

$$\Phi(R^\mu, \vec{r}) = \Phi(\vec{R}, \vec{r}) \cdot \exp(-iR^0E), \tag{35}$$

(and similarly for $\Psi$, $\tilde{\Psi}$ and $\Theta$) the equations for $\Phi(\vec{R}, \vec{r})$ and other functions simplify. In the frame of the centre-of-mass, $\vec{P} = 0$, they read

$$\left[\tilde{m}\beta^0 \otimes \mathbf{1} + m\mathbf{1} \otimes \beta^0 + p_k\Gamma^k - \frac{e\tilde{e}}{4\pi r}\beta^\mu \otimes \beta_\mu + \beta^0 \otimes \beta^0 E\right]\Phi = 0, \tag{36}$$

$$\left[-\tilde{M}\beta^0 \otimes \mathbf{1} + m\mathbf{1} \otimes \gamma^0 + p_kT^k - \frac{e\tilde{e}}{4\pi r}\beta^\mu \otimes \gamma_\mu + \beta^0 \otimes \gamma^0 E\right]\Psi = 0, \tag{37}$$

$$\left[\tilde{m}\gamma^0 \otimes \mathbf{1} - M\mathbf{1} \otimes \beta^0 + p_k\tilde{T}^k - \frac{e\tilde{e}}{4\pi r}\gamma^\mu \otimes \beta_\mu + \gamma^0 \otimes \beta^0 E\right]\tilde{\Psi} = 0, \tag{38}$$

$$\left[-\tilde{M}\gamma^0 \otimes \mathbf{1} - M\mathbf{1} \otimes \gamma^0 + p_kS^k - \frac{e\tilde{e}}{4\pi r}\gamma^\mu \otimes \gamma_\mu + \gamma^0 \otimes \gamma^0 E\right]\Theta = 0. \tag{39}$$

These are the two-body relativistic equations containing the relevant potentials which we were aiming at. We omitted here the terms $|\phi|^4$ in the Lagrangian since our main purpose was to derive lower order differential equations for the bilocal fields. We shall include all the relevant terms in case of the supersymmetric extension of QED which we are calculating at the moment.

## REFERENCES


[1] W. Garczyński, M. Klimek, *Acta Phys. Pol.* **B20**, 755 (1989).
[2] R. J. Duffin, *Phys. Rev.* **54**, 1114 (1938).
[3] N. Kemmer, *Proc. R. Soc. London* **A173**, 91 (1939).
[4] H. Umezawa, *Quantum Field Theory*, North-Holland Publish Co., Amsterdam 1956.
[5] A. O. Barut, S. Komy, *Fortschr. Phys.* **33**, 309 (1985).
[6] A. O. Barut, N. Ünal, *Fortschr. Phys.* **33**, 319 (1985).
[7] W. Królikowski, *Acta Phys. Pol.* **B10**, 739 (1979); **B12**, 793 (1981); **B14**, 97 (1983); **B14**, 109 (1983).
[8] W. Królikowski, A. Turski, *Acta Phys. Pol.* **B17**, 75 (1986).
[9] A. Turski, Ph. D. Thesis, ITP,Warsaw Univ. 1985, unpublished.
[10] O. W. Greenberg, J. Sucher, *Phys. Lett.* **99B**, 339 (1981).
[11] D. B. Lichtenberg, W. Namgung, E. Predazzi, J. G. Wills, *Phys. Rev. Lett.* **48**, 1653 (1982).
[12] G. T. Bodwin, D. R. Yennie, *Phys. Rep.* **C43**, 267 (1978).
[13] G. P. Lepage, *Phys. Rev.* **A16**, 863 (1977).
[14] W. W. Buck, F. Gross, *Phys. Rev.* **D20**, 2361 (1979).