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RELATIVISTIC QUANTUM TRANSPORT THEORY FOR SCALAR FIELDS

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The alternative definition of the many-particle Wigner functions for real and complex scalar fields is proposed. It leads to the consistent quantum transport theory. We obtain the exact hierarchy of the quantum transport equations. Its truncation in the kinetic case yields the relativistic Vlasov-type equation with the series of the quantum corrections. The hierarchy is truncated also beyond the kinetic case.

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1. *Introduction*

There is a great interest in attempts to describe our reality at the many-body level. Generally, there are two possible strategies. The first one is to investigate the modifications to the interaction which are introduced by the presence of other particles. For the Quantum Field Theory (QFT) this is represented by the QFT with finite temperature [1]. The other way is to abandon the study of the microscopic interactions and to concentrate on investigations of the quasimacroscopic behaviour of the system. This is the Non-Equilibrium Quantum Statistical Thermodynamics. It may be realized in the usual space-time or in the phase-space. The latter realization is called the Quantum Transport Theory (QTT) (for review in the nonrelativistic case see Ref. [2]). The quasimacroscopic QTT uses QFT as the microscopic theory of interaction only. Thus the full internal structure of QFT does not have to be regarded in QTT.

QTT was initiated by E.P. Wigner [3]. The author defined the quantum-mechanical analog of the classical probability distribution function spanned on the phase-space — Wigner Function (WF). The dynamical equation for WF was derived on the basis of the Schrödinger equation. The particular solution to the equation was given as the quantum correction to the classical equilibrium distribution.

There are known some QTTs for different models of QFTs [4, 5, 6, 7]. All of them are effective kinetic theories. The kinetic theory is a proper tool for the description

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of dilute systems. It uses the one-particle WF only, which means that the correlations are neglected. This method does not permit any examination of the role of the collective effects. In dense systems that role may be essential. The only known definition of the many--particle WF for QFT, the one for the real scalar field [4], is unsatisfactory.

The aim of the present paper is to introduce many-particle WFs, for real and complex scalar fields, and thus to reach the consistent QTT.

The paper is organized as follows. In Sect. 2 we collect the consistency conditions of WFs for QFTs. Next, we discuss the failure of the multiparticle production approach, then define our WFs and display their connections with the physical quantities of the field. In Sect. 3 we derive general transport equations. The obtained hierarchy of the transport equations is truncated in two approximations, in the kinetic case and beyond. In Sect. 4 we comment on the definition of the WFs for different scalar fields. In Sect. 5 we summarize our results.

2. *WF for quantum fields*

2a. Consistency conditions of WFs for QFTs

WF spanned on the quantum phase-space cannot be a proper probability distribution function. The uncertainty principles are the main obstacles. Thus for the theory in the quantum phase-space we do not have any *a priori* given probability distribution functions. This causes two problems. First one is that WFs, especially many-particle WFs, may be chosen in different ways (this is already the case with quantum mechanics [2]). The other one is that phenomenological derivations of the transport equations, based on the conservation of the probability densities, may fail. Still another problem is that since QTT is the quasimacroscopic theory and it uses WFs only, we want the field operators to vanish from the transport equations. In conclusion, while defining WFs for QFTs, the following consistency conditions should be satisfied:

(1°) we should make it possible to express the physical quantities of the field in terms of WFs;

(2°) the transport equations should be derived rigorously from the field equations;

(3°) the definition of the WFs should lead to the ϕ-closed transport equations (we will call an equation ϕ-closed if it does not contain any averaged field operators ϕ, which cannot be expressed in terms of (different) WFs).

The conditions (1°)–(3°) are very natural ones, but, existing definitions of WFs for QFTs mostly do not fulfil all of them. This is because those approaches are originally kinetic ones, and therefore the condition (3°) is not fulfiled. It is not enough to have the certain number of coupled field operators to say that the transport equations form the hierarchy. The hierarchy should be exact, i.e. be formed by WFs.

2b. WF for the real scalar field in the multiparticle production approach

The above difficulties should be cured, in the systematic way, by introducing many--particle WFs, but the only known definition does not fulfil the consistency conditions either. Carruthers and Zachariasen [4] have introduced their many-particle WFs, for the real scalar field, in the context of multiparticle production. Their N-particle WF reads

$$F_N(p_1R_1, \ldots, p_NR_N) = \int \prod_{j=1}^{N} dr_j \exp(ip_jr_j) \langle T(\phi(R_1-\tfrac{1}{2}r_1) \ldots \phi(R_N-\tfrac{1}{2}r_N))$$

$$\times T(\phi(R_1+\tfrac{1}{2}r_1) \ldots \phi(R_N+\tfrac{1}{2}r_N))\rangle. \tag{2.1}$$

The authors have shown that it is possible to connect the Fourier transform of the N-particle WF $\bar{F}_N$ with the inclusive probability for the production of N particles

$$P_{\text{incl.}} = \sum_X |S_{fi}(X)|^2$$

$$= \left\{ \prod_{j=1}^{N} \frac{[(p_j+\frac{1}{2}q_j)^2-\mu^2]\,[(p_j-\frac{1}{2}q_j)^2-\mu^2]}{2(2\pi)^3\omega_j} \bar{F}_N(p_1q_1, \ldots, p_Nq_N) \right\}_{q_j=0}, \tag{2.2}$$

where $\omega_j^2 = p^2+\mu^2$.

In spite of this attractive connection, the approach has one important shortcoming: the transport equations are not ϕ-closed. Even in the simplest case, the transport equation for the one-particle WF, so called pairing approximation is necessary to make the equation ϕ-closed. In the equation for the two-particle WF, the expressions that appear cannot be expressed or sensibly approximated in terms of WFs. The multiparticle production approach becomes a mere formality, due to the fact that the transport equations are not ϕ-closed.

2c. WF for scalar fields in the thermodynamical approach

We define our many-particle WFs in such a way to connect them with the thermodynamical quantities of the field. For the N-particle WF for the complex scalar field we take the averaged product of the one-particle Wigner operators [8]

$$f_N(p_1R_1, \ldots, p_NR_N) = \int \prod_{j=1}^{N} dr_j \exp(ip_jr_j) \langle\!\langle \phi^+(R_1+ar_1)\phi(R_1-ar_1)$$

$$\ldots \phi^+(R_N+ar_N)\phi(R_N-ar_N)\rangle\!\rangle_{\text{reg}}, \tag{2.3}$$

where $dr_j \equiv d^4r_j$ and the constant a will be chosen equal to $1/\sqrt{2}$ to get the proper classical limit of the transport equations. The average $\langle\!\langle \ldots \rangle\!\rangle$ takes into account both field and thermal degrees of freedom. The field operators are in the Heisenberg picture. The composite field operator in (2.3) is regularized by means of the Zimmermann ordering.

Let us consider the complex scalar field with the Lagrangian density:

$$\mathscr{L}(x) = \partial_\mu\phi^*\partial^\mu\phi - m^2(x)\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2, \tag{2.4}$$

with the effective mass $m^2(x) = \mu^2+\mu^2(x)$, where $\mu^2(x)$ is the form of the couplings of external fields. With the use of the definition (2.3) one can express the densities of the averaged physical quantities of the quantized field. The quantities of interest are the ones for non-equilibrium thermodynamical description. It is a straightforward calculation to check the identities for the energy-momentum tensor

$$\mathscr{T}_{\alpha\beta}(x) = \frac{a^{-2}}{2}\int dpp_\alpha p_\beta f_1(p, x) - \frac{a^{-2}}{2} g_{\alpha\beta}\int dpp^\mu p_\mu f_1(p, x)$$

$$+[\tfrac{1}{2}\partial_\alpha\partial_\beta + 2g_{\alpha\beta}m^2(x)]\int dpf_1(p, x) - 3\frac{\lambda}{2}g_{\alpha\beta}\int dp_1 dp_2 f_2(p_1, x, p_2, x), \tag{2.5}$$

for the Noether current associated with the global U(1) symmetry

$$j_\alpha(x) = a^{-1}\int dpp_\alpha f_1(p, x) \tag{2.6}$$

and for the nonconserved total number of particles density

$$n(x) = \int dpf_1(p, x). \tag{2.7}$$

Here $dp \equiv d^4p/(2\pi)^4$, $g_{\mu\nu} = (+, - - -)$ and the field equations in the quantum version have been used to get Eq. (2.5). One sees that for the field theory with the (self) interaction term in the Lagrangian there always is the need to introduce many-particle WFs.

3. The transport equations

3a. The hierarchy of general transport equations

The transport equations are derived by rewriting the field equations for the field operators present in the WFs. Using the quantum version of the fields equations in the form

$$(\Box_i - \Box_{i+1})\langle\phi^+(1)\phi(2)\ldots\phi^+(i)\phi(i+1)\ldots\phi^+(2N-1)\phi(2N)\rangle_{\rm reg}$$
$$= \langle\phi^+(1)\phi(2)\ldots j^+(i)\phi(i+1)\ldots\phi^+(2N-1)\phi(2N)\rangle_{\rm reg}$$
$$-\langle\phi^+(1)\phi(2)\ldots\phi^+(i)j(i+1)\ldots\phi^+(2N-1)\phi(2N)\rangle_{\rm reg} \tag{3.1}$$

and applying the identities

$$\partial_{(R_j)\mu}\partial^\mu_{(r_j)} \equiv a(\Box_{x_j} - \Box_{y_j}), \quad x_j \equiv R_j + ar_j, \quad y_j \equiv R_j - ar_j, \tag{3.2}$$

we arrive at the equation for WF without the Fourier transform in the definiton (2.3). Performing the Fourier transform we get the transport equation for the N-particle WF

$$\sum_{j=1}^N p_{j\alpha}\partial^\alpha_j f_N(p_1R_1, \ldots, p_NR_N) = ai\sum_{j=1}^N \int dr \exp(ip_jr)\int dp \exp(-ipr)$$

$$\times f_N(p_1R_1, \ldots, \overset{(j)}{p\,R_j}, \ldots, p_NR_N)\,[\mu^2(R_j+ar) - \mu^2(R_j-ar)]$$

$$-\lambda ai\sum_{j=1}^N \int dr \exp(ip_jr)\int dp'dp \exp(-ipr)$$

$$\times [f_{N+1}(p_1R_1, \ldots, \overset{(J)}{p' R_j+ar}, pR_j, \ldots, p_NR_N)$$

$$-f_{N+1}(p_1R_1, \ldots, pR_j, \overset{(J)}{p'R_j-ar}, \ldots, p_NR_N)]. \tag{3.3}$$

Above hierarchy of the transport equations links higher WFs with first two WFs, on which the physical quantities depend directly. The level of truncation of the hierarchy (3.3) will determine how precisely the physical quantities (2.5)–(2.7) will be known.

3b. The kinetic case

The simplest way to truncate the hierarchy (3.3) is to assume that particles are uncorrelated, i.e. that the n-particle WF is equal to the product of the one particle WFs. Then we have to deal with the one-particle WF only. This is the case of the kinetic theory. The transport equation is then

$$p_1^\alpha \partial_{1\alpha} f_1(p_1, R_1) - ai \int dr \exp(ip_1 r) \int dp \exp(-ipr) f_1(p, R_1)$$

$$\times \{\mu^2(R_1+ar) - \mu^2(R_1-ar) - \lambda \int dp' [f_1(p', R_1+ar) - f_1(p', R_1-ar)]\} = 0. \tag{3.4}$$

The collision term in the equation (3.4) is the very complicated integral. In order to write it in the quasilocal form we develop $\mu^2(x)$ and WFs in the Taylor series. Due to the form of the collision integral, the terms in this series appear as the quantum corrections

$$p_1^\alpha \partial_\alpha f_1(p_1, R_1) - \partial_\alpha \mu_{sh}^2(R_1) \partial_{(p)}^\alpha f_1(p_1, R_1)$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k(2k+1)!} \partial_{\alpha_1} \ldots \partial_{\alpha_{2k+1}} \mu_{sh}^2(R_1) \partial_{(p)}^{\alpha_1} \ldots \partial_{(p)}^{\alpha_{2k+1}} f_1(p_1, R_1) = 0, \tag{3.5}$$

where

$$\mu_{sh}^2(x) = \mu^2(x) - \lambda \int dp' f_1(p', x) \tag{3.6}$$

is the shifted effective mass. The k-term in the series is proportional to $\hbar^{2k}$. The transport equation in this form resemble the relativistic Vlasov-type equation with the series of the quantum corrections.

3c. Beyond the kinetic case

For the Lagrangian density (2.4) the physical quantities depend upon f_1 and f_2 only. To get these functions as the independent ones, one has to truncate the hierarchy at the level higher than that in the kinetic case. First truncation beyond the kinetic case neglects the three-body correlations. Then the equation for f_1 is strictly Eq. (3.3). The three-particle WFs, present in the equation for f_2, are approximated using f_1, f_2 and assuming the three--body correlation function vanishes. The resulting transport equation for f_2 complicates a lot. The equations in the full integral form are given in the Appendix.

The set of the coupled integro-differential equations (A.1) and (A.4) is the self-consistent one. The equations may be expressed in the semiclassical form too.

4. *Other scalar fields*

For the real scalar field the same definition of the WFs as (2.3) may be used. The Hermitean conjugations in (2.3) act trivially. The current density (2.6) vanishes identically.

For the case of the multicomponent scalar field $O(N)$ WFs should be supplied by extra indices.

5. *Summary*

We have given the alternative definition of the many-particle WFs for scalar fields. Our approach is based on the regularized composite operators. The definition fulfils the consistency conditions of WFs for QFTs: the quantities of non-equilibrium thermodynamics such as the energy-momentum, the current and the total number of particles densities may be expressed in terms of our WFs; the transport equations derived rigorously from the field equations are ϕ-closed. The transport equations form the exact hierarchy. The truncation of the hierarchy gives very complicated equations, but the equations may be expressed as the sum of the classical parts and the series of the quantum corrections.

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APPENDIX

In this appendix we give the transport equations truncated by neglecting three-body correlations.

The equation for f_1 reads

$$p_{1\alpha}\partial^{\alpha}f_1(p_1, R_1) = ai \int dr \exp(ip_1 r) \int dp \exp(-ipr) f_1(p, R_1)$$

$$\times [\mu^2(R_1+ar)-\mu^2(R_1-ar)] - \lambda ai \int dr \exp(ip_1 r) \int dp' dp \exp(-ipr)$$

$$* [f_2(p'R_1+ar, pR_1) - f_2(pR_1, p'R_1-ar)]. \tag{A.1}$$

The identities defining two-body correlations

$$f_2(\mathrm{I}, \mathrm{II}) = f_1(\mathrm{I})f_1(\mathrm{II}) + g_2(\mathrm{I}, \mathrm{II}) \tag{A.2}$$

and three-body correlations

$$f_3(\mathrm{I}, \mathrm{II}, \mathrm{III}) = f_1(\mathrm{I})f_1(\mathrm{II})f_1(\mathrm{III}) + f_1(\mathrm{I})g_2(\mathrm{II}, \mathrm{III})$$

$$+ f_1(\mathrm{II})g_2(\mathrm{I}, \mathrm{III}) + f_1(\mathrm{III})g_2(\mathrm{I}, \mathrm{II}) + g_3(\mathrm{I}, \mathrm{II}, \mathrm{III}), \tag{A.3}$$

imply the right hand side of Eq. (3.3) for $N = 2$

$$[p_{1\alpha}\partial_1^{\alpha} + p_{2\alpha}\partial_2^{\alpha}] f_2(p_1R_1, p_2R_2)$$

$$= \{ai \int dr \exp(ip_1 r) \int dp \exp(-ipr) f_2(pR_1, p_2R_2)$$

$$[\mu^2(R_1+ar) - \mu^2(R_1-ar)] + (1 \Leftrightarrow 2)\}$$

$$+\lambda ai \int dr \exp(ip_1 r) \int dp' dp \exp(-ipr) \{f_1(pR_1) [2f_1(p_2R_2) f_1(p'R_1+ar)$$
$$-2f_1(p_2R_2) f_1(p'R_1-ar) + f_2(p'R_1-ar, p_2R_2) - f_2(p'R_1+ar, p_2R_2)]$$
$$+f_1(p_2R_2) [f_2(pR_1, p'R_1-ar) - f_2(p'R_1+ar, pR_1)]$$
$$+f_2(pR_1, p_2R_2) [f_1(p'R_1-ar) - f_1(p'R_1+ar)]\}$$
$$+\lambda ai \int dr \exp(ip_2 r) \int dp' dp \exp(-ipr) \{f_1(pR_2) [2f_1(p_1R_1) f_1(p'R_2+ar)$$
$$-2f_1(p_1R_1) f_1(p'R_2-ar) + f_2(p_1R_1, p'R_2-ar) - f_2(p_1R_1, p'R+ar)]$$
$$+f_1(p_1R_1) [f_2(pR_2, p'R_2-ar) - f_2(p'R_2+ar, pR_2)]$$
$$+f_2(p_1R_1, pR_2) [f_1(p'R_2-ar) - f_1(p'R_2+ar)]\} \tag{A.4}$$

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