AXIAL ANOMALY IN RELATIVISTIC QUANTUM MECHANICS*

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The violation of chiral symmetry in massless QED is discussed within the framework of relativistic quantum mechanics.

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1. *Introduction*

There exist different approaches to the problem of axial (or chiral) anomalies. First of all, there is an euclidean formulation relating the quantum-mechanical symmetry breaking to the ultraviolet properties of the theory. In this approach one makes use of perturbation theory [1] or appeal to the powerful geometrical and topological tools like index theorems [2]. The Fujikawa method [3] is also the "ultraviolet" one.

In all the above approaches the physical origin of the effects caused by anomalies remains rather obscure. On the other hand, the axial anomalies (or more generally — the anomalies in global currents) do give rise to physical effects, the best known being $\pi^0 \to 2\gamma$ decay. As noticed in Ref. [4] it is hard to understand why this decay should depend on dynamics far below hadronic scale.

Fortunately, many authors [5] offered a somewhat different explanation which seems to be very appealing. Its main idea is as follows. First of all, one may confine himself to the case of external background gauge fields. Then the first quantized theory of fermions interacting with those fields can in principle be solved; the second quantized theory is obtained in a standard way by filling all negative energy levels. Then the following picture of axial anomaly emerges: the conservation law for chirality is violated due to the pair creation, the chirality of a pair being nonvanishing. The way this happens is very nice indeed. The chiral symmetry is of course present at the level of first quantization — the solutions to the Dirac equation carry the representation of chiral group and γ_5 charge is conserved. After second quantization this symmetry prevents the external field to excite

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the negative energy particle to the positive energy state of different chirality; the process depicted in Fig. 1 is forbidden. On the other hand the pair creation in the way that conserves chirality, as in Fig. 2, is allowed. This is just the way the pair production usually takes place. There is, however, one possibility to produce out of the vacuum the pair with non-vanishing chirality in the final state, which is perfectly consistent with the chiral symmetry: the whole tower of $\gamma_5 = 1$ negative states is shifted (say) up while the one corresponding to $\gamma_5 = -1$ — down; see Fig. 3. This is why the chirality may be not conserved in quantum theory still being the symmetry of the classical one.

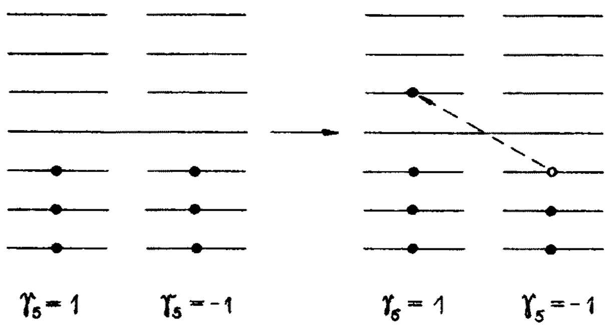

Fig. 1. The process not allowed by axial symmetry

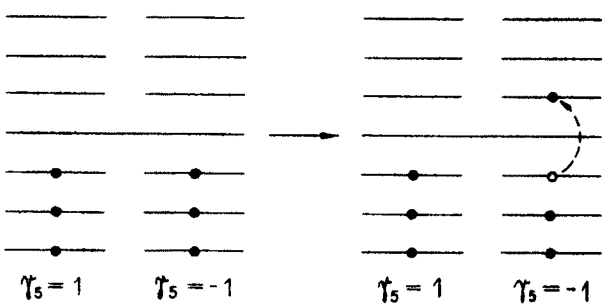

Fig. 2. The process allowed by axial symmetry

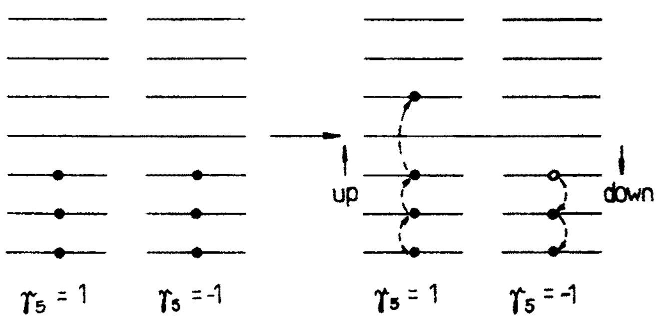

Fig. 3. The process leading to the change of chirality but allowed by axial symmetry

The trouble with this explanation is related to the fact that the Dirac sea is infinitely deep and one has to take care about book-keeping of large negative levels. This is the place where the ultraviolet properties of the theory come into play. One can check [4] that the gauge invariant point-splitting regularization of axial current gives the proper description of damping out the contribution from the depth of Dirac sea.

Now, the important point is the following[1]. We are faced with the problem of ultraviolet divergencies because we insists on having the detailed space-time description of chiral symmetry breaking. Then the ultraviolet properties become relevant — we concentrate on local operators like currents to be able to trace the behaviour in time of chiral charge. The problem of defining the current is related to ultraviolet divergencies.

However, we may recognize the existence and properties of anomalies after their ultimate effects. To this end we start with incoming vacuum in the distant past when the field is such that it is possible to define unambiguously the positive and negative energies (no pair creation). The chiral charge is then well defined in terms of the corresponding creation and annihilation operators. The chirality of the final state equals ${}_{\text{in}}\langle 0|Q^5_{\text{out}}|0\rangle_{\text{in}}$. It can be calculated in two ways. We may trace the behaviour of the interpolating charge $Q^5(t)$, i.e. consider the current with all ultraviolet problems related to it. Alternatively, we may proceed as follows. The charge Q^5_{out} is unitarily equivalent to Q^5_{in}. Therefore Q^5_{out} is unambiguously defined in terms of "out" creation and annihilation operators. To calculate the final chirality we have to express only $|0\rangle_{\text{in}}$ in the "out" basis. This second possibility has an advantage of skiping the problems with current. Moreover, it offers the chance to formulate the whole problem of axial symmetry breaking in the framework of relativistic quantum mechanical approach due to Feynman [6]. This is the aim of the present paper. The paper is organized as follows. In Sect. 2 we discuss the general framework. Time-dependent fields of some special type are considered in Sect. 3, while Sect. 4 is devoted to the problem of static fields. Sect. 5 contains short conclusions. Some information used in text is collected in the Appendices.

2. *The relativistic quantum mechanics*

We consider the massless charged fermions interacting with the external electromagnetic field. The corresponding Dirac equation reads

$$i\gamma^\mu(\partial_\mu+ieA_\mu)\psi = 0; \tag{1}$$

we adopt the choice of γ-matrices made in Ref. [13]. We choose the hamiltonian gauge $A_0 = 0$.

We assume that in the distant past and future the potentials A_i are such that they allow for unambiguous definition of positive and negative part of spectrum. In the massless case it means that there is at most magnetic field (although arbitrary) for $t \to \pm\infty$.

The Feynman approach to relativistic quantum mechanics is usually formulated in terms of propagators. However, in the case of external field which does not vanish for $t \to \pm\infty$ it is more convenient to adopt the equivalent approach based on the exact solu-

[1] strictly speaking, the picture sketched below is somewhat more complicated — see Sect. 5.

tions to the Dirac equation with external field [7]. Namely, we take the solutions $\psi_F^{\pm}(x|g)$ defined by the Feynman boundary conditions. For example, $\psi_F^{-}(x|g)$ is the solution containing only negative frequency part in the distant past while in the distant future the negative frequency part is g. Then the amplitude of finding f in the positive frequency part in future is just the amplitude of creating a pair with the particle and antiparticle in the states f and g, respectively. The corresponding probability is of course a relative one; to calculate the absolute probability one has to calculate the probability of vacuum-to-vacuum transition.

In a similar way one computes the amplitude of antiparticle scattering as well as (with the help of ψ_F^{+}) the pair annihilation and particle scattering.

It is easy to see that the pair production calculated as above corresponds to Fig. 1. In fact, we start with ψ_F^{-} having a definite chirality asymptotics in the past; then, due to the fact, that Dirac hamiltonian commutes with γ_5, both negative and positive frequency parts in future have the same chirality. Therefore, the total chirality of the pair produced equals zero. Chirality is conserved in Feynman approach.

Now, we have to identify the processes responsible for chirality violation. As it is well known one cannot in general impose the boundary conditions on the negative (positive) frequency parts both in the past and future. This is easily seen by taking the free Dirac equation. However, it may happen that there exist some special solutions which seem to be "overdetermined" from the point of view of the allowed boundary conditions. We claim that these are just the solutions responsible for violation of chirality conservation law. More precisely, let $\psi_F^{0}(x|g)$ be the solution with the following asymptotic behaviour: for the distant past there is only a negative frequency part while in the distant future there is *only* positive frequency part g. We interprete such a solution as a creation of particle in the state g with the probability one. This leads, of course, to the chirality production. To save the principle of charge conservation the corresponding antiparticle must be produced. We associate with such a process the solution to the Dirac equation containing only positive frequency part in the past and only negative one in future. The existence of such a solution accompanying the former one follows from the *CT*-symmetry. It follows also that the antiparticle carry the same chirality; there is no cancellation.

Now, what about the normalization of probabilities? First of all, let us note that the probabilities of creation of single particles (antiparticles) related to the solutions ψ_F^{0} should be considered to be the absolute ones; was it not so the charge would be conserved only statistically. There would be nonzero probability of creating only one kind of particle[2]. To determine the normalization of other processes let us note that now the probability for the vacuum to remain vacuum is exactly zero. However, we may resolve this trouble by demanding that the Feynman principle is still valid: the normalized total probability of scattering should equal one, i.e. there is no Klein paradox. Due to the Pauli principle there is no scattering to the final states occupied by the particles created with probability one. Therefore we can simply discard the states described by ψ_F^{0} and apply the standard normalization prescription to the all remaining processes.

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[2] Note that this is not possible for the processes described by ψ_F^{-}, there we calculate the amplitude of *pair to appear*.

3. *The creation of chirality*

Let us assume the following field configuration. There is a magnetic field parallel to the third axis, not depending on t and z

$$\hat{H} = H(x, y)\hat{z} \tag{2}$$

and the uniform electric field also parallel to it

$$\hat{E} = E(t)\hat{z} \tag{3}$$

such that $\hat{E} = 0$ for $|t| > T$. We choose the following gauge

$$\begin{gathered} A_x = A_x(x, y), \quad A_y = A_y(x, y), \quad A_z = A_z(t), \quad A_0 = 0, \\ A_z(t < -T) = A_-, \quad A_z(t > T) = A_+. \end{gathered} \tag{4}$$

For definiteness we assume in the sequel that $eA_+ \geqslant eA_-$. To solve the Dirac equation

$$[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi = 0$$

we take its square

$$\left[(\partial_\mu + ieA_\mu) + m - \frac{e}{2}\,\sigma^{\mu\nu}F_{\mu\nu}\right]\psi = 0 \tag{5}$$

and look for the solutions in the form [8]

$$\psi = \phi X, \tag{6}$$

where ϕ is some scalar function and X is an eigenvector of $\sigma^{\mu\nu}F_{\mu\nu}$. Below we list the eigenvalues and eigenvectors of this matrix.

$$a\colon\ 2(H - iE), \quad 2(H + iE), \quad 2(-H + iE), \quad 2(-H - iE),$$

$$X\colon \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}.$$

The solution to the Dirac equation reads then

$$\psi = [i\gamma^\mu(\partial_\mu + ieA_\mu) + m]\,(\Phi X). \tag{7}$$

With our choice of gauge Eq. (5) reads

$$\left[\partial_t^2 - (\partial_x + ieA_x)^2 - (\partial_y + ieA_y)^2 - (\partial_z + ieA_z)^2 + m^2 - \frac{ea}{2}\right]\phi = 0 \tag{8}$$

and admits the separation of variables. If we put

$$\phi = e^{ip_z z}f(t)g(x, y) \tag{9}$$

we arrive at the following set of equations

$$\left[-(\partial_x+ieA_x)^2-(\partial_y+ieA_y)^2-\frac{ea_H}{2}\right]g(x,y) = \Lambda g(x,y), \tag{10a}$$

$$\left[\partial_t^2+(p_z+eA_z)^2+m^2-\frac{ea_E}{2}+\Lambda\right]f(t) = 0, \tag{10b}$$

where $a_H = \pm 2H(x, y)$, $a_E = \pm 2iE(t)$ and Λ is the separation constant.

Both equations (10) are rather nice. The first one describes the planar system consisting of a charged nonrelativistic particle in the external magnetic field. As discussed in Appendix A it exhibits $N = 2$ supersymmetry. This allows for some definite statements concerning its spectral properties.

The second equation is the one-dimensional Schroedinger equation with complex potential. It is discussed in Appendix B where the identification $p_z+eA_z = V(t)$ has been made. This equation also exhibits some kind of $N = 2$ supersymmetry.

Having solved equations (10) we obtain four solutions to the Dirac equation corresponding to four choices of eigenvectors X. They read

$$\psi_1 = \begin{bmatrix} -\Sigma_+ + m \\ -\theta_- \\ \Sigma_+ + m \\ \theta_- \end{bmatrix} f(t)g(x, y)e^{ip_z z}, \tag{11a}$$

$$\psi_2 = \begin{bmatrix} \theta_+ \\ -\Sigma_+ + m \\ \theta_+ \\ -\Sigma_+ - m \end{bmatrix} f(t)g(x, y)e^{ip_z z}, \tag{11b}$$

$$\psi_3 = \begin{bmatrix} -\Sigma_- + m \\ \theta_- \\ -\Sigma_- - m \\ \theta_- \end{bmatrix} f(t)g(x, y)e^{ip_z z}, \tag{11c}$$

$$\psi_4 = \begin{bmatrix} -\theta_+ \\ -\Sigma_- + m \\ \theta_+ \\ \Sigma_- + m \end{bmatrix} f(t)g(x, y)e^{ip_z z}. \tag{11d}$$

The operators $\theta_\pm$, $\Sigma_\pm$ are defined in Appendices. Of course, for fixed f and g only two from the above solutions are linearly independent.

To obtain the solutions with the Feynman boundary conditions we have to select the solutions $f(t)$ to the Schroedinger equation (10b). In particular, to obtain ψ_F^- we have to take $f(t)$ containing only the negative frequencies for $t \to -\infty$. This gives a nice interpretation of the pair production and scattering in terms of reflection and transition in the one-dimensional complex barrier from Eq. (10b) (actually, one has to take the complex conjugated barrier due to the opposite time vs space dependence of positive energy vs momentum waves).

With the assumptions about magnetic field made in Appendix A the spectra of the hamiltonians on the left hand sides of Eq. (10a) are purely discrete. Let us call Λ_n, $n = 0, 1, \ldots, \Lambda_0 = 0$, $\Lambda_n > 0$ for $n > 0$ their eigenvalues and $g_n^{\pm}$ — their eigenfunctions; one must keep in mind that $g_0^- = 0$ and $\theta_- g_0^+ = 0$. The functions ψ_1 and ψ_3 (ψ_2 and ψ_4) correspond to the upper (lower) parts of the supersymmetric hamiltonian (A.7). On the other hand the functions ψ_1 and ψ_2 (ψ_3 and ψ_4) correspond to the lower (upper) part of the hamiltonian (B.3).

Let us now take the massless case, $m = 0$. First we investigate the effect of $n > 0$ levels on chirality production. The two linearly independent Feynman solutions $\psi_F^-(x)$ may be obtained for example, from ψ_3 and ψ_4 by choosing $f(t)$ to be the solution f_n^+ to (B.4a) with $E = \Lambda_n$, with the following asymptotic behaviour

$$f_n^+ \underset{t\to-\infty}{\simeq} B_n(p_z)e^{i\nu_n^- t},$$

$$f_n^+ \underset{t\to-\infty}{\simeq} A_n(p_z)e^{-i\nu_n^+ t}+e^{i\nu_n^+ t} \tag{12}$$

here $\nu_n^{\pm} = (\Lambda_n+(p_z+eA_{\pm})^2)^{1/2}$. According to the discussion in Sect. 2 such solutions do not give rise to chirality nonconservation. They correspond to the processes pictured in Fig. 2. To find the contribution to the chirality violation from the levels under consideration we have to look for the solutions with the asymptotics

$$f_n^+ \underset{t\to-\infty}{\simeq} e^{i\nu_n^- t}, \quad f_n^+ \underset{t\to\infty}{\simeq} C_n(p_z)e^{-i\nu_n^+ t}. \tag{13}$$

However, we do not need to consider the question whether such solutions do exist. To see this we note that any such f_n^+ provides simultaneously the solutions in the form ψ_3 and ψ_4. Now, ψ_3 and ψ_4 correspond to opposite chiralities and they cancel. This cancellation is complete because the degeneracies of the excited states of supersymmetric partners are the same.

It is therefore enough to consider the case of $\Lambda_0 = 0$. Then only ψ_1 and ψ_3 survive and take the following form

$$\psi_1 = \begin{bmatrix} -g_0^+\Sigma_+ f_- \\ 0 \\ g_0^+\Sigma_+ f_- \\ 0 \end{bmatrix} e^{ip_z z}, \qquad \psi_3 = \begin{bmatrix} -g_0^+\Sigma_- f_+ \\ 0 \\ -g_0^+\Sigma_- f_+ \\ 0 \end{bmatrix} e^{ip_z z}. \tag{14}$$

The functions $\Sigma_{\pm} f_{\mp} = h_{\mp}$ fulfil the equations

$$\Sigma_{\pm} h_{\pm} = 0. \tag{15}$$

The solutions to Eq. (15) read

$$h_{\pm}(t) = \text{const}\times\exp\left[\mp i \int^{t} d\tau(p_z+eA_z(\tau))\right]. \tag{16}$$

For $-eA_- > p_z > -eA_+$ $h_+(t)$ contains only negative (positive) frequencies for $t \to -\infty (t \to \infty)$. The number of states in the interval dp_z is $dp_z L/2\pi$ times the number of zero modes g_0. It equals, for large L, $e\phi(S)/2\pi$, where $S = L^2$ and $\Phi(S)$ is the flux of magnetic field through S.

The chirality of each particle produced in the state (which corresponds to $h_+(t)$ equals 1, so the total change of chirality is

$$\Delta\kappa = \frac{e\Phi(S)L}{(2\pi)^2} \int_{-eA_+}^{-eA_-} dp_z = \frac{e^2\Phi(S)L}{(2\pi)^2} \int_{-T}^{T} dt E(t). \tag{17}$$

The above process is accompanied by the antiparticle production. To see that this is the case consider $h_-(t)$ (which corresponds to ψ_1). For $-eA_- > p_z > -eA_+ h_-(t)$ contains only positive (negative) frequencies for $t \to -\infty$ $(t \to \infty)$. Tfe chirality of ψ_1 is -1 which corresponds to the positive chirality of antiparticle produced. The total contribution is then

$$\Delta\kappa = \frac{e^2}{2\pi^2} \int_{L^3} dV \int_{-T}^{T} dt H(x, y) E(t). \tag{18}$$

This is exactly the quantity one gets from the anomalous divergence equation

$$\partial_\mu j_5^\mu = \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}.$$

Note that the solutions ψ_1 and ψ_3 contributing to antiparticle and particle creation, respectively, are related by CT-transformation, as indicated in Sect. 3.

3. *Static fields*

The case of static fields is especially interesting. The problem we are faced here is that the external field is not switched off in the past and future; therefore, the definition of the positive and negative frequencies possess a nontrivial problem. One way to solve it is to find the solutions which reduce to the ones with definite frequency after adiabating switching off the interaction in the past and future [9]. However, there exists another very nice method of selecting the suitable solutions [10]. One expects the motion of charged particle in the constant uniform electric field to be quasiclassical for $t \to \pm\infty$. The wave function should take the form $\psi \simeq \exp iS$ with the classical action S. We select the solutions by solving the Hamilton-Jacobi equation and comparing the asymptotic behaviour in t of S with that of the wave function. The last problem is to choose the gauge. The point is that in the free particle case the energy is related in a simple way to the time dependence of the wave function only in the gauge $A_0 = 0$. Consequently, we expect such a gauge to be the suitable for our purposes[3]. Therefore we choose the same gauge as in Sect. 2, i.e. $A_0 = 0$,

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[3] Obviously the solutions to the H-J equations in different gauges are gauge-related. However, this is not the case for the solutions obtained by separating variables in different gauges.

$A_z = Et$. To solve the Hamilton-Jacobi equation we put $H = 0$ because the magnetic field cannot remove the gap between particle and antiparticle states. The H-J equation reads

$$((\partial_x S)^2 + (\partial_y S)^2 + (\partial_z S + eEt)^2 + m^2)^{\frac{1}{2}} + \partial_t S = 0.$$

The solution to it is

$$S = p_x x + p_y y + p_z z - \frac{\sigma}{2}\sqrt{\sigma^2+\kappa^2} - \frac{\kappa}{2}\,\mathrm{arcsh}\,\frac{\sigma}{\sqrt{\kappa}}$$

with $\kappa \equiv \dfrac{p_x^2+p_y^2+m^2}{eE}$, $\sigma = \dfrac{eEt+p_z}{\sqrt{eE}}$. The asymptotic behaviour of S reads

$$S \xrightarrow[|\sigma|\to\infty]{} -(\mathrm{sgn}\,\sigma)\frac{\sigma^2}{2}. \tag{19}$$

To calculate the rate of chirality production we take for definiteness the uniform magnetic field H (in the case $\hat{H} = H(x, y)\hat{z}$ everything goes through without changes). With the gauge choice $A_y = -Hx$, $A_x = 0$ the two linearly independent solutions ψ_F^- are

$$\psi_{3,4} = \left[u_{3,4} D_{-\frac{i\lambda_n^{3,4}}{2}}(-(1+i)\sigma) - \left(\frac{1+i}{2}\right) v_{3,4} \times D_{-\frac{i\lambda_n^{3,4}}{2}-1}(-(1+i)\sigma)\right] e^{-\varrho^2/2} e^{i(p_y y + p_z z)}, \tag{20}$$

with

$$\varrho \equiv \frac{eHx - p_y}{\sqrt{eH}}, \qquad \lambda_n^3 = \frac{2neH+m^2}{eH}, \qquad \lambda_n^4 = \frac{2(n+1)eH+m^2}{eE}$$

and

$$u_3 = \begin{bmatrix} \sqrt{eE}\,H_n(\varrho) \\ 0 \\ \sqrt{eE}\,H_n(\varrho) \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} mH_n(\varrho) \\ -\sqrt{eH}\,H_n'(\varrho) \\ -mH_n(\varrho) \\ -i\sqrt{eH}\,H_n'(\varrho) \end{bmatrix},$$

$$u_4 = \begin{bmatrix} 0 \\ \sqrt{eE}\,H_n(\varrho) \\ 0 \\ -\sqrt{eE}\,H_n(\varrho) \end{bmatrix}, \qquad v_4 = \begin{bmatrix} -i\sqrt{eH}\,H_{n+1}(\varrho) \\ mH_n(\varrho) \\ i\sqrt{eH}\,H_{n+1}(\varrho) \\ mH_n(\varrho) \end{bmatrix}.$$

$H_n(\varrho)$ are the Hermite polynomials while D_α — the parabolic cylinder functions. From the asymptotic behavior

$$\psi_{3,4} \underset{\sigma\to-\infty}{\simeq} u_{3,4} e^{-\varrho^2/2} e^{i(p_y y + p_z z)} e^{-i\sigma^2/2} [-(1+i)\sigma]^{-i\lambda_n/2}$$

$$\psi_{3,4} \underset{\sigma\to\infty}{\simeq} \left\{ u_{3,4} e^{-i\sigma^2/2}[-(1+i)\sigma]^{-i\lambda_n/2} - \left(\frac{1+i}{2}\right) v_{3,4} \right.$$

$$\left. \times \frac{2\pi e^{-\pi\lambda_n/2}}{\Gamma\left(1+\frac{i\lambda_n}{2}\right)} e^{i\sigma^2/2}[-(1+i)\sigma]^{i\lambda_n/2} \right\} \times e^{-\varrho^2/2} e^{i(p_y y + p_z z)} \tag{21}$$

we get the well-known result [10] for the relative probability of pair production

$$\omega_i(p_y, p_z, n) = (e^{\pi\lambda_n^i} - 1)^{-1}, \qquad i = 3, 4. \tag{22}$$

Now take $m = 0$. It follows from Eq. (21) that for any $n > 0$ we get the usual Feynman function ψ_F^-. For $n = 0$ only the solution ψ_3 exists. According to the formula (22) $\omega_3(n = 0) = \infty$. That means that the absolute probability equals one. However, in this case we are dealing with ψ_F^0 rather than ψ_F^-. In fact, we see that $v_3 = 0$ and, according to the formula (21) only the positive frequency part in the future survives. Our solution describe the particle production. Taking into account that the density of final states is $L^2 dp_y dp_z/(2\pi)^2$ while p_y- and p_z-integrations give eHL and eET, respectively, we get for the change of chirality $e^2HEL^3T/4\pi^2$. Adding the contribution from ψ_1 we have

$$\frac{d\kappa}{dt dV} = \frac{e^2}{2\pi^2} HE \tag{23}$$

again in the perfect agreement with anomaly equation.

5. *Conclusions*

The problem of violation of axial symmetry may be understood within a framework of relativistic quantum mechanics. The processes responsible for the change of chirality are described by the functions ψ_F^0. They provide a special kind of solutions which exist only in some circumstances. No local operators are involved and no ultraviolet problems encountered.

However, it would not be fair to say in general that there are no ultraviolet aspects involved. Let us note the following. Due to the conservation of electric charge the processes leading to violation of axial symmetry must occur with the probability one or zero. Therefore the expectation value of chiral charge should be an integer for $t \to \infty$. But integrating the anomaly one may get any real number. This contradiction is due to the fact that there are some ultraviolet divergencies to be taken into account. They manifest itself as a regularization dependence of the current definition. The current has a c-number part depending on external field, which gives rise to the vacuum charge and causes the unitary inequivalnce of “in“ and “out“ charges.

The problem sketched here will be discussed in some detail elsewhere.

It is a pleasure to acknowledge a helpful discussions with Dr S. Giler and Dr. P. Maślanka.

APPENDIX A [12]

The hamiltonian (actually — twice the hamiltonian) of a planar system consisting of a charged particle in an external magnetic field $H = H(x, y)$ reads

$$\mathcal{H} = P_x^2+P_y^2-eH\sigma_z, \tag{A1}$$

with $P_{x,y} = -i\partial_{x,y}+eA_{x,y}$. In spite of the fact that the number of bosonic degrees of freedom is two while that of fermionic ones — one only, the theory exhibits $N = 2$ supersymmetry. To see this we define two supercharges

$$Q_1 = \sigma_x P_x+\sigma_y P_y, \quad Q_2 = \sigma_y P_x-\sigma_x P_y. \tag{A2}$$

The SUSY algebra reads

$$\{Q_i, Q_k\} = 2\mathcal{H}\delta_{ik}. \tag{A3}$$

As usual one defines the charges

$$Q_\pm = \tfrac{1}{2}(Q_1 \pm iQ_2), \tag{A4}$$

$$Q_+ = \begin{pmatrix} 0 & P_x-iP_y \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \theta_+ \\ 0 & 0 \end{pmatrix},$$

$$Q_- = \begin{pmatrix} 0 & 0 \\ P_x+iP_y & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ \theta_- & 0 \end{pmatrix}, \tag{A5}$$

with the properties

$$Q_+^\dagger = Q_-, \quad Q_+^2 = Q_-^2 = 0,$$

$$\{Q_+, Q_-\} = \mathcal{H}. \tag{A6}$$

Therefore one can write

$$\mathcal{H} = \begin{pmatrix} P_x^2+P_y^2-eH & 0 \\ 0 & P_x^2+P_y^2+eH \end{pmatrix} \equiv \begin{pmatrix} \mathcal{H}_+ & 0 \\ 0 & \mathcal{H}_- \end{pmatrix} \equiv \begin{pmatrix} \theta_+\theta_- & 0 \\ 0 & \theta_-\theta_+ \end{pmatrix}. \tag{A7}$$

As it is always the case for $N = 2$ SUSY all excited states are doubled. In fact, we have the following relations

$$\mathcal{H}_+\chi_+ = E\chi_+, \quad \mathcal{H}_-\chi_- = E\chi_-, \quad E > 0$$

$$\chi_+ = \frac{1}{\sqrt{E}}\theta_+\chi_-, \quad \chi_- = \frac{1}{\sqrt{E}}\theta_-\chi_+. \tag{A8}$$

The spectrum depends strongly on the configuration of magnetic field. If the total flux $\Phi = \int dxdyH(x, y)$ is finite, there exist in general bound as well as scattering states and the structure of spectrum is quite complicated. Here we consider the simplest generalization of the case of uniform field: $H = H_0+\tilde{H}(x, y)$ and the total flux of $\tilde{H}(x, y)$ is finite. For definiteness we assume that $eH_0 > 0$. Then the spectrum is purely discrete. To solve the problem of ground states we choose the gauge $\partial_x A_x + \partial_y A_y = 0$. Then $A_x = \partial_y \phi$, $A_y = -\partial_x \phi$ and ϕ obeys the equation

$$(\partial_x^2 + \partial_y^2)\phi = H(x, y). \tag{A9}$$

The possible candidates for ground states are $\psi_+ = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}$ and $\psi_- = \begin{pmatrix} 0 \\ \chi_- \end{pmatrix}$ with

$$(P_x + iP_y)\chi_+ \equiv \theta_- \chi_+ = 0, \tag{A10a}$$

$$(P_x - iP_y)\chi_- \equiv \theta_+ \chi_- = 0. \tag{A10b}$$

We look for the solution to (A10a) in the form

$$\chi_+ = \exp(-e\phi)f. \tag{A11}$$

Then it appears that f is an analytic function of $z = x + iy$ (in fact an entire one). To answer the question whether (A11) is normalizable or not one has to determine the asymptotic behaviour of ϕ. To this end we note that the asymptotic behaviour of the solution to Eq. (A9) is

$$\phi \underset{|z| \to \infty}{\simeq} \tfrac{1}{4} H_0 |z|^2 \tag{A12}$$

so that χ_+ from Eq. (A10a) is normalizable for any polynomial $f(z)$. To determine the number of ground states in some large "volume" one proceeds along the same lines as for the case of uniform field. The wave function $(\chi_+)_N = z^N \exp(-e\phi)$ behaves like $|z|^N \exp\left(\frac{-eH_0}{4}|z|^2\right)$ for large $|z|$. It follows then that for large N, $\langle |z| \rangle \sim \left(\frac{2N}{eH_0}\right)^{1/2}$. If we include all states with $\langle |z| \rangle < R$ then the number of them in the disc of radius R will be $eH_0 R^2/2$. The density ν of states is $eH_0/2\pi$ or, taking into account that the flux of $\tilde{H}$ is finite

$$\nu = \frac{e}{2\pi S} \int_S dx dy H(x, y) \equiv \frac{e\Phi(S)}{2\pi S}. \tag{A13}$$

Eq. (A10b) does not posses the normalizable solutions.

APPENDIX B

Let us consider the one-dimensional hamiltonian with the complex potential

$$\mathscr{H} = -\frac{d^2}{dt^2} - V^2(t) + i\frac{dV(t)}{dt}\sigma_3 = \begin{pmatrix} -\frac{d^2}{dt^2} - V^2 + iV' & 0 \\ 0 & -\frac{d^2}{dt^2} - V^2 - iV' \end{pmatrix} \tag{B1}$$

is of course non-hermitean but still enjoys $N = 2$ supersymmetry. Introducing the operators

$$\Sigma_\pm \equiv -\frac{id}{dt} \pm V(t), \quad \Sigma_+^+ = \Sigma_+, \quad \Sigma_-^+ = \Sigma_- \tag{B2}$$

we write

$$\mathscr{H} = \begin{pmatrix} \Sigma_+\Sigma_- & 0 \\ 0 & \Sigma_-\Sigma_+ \end{pmatrix}. \tag{B3}$$

Let us consider the Schroedinger equations with real energy

$$\left[-\frac{d^2}{dt^2} - V^2(t) + iV'(t)\right] f_E^+(t) = Ef_E^+(t), \tag{B4a}$$

$$\left[-\frac{d^2}{dt^2} - V^2(t) - iV'(t)\right] f_E^-(t) = Ef_E^-(t). \tag{B4b}$$

We are interested in the solutions with $E \geqslant 0$. The asymptotic expressions read

$$f(t) \underset{t\to\infty}{\simeq} \exp(\pm iv_+ t), \quad v_\pm = \sqrt{E + V_\pm^2},$$

$$f(t) \underset{t\to-\infty}{\simeq} \exp(\pm iv_- t), \quad V_\pm = \lim_{t\to\pm\infty} V(t).$$

The standard current is not conserved for Eqs. (B4) because the potential is complex. However, it is not difficult to construct another current which is conserved. To see this we write (B4a) as

$$\Sigma_+\Sigma_- f_E^+ = Ef_E^+.$$

Now, $(\Sigma_\pm)^* = -\Sigma_\mp$ and consequently

$$\Sigma_-\Sigma_+(f_E^+)^* = E(f_E^+)^*.$$

Multiplying the above equation by Σ_+ we see that $\Sigma_+(f_E^+)^*$ fulfils the same equation as f_E^+ do. Their Wronskian

$$W = f_E^+ \frac{d}{dt}(\Sigma_+(f_E^+)^*) - \Sigma_+(f_E^+)^* \frac{df_E^+}{dt}$$

provides the conserved current. Then

$$f_E^+ \left(\frac{d^2}{dt^2}(f_E^+)^* + V^2(f_E^+)^* + \frac{idV}{dt}(f_E^+)^* + E(f_E^+)^*\right) - iW = E|f_E^+|^2 + |\Sigma_- f_E^+|^2$$

is also conserved. For Eq. (B4b) the conserved current reads

$$E|f_E^-|^2 + |\Sigma_+ f_E^-|^2$$

The existence of these currents is equivalent to the fact that Dirac equation preserves the scalar product.

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