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DYNAMICAL MEANING OF THE ADIABATIC QUANTUM PHASE*

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The interaction between the "macrosystem" (corresponding to the so-called slow variables) and the "microsystem" (corresponding to the so-called fast variables) is considered. By using the Born-Oppenheimer adiabatic approximation it is shown that such interaction modifies dynamics of the "macrosystem". We find the form of modified Poisson brackets for the "macrosystem". We also show how symmetries and laws of conservation of the "macrosystem" are influenced by this interaction.

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1. Introduction

In quantum mechanics one usually deals with systems composed of several subsystems interacting with each other. An often used method to describe the interaction between subsystems is based on the so-called adiabatic theorem. In general, this theorem can be stated in the following way [1]: Let us consider two interacting systems and let us freeze out the motion of one of them (the "macrosystem"), then the dynamical variables of the frozen system appear as external parameters in the Hamiltonian of the remaining ("micro") system. If we solve the Schrödinger equation corresponding to the "microsystem" with the above assumptions, we find that a slow evolution of external parameters does not cause any quantum transition between different energetic levels. Of course, the assumptions of the adiabatic theorem are not fully satisfied in real physical processes taking place at a finite time interval. However, the notion of the adiabatic change provides a useful hypothesis for discussing the wide class of quantum processes including quantum field theory.

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It has been recently shown that the traditional quantum adiabatic theorem is incorrect in some special cases: namely in the case when external parameters change along a closed loop, the wave function of the "microsystem" should be multiplied by certain phase term. This term is of topological character and depends on the loop configuration in the phase--space of external parameters [2–5]. It is associated with the first Chern class of natural Hermitian connection in the Hilbert fibre bundle over the manifold of parameters. This discovery, due to M. V. Berry, proves to be useful in a variety of situations. For example, it has been applied to investigate the problem of semiclassical quantization [6], as well as in the case of problems associated with the interpretation of gauge anomalies [7, 8]; by means of it anomalous commutators in chiral gauge theories have been determined [9–11]. A review of problems connected with the adiabatic theorem can be found in two papers by R. Jackiw [12, 13].

The aim of the present paper is to show how the interaction of "macro" and "micro" systems, within the frame of Born-Oppenheimer adiabatic approximation, modifies dynamics of the "macrosystem". In Section 2, we calculate equations of motion modified by this interaction for the classical version of the "macrosystem". We find modified Poisson brackets along with the symplectic form corresponding to the "macrosystem". By choosing a special case of the "microsystem", we show in Section 3, how the assumed spherical symmetry of the "macrosystem" is modified by interactions with "microsystem". Section 4 contains conclusions and suggestions concerning the application of our results in the field of chiral gauge theories.

2. *Modification of the symplectic structure*

Let us consider a quantum-mechanical system described by the hamiltonian

$$\hat{\mathbf{H}} = \hat{H}_0(\hat{P}, \lambda) + \hat{H}(\hat{p}, x, \lambda), \tag{2.1}$$

where

$$\hat{P}_\mu = -i\hbar \frac{\partial}{\partial \lambda^\mu}; \quad \mu = 1, \ldots, N$$

$$\hat{p}_i = -i\hbar \frac{\partial}{\partial x^i}; \quad i = 1, \ldots, M,$$

The system described by the Hamiltonian $\hat{H}_0$ will be called "macrosystem", and the system described by the Hamiltonian $\hat{H}$ — "microsystem". Let us assume that there exists a complete set of functions $\{u_m(x, \lambda)\}$ such that

$$\hat{H}(\hat{p}, x, \lambda)u_m(x, \lambda) = E_m(\lambda)U_m(x, \lambda)$$

and

$$(u_m, u_n) = \int d^M x u_m^*(x, \lambda) u_n(x, \lambda) = \delta_{mn}, \quad \text{for arbitrary } \lambda.$$

Moreover, we assume that the Hamiltonian $\hat{H}(p, x, \lambda)$ is nondegenerate. The wave function ψ of the system described by Hamiltonian $|\hat{\mathbf{H}}|$ can be presented in terms of a complete set of functions $\{u_m\}$

$$\psi(\lambda, x; t) = \sum_m \Phi_m(\lambda, t) u_m(x, \lambda). \tag{2 2}$$

For $m \neq n$, we have:

$$\left(u_m, \frac{\partial u_n}{\partial \lambda^\mu}\right) = \frac{\left(u_m, \dfrac{\partial \hat{H}}{\partial \lambda^\mu} u_n\right)}{E_n - E_m}. \tag{2.3}$$

We assume that the Hamiltonian $\hat{H}(\hat{p}, x, \lambda)$ is a slow variable in λ^μ-parameters; this means that $\left|\left(u_m, \frac{\partial \hat{H}}{\partial \lambda^\mu} u_n\right) \Delta\lambda^\mu_{(m,n)}\right| \cdot |E_n - E_m|^{-1} \ll 1$, where $\Delta\lambda^\mu_{(m,n)}$ corresponds to a characteristic shift of parameters λ^μ during the Bohr time interval $\Delta T_{(m,n)} = h|E_n - E_m|^{-1}$. With the above assumptions the terms $\left(u_n, \frac{\partial u_m}{\partial \lambda^\mu}\right)$ can be neglected, for $m \neq n$. This is the basic assumption of the Born-Oppenheimer adiabatic approximation. By using this assumption we shall now determine equations satisfied by wave functions $\Phi_n(\lambda, t)$ of the "macrosystem". Because the wave function $\psi(\lambda, x; t)$ satisfies the Schrödinger equation, one has

$$i\hbar \sum_m u_m \frac{\partial}{\partial t} \Phi_m(\lambda, t) = \sum_m [\hat{H}_0(\hat{P}, \lambda) + \hat{H}(\hat{p}, x, \lambda)] \Phi_m u_m. \tag{2.4}$$

Let us project this equation onto a fixed wave function $u_n(x, \lambda)$

$$i\hbar \frac{\partial}{\partial t} \Phi_n(\lambda. t) = \sum_m (u_n, \hat{H}_0(\hat{P}, \lambda) \Phi_m(\lambda, t) u_m) + E_n(\lambda) \Phi_n(\lambda, t).$$

We must calculate the term $\sum_m (u_n, \hat{H}_0(\hat{P}, \lambda)\Phi_m(\lambda, t)u_m)$. In order to do this, let us assume that the Hamiltonian $\hat{H}_0(\hat{P}, \lambda)$ has the following general form:

$$\hat{H}_0\left(-i\hbar \frac{\partial}{\partial \lambda^\mu}, \lambda\right) = V(\lambda) + \sum_n V(\lambda)^{\mu_1 \ldots \mu_z} \left(-i\hbar \frac{\partial}{\partial \lambda^{\mu_1}}\right) \circ \ldots \circ \left(-i\hbar \frac{\partial}{\partial \lambda^{\mu_n}}\right). \tag{2.5}$$

We shall show, by using the Born-Oppenheimer approximation, that the following estimation is valid:

$$\sum_m \left(u_n, -i\hbar \frac{\partial}{\partial \lambda^{\mu_1}} \circ \ldots \circ -i\hbar \frac{\partial}{\partial \lambda^{\mu_k}} (\Phi_m(\lambda, t) u_m)\right)$$

$$\cong \left(-i\hbar \frac{\partial}{\partial \lambda^{\mu_1}} - \hbar A^{(n)}_{\mu_1}\right) \circ \ldots \circ \left(-i\hbar \frac{\partial}{\partial \lambda^{\mu_k}} - \hbar A^{(n)}_{\mu_k}\right) \Phi_n(\lambda, t). \tag{2.6}$$

where

$$A^{(n)}_{\mu_j}(\lambda) = i\left(u_n, \frac{\partial}{\partial\lambda^{\mu_j}} u_n\right) \tag{2.7}$$

is the so-called Berry connection.

We shall prove the equation (2.6) by induction with respect to k. For $k = 1$, we have:

$$\sum_m \left(u_n, -i\hbar \frac{\partial}{\partial\lambda^\mu} (\Phi_m(\lambda, t)u_m)\right) = \sum_m \left(u_n, \left(-i\hbar \frac{\partial}{\partial\lambda^\mu} \Phi_m\right) u_m\right)$$

$$-i\hbar \sum_m \left(u_n, \frac{\partial u_m}{\partial\lambda^\mu}\right) \Phi_m(\lambda, t) \cong \left(-i\hbar \frac{\partial}{\partial\lambda^\mu} - \hbar A^{(n)}_\mu\right) \Phi_n(\lambda, t). \tag{2.8}$$

If the equation (2.6) is correct for k, then for $(k+1)$ we obtain:

$$\sum_m \left(u_n, -i\hbar \frac{\partial}{\partial\lambda^{\mu_1}} \circ \ldots \circ -i\hbar \frac{\partial}{\partial\lambda^{\mu_{k+1}}} (\Phi_m u_m)\right) = \sum_m \left(u_n, \left[-i\hbar \frac{\partial}{\partial\lambda^{\mu_1}}\right.\right.$$

$$\left.\left.\circ \ldots \circ -i\hbar \frac{\partial}{\partial\lambda^{\mu_k}} \left(-i\hbar \frac{\partial\Phi_m}{\partial\lambda^{\mu_{k+1}}}\right)\right] u_m\right) + \sum_m \left(u_n, \left[-i\hbar \frac{\partial}{\partial\lambda^{\mu_1}}\right.\right.$$

$$\left.\left.\circ \ldots \circ -i\hbar \frac{\partial}{\partial\lambda^{\mu_k}} \Phi_m\right]\left(-i\hbar \frac{\partial u_m}{\partial\lambda^{\mu_{k+1}}}\right)\right).$$

It is known that:

$$-i\hbar \frac{\partial}{\partial\lambda^{\mu_{k+1}}} u_m = \sum_r u_r \left(u_r, -i\hbar \frac{\partial u_m}{\partial\lambda^{\mu_{k+1}}}\right) \cong u_m \left(u_m, -i\hbar \frac{\partial u_m}{\partial\lambda^{\mu_{k+1}}}\right),$$

hence:

$$\sum_m \left(u_n, -i\hbar \frac{\partial}{\partial\lambda^{\mu_1}} \circ \ldots \circ -i\hbar \frac{\partial}{\partial\lambda^{\mu_{k+1}}} [\Phi_m u_m]\right)$$

$$= \left(-i\hbar \frac{\partial}{\partial\lambda^{\mu_1}} - \hbar A^{(n)}_{\mu_1}\right) \circ \ldots \circ \left(-i\hbar \frac{\partial}{\partial\lambda^{\mu_k}} - \hbar A^{(n)}_{\mu_k}\right)\left[-i\hbar \frac{\partial\Phi_n}{\partial\lambda^{\mu_{k+1}}}\right]$$

$$+ \sum_m \left(u_n, -i\hbar \frac{\partial}{\partial\lambda^{\mu_1}} \circ \ldots \circ -i\hbar \frac{\partial}{\partial\lambda^{\mu_k}} \left[\Phi_m u_m \left(u_m, -i\hbar \frac{\partial u_m}{\partial\lambda^{\mu_{k+1}}}\right)\right]\right)$$

$$= \left(-i\hbar \frac{\partial}{\partial\lambda^{\mu_1}} - \hbar A^{(n)}_{\mu_1}\right) \circ \ldots \circ \left(-i\hbar \frac{\partial}{\partial\lambda^{\mu_{k+1}}} - \hbar A^{(n)}_{\mu_{k+1}}\right) \Phi_n,$$

which proves the equation (2.6).

This implies that:

$$\sum_m (u_n, \hat{H}_0(\hat{P}, \lambda)\Phi_m(\lambda, t)u_m) = \hat{H}_0\left(-i\hbar \frac{\partial}{\partial\lambda^\mu} - \hbar A_\mu^{(n)}, \lambda\right)\Phi_n(\lambda, t). \tag{2.9}$$

Hence the wave function $\Phi_n(\lambda, t)$ describing the "macrosystem" satisfies the following effective Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\Phi_n(\lambda, t) = \left[\hat{H}_0\left(-i\hbar \frac{\partial}{\partial\lambda^\mu} - \hbar A_\mu^{(n)}, \lambda\right) + E_n(\lambda)\right]\Phi_n(\lambda, t). \tag{2.10}$$

We see that the interaction between "macro" and "micro-systems", within the frame of the Born-Oppenheimer approximation, leads to the following modification of the Hamiltonian corresponding to the "macrosystem":

$$\hat{H}_0\left(-i\hbar \frac{\partial}{\partial\lambda^\mu}, \lambda\right) \to \hat{H}_0\left(-i\hbar \frac{\partial}{\partial\lambda^\mu} - \hbar A_\mu^{(n)}, \lambda\right) + E_n(\lambda) \equiv \hat{H}_{\text{eff}}. \tag{2.11}$$

Now, let us assume that there exists a classical system described by the Lagrangian L_0, which is equivalent to the system described by the Hamiltonian $\hat{H}_0$.

We shall find the form of the Lagrangian L_{eff} of this classical system corresponding to the quantum system described by the Hamiltonian $\hat{H}_{\text{eff}}$. This Lagrangian is the following:

$$L_{\text{eff}} = L_0 + \hbar A_\mu^{(n)}\dot{\lambda}^\mu - E_n(\lambda). \tag{2.12}$$

This is so because the classical Hamiltonian corresponding to this Lagrangian is:

$$H_{\text{eff}} = \tilde{P}_\mu\dot{\lambda}^\mu - L_{\text{eff}},$$

where

$$\tilde{P}_\mu = \frac{\partial L_{\text{eff}}}{\partial\dot{\lambda}^\mu} = P_\mu + \hbar A_\mu^{(n)} \tag{2.13}$$

and

$$P_\mu = \frac{\partial L_0}{\partial\dot{\lambda}_\mu}. \tag{2.14}$$

Thus

$$\begin{aligned} H_{\text{eff}} &= (P_\mu + \hbar A_\mu^{(n)})\dot{\lambda}^\mu - L_0 - \hbar A_\mu^{(n)}\dot{\lambda}^\mu + E_n(\lambda) \\ &= P_\mu\dot{\lambda}^\mu - L_0 + E_n(\lambda) = H_0 + E_n(\lambda), \end{aligned} \tag{2.15}$$

where $H = P_\mu\dot{\lambda}^\mu - L_0$ is a classical Hamiltonian of the system described by the Lagrangian L_0. We see that

$$H_{\text{eff}} = H_0(P_\mu, \lambda) + E_n(\lambda) = H_0((\tilde{P}_\mu - \hbar A_\mu^{(n)}), \lambda) + E_n(\lambda). \tag{2.16}$$

By replacing the generalized momenta $\tilde{P}_\mu$ by operators $-i\hbar \frac{\partial}{\partial \lambda^\mu}$ one obtains the quantum theory from a classical one described by the Lagrangian L_{eff}. Hence the Hamiltonian operator of the system described at the classical level by L_{eff} is

$$\hat{H}_{\text{eff}} = \hat{H}_0\left(\left(-i\hbar \frac{\partial}{\partial \lambda^\mu} - \hbar A_\mu^{(n)}\right), \lambda\right) + E_n(\lambda)$$

which proves the correctness of (2.12).

Our consideration can be summarized in the form of the following diagram:

Classical level "macrosystem" $L_0(\lambda, \dot{\lambda})$	"+"	Quantum level adiabatic interaction with "microsystem" (Born-Oppenheimer approximation) $\hat{H}_{\text{eff}}$

↙

Classical level "macrosystem", $L_{\text{eff}}(\lambda, \dot{\lambda}, A_\mu^{(n)})$

Let us discuss the problem at the classical level. We have shown that the adiabatic interaction between "macro" and "micro-systems" modifies the Lagrangian of the "macrosystem":

$$L_0(\lambda, \dot{\lambda}) \rightarrow L_{\text{eff}}(\lambda, \dot{\lambda}, A_\mu^{(n)}).$$

Let us find the equations of motion of the system described by the Lagrangian $L_{\text{eff}}(\lambda, \dot{\lambda}, A_\mu^{(n)})$. It is well known that the Lagrange-Euler equations for the functional

$$\int dt [P_\mu \dot{\lambda}^\mu - H_0(P, \lambda) + \hbar A_\mu^{(n)} \dot{\lambda}^\mu - E_n(\lambda)]$$

in $2N$-dimensional space (λ^μ, P_ν) $(\mu, \nu = 1, \ldots, N)$, are equivalent to the Hamilton equations. These equations are:

$$\dot{\lambda}^\mu = \frac{\partial H_{\text{eff}}}{\partial P_\mu},$$

$$\dot{P}_\mu = -\frac{\partial H_{\text{eff}}}{\partial \lambda^\mu} + \hbar[\partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}]. \tag{2.17}$$

Having defined Berry's curvature two-form:

$$F_{\mu\nu}^{(n)} = \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)} \tag{2.18}$$

we arrive at the following Hamilton equations:

$$\dot{\lambda}^\mu = \frac{\partial H_{\text{eff}}}{\partial P_\mu},$$

$$\dot{P}_\mu = -\frac{\partial H_{\text{eff}}}{\partial \lambda^\mu} + \hbar F_{\mu\nu}^{(n)} \frac{\partial H_{\text{eff}}}{\partial P_\nu}. \tag{2.19}$$

Let us introduce the following notation for variables (λ^μ, P_ν)

$$(y^1, \ldots, y^{2N}) = (\lambda^1, \ldots, \lambda^N, P_1, \ldots, P_N)$$

and let us write down the Hamilton equations in the form:

$$\dot{y}^i = g^{rs}\frac{\partial y^i}{\partial y^r}\frac{\partial H_{\rm eff}}{\partial y^s} = g^{is}\frac{\partial H_{\rm eff}}{\partial y^s}(i, r, s = 1, \ldots, 2N). \tag{2.20}$$

We see that:

$$g^{rs} = \begin{bmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & \hbar F^{(n)} \end{bmatrix}. \tag{2.21}$$

Hence the Poisson brackets of two arbitrary dynamical variables $f(\lambda^\mu, P_\nu)$ and $h(\lambda^\mu, P_\nu)$ have the following form:

$$\{f, h\} = g^{rs}\frac{\partial f}{\partial y^r}\frac{\partial h}{\partial y^s} = \frac{\partial f}{\partial \lambda^\mu}\frac{\partial h}{\partial P_\mu} - \frac{\partial f}{\partial P_\mu}\frac{dh}{\partial \lambda^\mu} + \hbar F^{(n)}_{\mu\nu}\frac{\partial f}{\partial P_\mu}\frac{\partial h}{\partial P_\nu}. \tag{2.22}$$

We see that the Poisson brackets corresponding to the theory described by the Lagrangian $L_{\rm eff}$ differ from Poisson brackets corresponding to the theory described by the Lagrangian L_0:

$$\{f, h\}_0 = \frac{\partial f}{\partial \lambda^\mu}\frac{\partial h}{\partial P_\mu} - \frac{\partial f}{\partial P_\mu}\frac{\partial h}{\partial \lambda^\mu}. \tag{2.23}$$

The "anomalous" term in $\{,\}$-brackets is determined by Berry's curvature two-form $F^{(n)}_{\mu\nu}$. For example, Poisson brackets of momentum components are non-zero

$$\{P_\mu, P_\nu\} = \hbar F^{(n)}_{\mu\nu}. \tag{2.24}$$

By direct but tiresome calculation, we obtain that:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$$

$$= -\hbar\left[\partial_\alpha F^{(n)}_{\beta\gamma} + \partial_\beta F^{(n)}_{\gamma\alpha} + \partial_\gamma F^{(n)}_{\alpha\beta}\right]\frac{\partial f}{\partial P_\alpha}\frac{\partial g}{\partial P_\beta}\frac{\partial h}{\partial P_\gamma}. \tag{2.25}$$

Hence the Jacobi identity for $\{,\}$-brackets is broken by Berry's curvatures at point where this form is not closed. It takes place at points at which energy levels of the Hamiltonian $\hat{H}(\hat{p}, x, \lambda)$ are degenerated [2]. For example,

$$\{P_\alpha\{P_\beta, P_\gamma\}\} + \{P_\beta, \{P_\gamma, P_\alpha\}\} + \{P_\gamma, \{P_\alpha, P_\beta\}\}$$

$$= -\hbar\left[\partial_\alpha F^{(n)}_{\beta\gamma} + \partial_\beta F^{(n)}_{\gamma\alpha} + \partial_\gamma F^{(n)}_{\alpha\beta}\right]. \tag{2.26}$$

At the points where energy levels are degenerated the Berry's curvature is not well defined. Therefore these are points at which Poisson brackets for dynamical variables cannot be well defined. A well-defined Poisson bracket {,} satisfies automatically the Jacobi identity at any point.

Now, we shall find the symplectic form Ω corresponding to the system described by L_{eff}. It is determined by an antisimmetric matrix g_{ij} such that $g_{ij}g^{jk} = \delta_i^k$. We see that:

$$g_{ij} = \begin{bmatrix} \hbar F^{(n)} & -\mathbf{1}_N \\ \mathbf{1}_N & 0 \end{bmatrix}. \tag{2.27}$$

Hence,

$$\begin{aligned} \Omega &= \tfrac{1}{2} g_{ij} dy^i \wedge dy^j \\ &= dP^\mu \wedge d\lambda_\mu + \frac{\hbar}{2} F_{\mu\nu}^{(n)} d\lambda^\mu \wedge d\lambda^\nu. \end{aligned} \tag{2.28}$$

Thus the symplectic form of the system described by L_{eff} differs from the symplectic form $\Omega_0 = dP^\mu \wedge d\lambda_\mu$ corresponding to the system described by L_0 of an "anomalous" term, which is proportional to Berry's curvature two-form. By virtue of Darboux theorem we know that if $\underbrace{\Omega \wedge \ldots \wedge \Omega}_{N\text{-times}} \neq 0$ and $d\Omega = 0$ then one can always choose such local coordinates $(x^1, \ldots, x^N, \pi_1 \ldots, \pi_N)$ that the form Ω has a canonical form $\Omega = d\pi_\mu \wedge dx^\mu$. Not always this can be done globally, however see [5]. Passing from the classical theory described by the Lagrangian L_{eff} to the quantum theory, we change well-defined Poisson brackets to commutators $[\hat{f}, \hat{g}] = i\hbar\{f, g\}$, where $\hat{f}$, $\hat{g}$ are operators corresponding to the dynamical variables f and g. We see that the commutators are modified by Berry's curvature form. For example, $[\hat{P}_\mu, \hat{P}_\nu] = i\hbar^2 F_{\mu\nu}^{(n)}$. Similar results have been obtained by H. Kuratsuji and Iida by using Feynman path-integrals [6, 14].

3. *Modification of symmetries and conservation laws*

We shall discuss the problem of modification of symmetries and conservation laws for a special case. We assume that the "macrosystem" corresponding to lagrangian L_0 at classical level, has the 6-dimensional phase-space ($y^i = (\lambda^1, \lambda^2, \lambda^3, P_1, P_2, P_3)$) and that it is spherically symmetric. Moreover, let us assume that at the quantum level it interacts with another system described by the Hamiltonian $H(\lambda) = \mu\vec{\lambda} \cdot \vec{\sigma}$, where $\vec{\sigma}$ are the standard Pauli matrices. The Hamiltonian $H(\lambda)$ has eigenvalues $E_\pm(\lambda) = \pm\mu|\vec{\lambda}|$, where $|\vec{\lambda}| = \sqrt{(\lambda^1)^2 + (\lambda^2)^2 + (\lambda^3)^2}$. The degeneracy of energy levels takes place at the point $\vec{\lambda} = (0, 0, 0)$. By introducing spherical coordinates $(\lambda^1, \lambda^2, \lambda^3) = |\vec{\lambda}|$ $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ we can easily show that eigenstates $\Psi_\pm$, corresponding to energy levels $E_\pm$ (respectively), are

$$\psi_+^{(1)} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{bmatrix} \quad \psi_-^{(1)} = \begin{bmatrix} -\sin(\theta/2)e^{-i\phi} \\ \cos(\theta/2) \end{bmatrix}$$

$$\psi_+^{(2)} = \begin{bmatrix} \cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2) \end{bmatrix} \quad \psi_-^{(2)} = \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2)e^{i\phi} \end{bmatrix}.$$

Vectors $\Psi_{\pm}^{(1)}$ are well defined for all values of angles (θ,ϕ) except for the southern pole $(\theta = \pi)$. We consider the sphere S^2 as embedded in R^3 with the help of the mapping $(\theta,\phi) \to (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$. The image of (π,ϕ), for any ϕ, turns out to be a single point $(0, 0, -1)$, which is the southern pole. With this point there are associated spinors labeled by the variable ϕ:

$$\psi_{-}^{(1)} = \begin{bmatrix} 0 \\ e^{i\phi} \end{bmatrix}, \quad \psi_{-}^{(1)} = \begin{bmatrix} -e^{-i\phi} \\ 0 \end{bmatrix}.$$

Therefore we do not have well-defined spinors $\Psi_{\pm}^{(1)}$ at the southern pole. Similary vectors $\Psi_{\pm}^{(2)}$ are well defined for all angles (θ,ϕ) except for the northern pole $(\theta = 0)$. We have: $\Psi_{+}^{(2)} = e^{-i\phi}\Psi_{+}^{(1)}$ and $\Psi_{-}^{(2)} = e^{i\phi}\Psi_{-}^{(1)}$. Thus, the coordinates $(\theta,\phi,\Psi_{+}^{(1)})$ and $(\theta,\phi,\Psi_{+}^{(2)})$ can be thought of as coordinates in a vector fiber bundle corresponding to the level $E_{+}(\lambda)$.

The S^2-space is the base of this fiber bundle, the complex space C^2 is a typical fiber and U(1) is a structural group. Coordinates $(\theta,\phi,\Psi_{+}^{(1)})$ correspond to the upper hemisphere of S^2-sphere, and the $(\theta,\phi,\Psi_{+}^{(2)})$ — coordinates correspond to the lower hemisphere.

Analogously, we can construct the vector fiber bundle corresponding to the energy level $E_{-}(\lambda)$. Obviously, the spherical coordinates on S^2 are singular at the points $\theta = 0, \pi$; however we use them for the sake of visuality.

In this fiber bundle, Berry's connection assumes the following form:

$$A_{\pm}^{(1)} = i(\psi_{\pm}^{(1)}, d\psi_{\pm}^{(1)}) = \pm\tfrac{1}{2}(\cos\theta - 1)d\phi,$$

$$A_{\pm}^{(2)} = i(\psi_{\pm}^{(2)}, \quad d\psi_{\pm}^{(2)}) = A_{\pm}^{(1)} \pm d\phi. \tag{3.1}$$

The Berry's curvature form is

$$F_{\pm} = dA_{\pm}^{(1)} = dA_{\pm}^{(2)} = \mp\tfrac{1}{2}\sin\theta d\theta \wedge d\phi. \tag{3.2}$$

Because the first Chern number for both fiber bundles is nonzero: $\frac{1}{2\pi}\int_{S^2} F_{\pm} = \mp 1$, these fiber bundles are nontrivial. The forms $A_{\pm}^{(1)}$ and $A_{\pm}^{(2)}$, in coordinates $(\lambda^1, \lambda^2, \lambda^3)$ are:

$$A_{\pm}^{(1)} = \pm\tfrac{1}{2}\frac{\lambda^2 d\lambda^1 - \lambda^1 d\lambda^2}{|\vec{\lambda}|(\lambda^3 + |\vec{\lambda}|)}$$

$$A_{\pm}^{(2)} = \pm\tfrac{1}{2}\frac{\lambda^2 d\lambda^1 - \lambda^1 d\lambda^2}{|\vec{\lambda}|(\lambda^3 - |\vec{\lambda}|)}. \tag{3.3}$$

We see that forms $A_{\pm}^{(1)}$ have a string-like singularity along the negative semiaxis λ^3, whereas $A_{\pm}^{(2)}$ forms have a string-like singularity along the positive semiaxis λ^3. The forms $A_{+}^{(a)}$ and $A_{-}^{(a)}$, where $a = 1, 2$, correspond to a magnetic monopole with charge of $\mp 1/2$, respectively, placed at the origin of the coordinate system. The Berry's curvature forms, in coordinates $(\lambda^1, \lambda^2, \lambda^3)$, are

$$F_{\pm} = \mp\tfrac{1}{2}\frac{1}{|\vec{\lambda}|^3}[\lambda^1 d\lambda^2 \wedge d\lambda^3 + \lambda^2 d\lambda^3 \wedge d\lambda^1 + \lambda^3 d\lambda^1 \wedge d\lambda^2]. \tag{3.4}$$

Therefore the "magnetic field" $(B_\pm)_i = \frac{1}{2}\varepsilon_{ijk}F_{\pm jk}$, where ε_{ijk} is the completely antisymmetric symbol, is

$$\vec{B}_\pm = \mp\tfrac{1}{2}\frac{\vec{\lambda}}{|\vec{\lambda}|^3}. \tag{3.5}$$

The fact that the system described by the Hamiltonian H_0 is spherically symmetric means that:

$$\{L_i, H_0\}_0 = \frac{\partial L_i}{\partial\lambda^k}\frac{\partial H_0}{\partial P_k} - \frac{\partial L_i}{\partial P_k}\frac{\partial H_0}{\partial\lambda^k} = 0,$$

where $L_i = \varepsilon_{ijk}\lambda^j P^k$ is the i-th component of the angular momentum, i.e. it is the rotation generator.

We have just seen that the adiabatic interaction of the "macro" and "micro-systems" modifies the Hamiltonian H_0 at the classical level: $H_0 \to H_0 + E_n(\lambda) = H_{eff}$. If $E_n(\lambda) = E_n(|\vec{\lambda}|)$, then $\{L_i, H_{eff}\}_0 = 0$. In our case, $E_\pm(\lambda) = \pm\mu|\vec{\lambda}|$ and therefore $\{L_i, H_{eff}\}_0 = 0$. However, this interaction modifies the Poisson brackets, and we must check if $\{L_i, H_{eff}\} = 0$. The direct calculation shows that:

$$\{L_i, H_{eff}\} = hF_{rs}\frac{\partial L_i}{\partial P_r}\frac{\partial H_{eff}}{\partial P_s}$$

$$= h\left\{(B_j\lambda^j)\frac{\partial H_{eff}}{\partial P_i} - B_i\left(\lambda^j\frac{\partial H_{eff}}{\partial P_j}\right)\right\}. \tag{3.6}$$

Hence, the components of the angular momentum L_i are not conserved in the system corresponding to L_{eff}. However, we can search for modified rotation generators $L_i \to M_i^\pm$ for which:

$$\{M_i^\pm, H_{eff}\} = 0, \text{ and } \{M_i^\pm, M_j^\pm\} = \varepsilon_{ijk}M_k^\pm. \tag{3.7}$$

Let

$$M_i^\pm = L_i + W_i^\pm(\lambda). \tag{3.8}$$

The condition that $\{M_i^\pm, H_{eff}\} = 0$ provides the equation for $W_i^\pm(\lambda)$:

$$\frac{\partial W_i^\pm}{\partial\lambda^r} = h[B_\pm^i\lambda^r - (B_\pm^j\lambda^j)\delta_{ir}]. \tag{3.9}$$

Having in mind that in our case $B_\pm^i = \mp\dfrac{1}{2}\dfrac{\lambda^i}{|\vec{\lambda}|^3}$, one obtains

$$\frac{\partial W_i^\pm}{\partial\lambda^r} = \pm\frac{h}{2}\frac{1}{|\vec{\lambda}|}\left(\delta_{ir} - \frac{\lambda_i\lambda_r}{|\vec{\lambda}|^2}\right). \tag{3.10}$$

Hence,

$$W_i^\pm = \pm\frac{h}{2}\frac{\lambda_i}{|\vec{\lambda}|}. \tag{3.11}$$

We can show, by direct calculation, that the Poisson bracets of $M_i^{\pm} = L_i \pm \frac{\hbar}{2} \frac{\lambda_i}{|\vec{\lambda}|}$ are $\{M_i^{\pm}, M_j^{\pm}\} = \varepsilon_{ijk} M_k^{\pm}$. Hence, the quantities $M_i^{\pm}$, and not L_i are generators of rotation. We meet a similar situation if we consider the motion of a charged particle in the field of a magnetic monopole [15].

4. *Conclusion*

We have shown that the interaction of the "macro" and "micro-systems" in the Born--Oppenheimer approximation modifies the structure of the phase-space of the "macro-system". The Poisson brackets are affected by this modification. Therefore, commutators with "anomalous" terms, proportional to Berry's curvature form, appear in the corresponding quantum theory. We have also shown that this interaction modifies the spherical symmetry of the "macrosystem". A certain term must be added to the angular momentum; this term is proportional to Berry's curvature two-form.

The procedure, which we have applied in the case of finite degrees of freedom can be extended to a quantum field theory in a natural way. It can be used to discuss the problems associated with the interaction of gauge fields and chiral fermionic fields. Then the "macrosystem" would correspond to gauge fields and the "microsystem" to fermionic fields. In this case, one usually quantizes fermionic fields first, and then the gauge fields. Hence, it is important to determine the fermionic effective action corresponding to the effective lagrangian (2.12). The technique proposed in the present paper can also be used to discuss gauge anomalies. Our approach is based on the adiabatic approximation and it may appear to be inadequate to describe anomalies since they are independent of the approximation scheme. However, the anomalous commutators result from the reaction of physical states to gauge transformations, and the realization of gauge transformation on the Fock vacuum, corresponding to fermionic fields [7, 11], can always be described in terms of an adiabatic change.

The modification of the "macrosystem" symmetry can provide suggestions how to quantize theories with anomalies.

The modern approach to investigate the early Universe involves the quantum field theory in a curved spacetime, and all calculations are performed with the help of quasi--static assumption. This means that the quantum processes are so fast that the spacetime can be regarded as static. This corresponds to the adiabatic theorem. The results obtained in the present paper can also be used as a suggestion how to develop quantum field theory in curved spacetime.

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