ON THE THEORY OF FIELDS IN FINSLER SPACES — V

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Some structural considerations are made on the field equations in the theory of fields in Finsler spaces. In particular, much attention is paid to the Finslerian gravitational field equations proposed by Miron.

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1. *Introduction*

As has been mentioned in a previous paper [1], the Finslerian gravitational field may be regarded as a unified field between the Einstein gravitational field spanned by points $\{x\}$ and the internal field spanned by vectors $\{y\}$. The former, which will be called the (x)-field, is, of course, dominated by the Riemann metric $\gamma_{\lambda\kappa}(x)$ $(\kappa, \lambda = 1, 2, 3, 4)$, while the latter, which will be called the (y)-field, is governed by the Riemann metric $h_{ij}(y)$ $(i, j = 1, 2, 3, 4)$, in general. In other words, the Finslerian field is "nonlocalized" and "multi-dimensionalized" by the vector y (cf. [2]). In the previous paper [1], in order to reduce the dimension number 8 of the unified field to 4, a compactification process of the internal (y)-field has been considered, where the (y)-field is mapped on the external (x)-field by means of the so-called N-mapping. The quantity $N = (N^{\kappa}_{i}, N^{i}_{\lambda})$ denotes the nonlinear connection, which plays the most important role in the theory of vector bundles (cf. [3], see below). As a result, a new Finsler metric

$$g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y);$$
$$h_{\lambda\kappa}(x, y) \equiv N^{i}_{\lambda} N^{j}_{\kappa} h_{ij}(y) \tag{1.1}$$

has been induced for the Finslerian gravitational field (see (2.6) of [1]). The term $h_{\lambda\kappa}$ may be interpreted in various ways such as the metrical fluctuation, the non-gravitational or material effect, etc. (cf. [4]). It should be remarked that N cannot be regarded as the vierbein, because N combines h_{ij} with $h_{\lambda\kappa}$, not with $g_{\lambda\kappa}$.

On the other hand, from the vector bundle-like standpoint [3, 5], the (y)-field is regarded as the fibre at the point x of the base (x)-field. Therefore, the unified field mentioned above may be compared to the total space of this vector bundle, which has an eight-dimensional Riemannian structure with the Riemann metric $G_{AB}(X^A)$ $(X^A = (x^\kappa, y^i);\ A = (\kappa, i) = 1, 2, 3, \ldots, 8)$. In this paper, we shall make some structural considerations on the field equations for the Finslerian gravitational field on the basis of differential geometry of the total space (cf. [3]), without taking account of the dimension-reduction-process represented by the N-mapping. (The dimension-reduction-process corresponds to the compactification of the (y)-field.) In particular, we shall focus our attention on the Miron field equations [3, 5], which are obtained from the Einstein field equation for the total space.

2. *On the geometrical structures*

As mentioned above, the internal (y)-field is unified with the external (x)-field and the resulting unified field has an eight-dimensional Riemannian structure. In this unified field, the following adapted frame is set from a general standpoint (cf. [3, 5]):

$$\left(\frac{\partial}{\partial\zeta^A}\right) \equiv \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\right),$$

$$(\delta\zeta^A) \equiv (dx^\kappa, \delta y^i = dy^i + N^i_\lambda dx^\lambda), \tag{2.1}$$

where N is geometrically fixed as the horizontal distribution supplementary to the vertical distribution in the tangent space of the vector bundle. From (2.1), the so-called decomposition factors are defined by (cf. (2.4) of [1])

$$A^B_\lambda = (\delta^\kappa_\lambda, -N^i_\lambda), \quad A^\kappa_B = (\delta^\kappa_\lambda, 0),$$

$$B^A_i = (0, \quad \delta^j_i), \quad B^i_A = (N^i_\lambda, \delta^i_j). \tag{2.2}$$

The metric component $g_{\lambda\kappa}$ of (1.1) is, therefore, given in this form by decomposing G_{AB} by means of (2.2), i.e., $g_{\lambda\kappa} = A^A_\lambda A^B_\kappa G_{AB}$ (cf. [6]), under the assumption that

$$G_{AB} = \begin{pmatrix} \gamma_{\lambda\kappa}(x) & 0 \\ 0 & h_{ij}(y) \end{pmatrix}.$$

In our theory [1], however, N has been determined physically from the inherent law of the internal variable y such as the rotation $\bar{y}^i = K^i_j(x)y^j$, which is reformulated in the form of intrinsic parallelism as follows: $\delta y^i = dy^i + N^i_\lambda dx^\lambda = 0$, where $N^i_\lambda \equiv K^i_{j\lambda}y^j$ and $K^i_{j\lambda} \equiv -\dfrac{\partial K^i_j}{\partial x^\lambda}$ (see (2.1) of [1]). This rotation may be regarded as the gauge transformation, different from the Lorentz transformation or the coordinate transformation $\bar{x}^\kappa = \bar{x}^\kappa(x^\lambda)$. As is understood from the above, N prescribes, in general, the interaction between the (x)- and (y)-fields, so that it plays also a role of unified gauge field.

From (2.1), for example, the metric tensor G_{AB} of the total space is written as [3, 5]

$$G(\delta\zeta, \delta\zeta) = G_{AB}\delta\zeta^A\delta\zeta^B$$

$$= g_{\lambda\kappa}(x, y)dx^\kappa dx^\lambda + g_{ij}(x, y)\delta y^i \delta y^j. \tag{2.3}$$

The quantities $g_{\lambda\kappa}$ and g_{ij} express the two kinds of unified Finsler metrices, whose concrete forms such as (1.1) can be discussed by prescribing suitably the concrete form of G_{AB}, as mentioned above.

Now, the connection coefficient Γ^A_{BC} in the total space is stipulated formally by the covariant derivative $\frac{\nabla_\partial}{\partial\zeta^C}\frac{\partial}{\partial\zeta^B} = \Gamma^A_{BC}\frac{\partial}{\partial\zeta^A}$, where the components are denoted by (see (2.2) of [1]) $\Gamma^A_{BC} \equiv (F^\kappa_{\lambda\mu}, F^i_{j\lambda}, \Theta^\kappa_{\lambda l}, \Theta^i_{jk})$. That is to say, the covariant derivatives of a vector $V^A = (V^\kappa, V^i)$ are defined by (see (2.3) of [1])

$$V^\kappa_{|\mu} = \frac{\delta V^\kappa}{\delta x^\mu} + F^\kappa_{\lambda\mu}V^\lambda, \quad V^i_{|\mu} = \frac{\delta V^i}{\delta x^\mu} + F^i_{j\mu}V^j;$$

$$V^\kappa_{|k} = \frac{\partial V^\kappa}{\partial y^k} + \Theta^\kappa_{\lambda k}V^\lambda, \quad V^i_{|k} = \frac{\partial V^i}{\partial y^k} + \Theta^i_{jk}V^j. \tag{2.4}$$

On the other hand, the almost Hermitian structure such as

$$G(JX, JY) = G(X, Y) \tag{2.5}$$

can be easily introduced by means of the almost complex structure $J\circ J = -I$. However, it should be remarked that there exist two kinds of almost Hermitian structures [7]: One is

$$J = J^\kappa_\lambda(x, y)\frac{\delta}{\delta x^\kappa}\otimes dx^\lambda + J^i_j(x, y)\frac{\partial}{\partial y^i}\otimes\delta y^j,$$

$$J^\kappa_\mu J^\lambda_\nu g_{\kappa\lambda} = g_{\mu\nu}, \quad J^i_k J^j_l g_{ij} = g_{kl} \tag{2.6}$$

and the other is

$$J = J^\kappa_i\frac{\delta}{\delta x^\kappa}\otimes\delta y^i + J^i_\lambda\frac{\partial}{\partial y^i}\otimes dx^\lambda;$$

$$J^\kappa_i J^\lambda_j g_{\kappa\lambda} = g_{ij}, \quad J^i_\kappa J^j_\lambda g_{ij} = g_{\kappa\lambda}. \tag{2.7}$$

Generally, the two Finsler metrices $g_{\kappa\lambda}$ and g_{ij} have no relation with each other, so that the case of (2.6) seems to be suitable. More specialized structures such as Hermitian, almost Kählerian or Kählerian, etc. can also be introduced, if necessary.

3. *On the field equations*

Since the total space has an eight-dimensional Riemannian structure with the unified metric G_{AB}, as mentioned before, its field equation may be written in the same form as the Einstein field equation, i.e.,

$$\mathscr{R}_{AB} - \tfrac{1}{2}\mathscr{R}G_{AB} = \tau_{AB}, \tag{3.1}$$

where $\mathscr{R}_{AB}$ ($\equiv \mathscr{R}^C_{ABC}$) and $\mathscr{R}$ ($\equiv \mathscr{R}_{AB}G^{AB}$) denote the Ricci-tensor and the total scalar curvature respectively, and τ_{AB} is the energy-momentum tensor. Starting from (3.1), Miron [3, 5] proposed the following field equations for the Finslerian gravitational field by decomposing (3.1) based on (2.1):

$$\begin{aligned} R_{\nu\lambda}-\tfrac{1}{2}(R+S)g_{\nu\lambda} &= \overset{1}{\tau}_{\nu\lambda}, \\ \overset{1}{P}_{j\lambda} &= \overset{2}{\tau}_{j\lambda}, \\ \overset{2}{P}_{\nu j} &= -\overset{3}{\tau}_{\nu j}, \\ S_{jk}-\tfrac{1}{2}(R+S)g_{jk} &= \overset{4}{\tau}_{jk}, \end{aligned} \tag{3.2}$$

where $\tau_{AB} \equiv (\overset{1}{\tau}_{\nu\lambda}, \overset{2}{\tau}_{j\lambda}, \overset{3}{\tau}_{\nu j}, \overset{4}{\tau}_{jk})$. To reach (3.2), it is necessary to obtain the components of $\mathscr{R}^C_{ABD}$, $\mathscr{R}_{AB}$ and $\mathscr{R}$. We shall here describe only the results for the sake of simplicity (For details, see [3, 5]):

The curvature tensor formed with Γ^A_{BC}:

$$\mathscr{R}^C_{ABD} \equiv (R^\kappa_{\nu\lambda\mu}, R^i_{j\lambda\mu}; P^i_{j\lambda k}, P^\kappa_{\nu\lambda j}; S^\kappa_{\nu ij}, S^i_{jkl}). \tag{3.3}$$

The Ricci tensor $\mathscr{R}_{AB}$:

$$\mathscr{R}_{AB} \equiv (R_{\nu\lambda} \equiv R^\kappa_{\nu\lambda\kappa}, \overset{1}{P}_{j\lambda} \equiv P^i_{j\lambda i}, \overset{2}{P}_{\nu j} \equiv -P^\kappa_{\nu\kappa j}, S_{jk} \equiv S^i_{jki}). \tag{3.4}$$

The total scalar $\mathscr{R}$:

$$\mathscr{R} \equiv R_{\nu\lambda}g^{\nu\lambda}+S_{jk}g^{jk} \equiv R+S. \tag{3.5}$$

Hitherto, the Finslerian field equations have been derived under quite special conditions (cf. [8]): For example, the field equation for the indicatrix $R_{ab}-\frac{1}{2}Rg_{ab} = \tau_{ab}$ $(a, b = 1, 2, 3)$ is extended to the whole field in the form $S_{\nu\lambda}-\frac{1}{2}Sg_{\nu\lambda}-g_{\nu\lambda} = \tau_{\nu\lambda}$ by means of the Gauss equation; Or, from the conservation law $(R_\mu{}^\kappa-\frac{1}{2}R\delta^\kappa_\mu-\frac{1}{2}KSy_\mu y^\kappa)_{|\kappa} = 0$ in the case of constant curvature K, a field equation such as $R_{\mu\kappa}-\frac{1}{2}Rg_{\mu\kappa}-\frac{1}{2}KSy_\mu y_\kappa = \tau_{\mu\kappa}$ is introduced; etc. Therefore, Miron's method seems to be quite systematic and his results (3.2) are very instructive. In the following, we shall pick up some interesting features underlying (3.2).

The decomposition of (3.1) can be considered in various ways according to the different frames, instead of (2.1). But the adapted frame (2.1) seems to be the most simple one and does not lose any physical essence.

The components of energy-momentum tensor τ_{AB} have different meanings, so that $\overset{1}{\tau}_{\nu\lambda} = 0$ and $\overset{4}{\tau}_{jk} = 0$, for example, do not mean the same vacuum states.

It should be noticed that these field equations contain the eight-dimensional effects, which are explicitly embodied in the S-term in $(3.2)_1$ or the R-term in $(3.2)_4$. The former may be compared to the cosmological term summarizing the contributions of the internal field.

If the compactification of the internal (y)-field is taken into account, then the R-term in $(3.2)_4$ must vanish and then $(3.2)_1$ becomes formally $R_{\nu\lambda}-\frac{1}{2}Sg_{\nu\lambda} = \overset{1}{\tau}_{\nu\lambda}$ ($= 0$ or $\neq 0$). In this case, $(3.2)_2$ and $(3.2)_3$ cannot be treated precisely, but they may be supposed to vanish, because the total space may approach a direct-product structure of the (x)- and (y)-fields. These situations provide a geometrical background for the complete compactification of the internal space (cf. [9]). In future, some relation with the theory of Inflation Universe should be considered under these situations.

By the way, Miron [3, 5] also proposed the following conservation laws by decomposing $\frac{\nabla_\partial}{\partial\zeta^B}(\mathcal{R}_A{}^B-\frac{1}{2}\mathcal{R}\delta^B_A) = 0$:

$$[R_\nu{}^\lambda - \tfrac{1}{2}(R+S)\delta^\lambda_\nu]_{|\lambda}+\overset{1}{P}_\nu{}^i{}_{|i} = 0,$$

$$[S_j{}^k-\tfrac{1}{2}(R+S)\delta^k_j]|_k-\overset{2}{P}_j{}^\lambda{}_{|\lambda} = 0. \tag{3.6}$$

From the standpoint of (3.2), such special cases as $\overset{1}{P}_\nu{}^i{}_{|i} = 0$ and $\overset{2}{P}_j{}^\lambda{}_{|\lambda} = 0$ or $\overset{1}{P}_\nu{}^i = 0$ and $\overset{2}{P}_j{}^\lambda = 0$ seem to be physically suitable. (But these specializations should be motivated by some physical conditions).

4. *Other comments*

In this Section, we shall consider one interesting special case by taking $y^i \equiv y^1 \equiv x^0$ (independent scalar). Then, we can rewrite (2.1), (2.3), (3.4) and (3.5) as follows (For details, see [3]):

$$\frac{\partial}{\partial\zeta^A} \equiv \left(\frac{\delta}{\delta x^\lambda} = \frac{\partial}{\partial x^\lambda} - N^0_\lambda\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right),$$

$$\delta\zeta^A \equiv (dx^\kappa, \quad \delta x^0 = dx^0+N^0_\lambda dx^\lambda). \tag{4.1}$$

$$G = G_{AB}\delta\zeta^A\delta\zeta^B = g_{\lambda\kappa}(x^\kappa, x^0)dx^\kappa dx^\lambda+g_{00}(x^\kappa, x^0)\delta x^0\delta x^0. \tag{4.2}$$

$$\mathcal{R}_{AB} \equiv (R_{\nu\lambda}, \overset{1}{P}_{0\lambda}, \overset{2}{P}_{\nu 0}, S_{00} = 0). \tag{4.3}$$

$$\mathcal{R} \equiv R, (S = 0). \tag{4.4}$$

The decomposition factors introduced by (2.2) are now reduced to

$$A^A_\lambda = (\delta^\kappa_\lambda, -N^0_\lambda), \quad A^\kappa_A = (\delta^\kappa_\lambda, 0),$$

$$B^A_0 = (0, 1), \qquad B^0_A = (N^0_\lambda, 1). \tag{4.5}$$

Therefore, if we apply (4.5) to G_{AB} as in (1.1), then we can obtain, e.g., $g_{\lambda\kappa}(x^\kappa, x^0) = \gamma_{\lambda\kappa}(x^\kappa + N^0_\lambda N^0_\kappa h_{00}\ (x^0)$, where $g_{00} \equiv h_{00}$ in this case.

With the aid of the relations (4.1)–(4.4), (3.2) is reformulated as

$$\begin{aligned} R_{\nu\lambda} - \tfrac{1}{2} R g_{\nu\lambda} &= \overset{1}{\tau}_{\nu\lambda}, \\ \overset{1}{P}_{0\lambda} &= \overset{2}{\tau}_{0\lambda}, \\ \overset{2}{P}_{\nu 0} &= -\overset{3}{\tau}_{\nu 0}, \\ -\tfrac{1}{2} R g_{00} &= \overset{4}{\tau}_{00}. \end{aligned} \tag{4.6}$$

Concerning (4.6), the same things as mentioned at the end of Section 3 can also be said. From $(4.6)_4$, if $\overset{4}{\tau}_{00} = 0$, then $R = 0$ or $g_{00} = 0$ must be satisfied: If $R = 0$, then $(4.6)_1$ becomes $R_{\nu\lambda} = \overset{1}{\tau}_{\nu\lambda}$ ($= 0$ or $\neq 0$).

In order to find the physical meaning of x^0 and N^0_λ, we shall here compare (4.2) with the square of arc length in the classical Kaluza-Klein theory (cf. [10]), i.e.,

$$\begin{aligned} G = G_{AB}\delta\zeta^A\delta\zeta^B &= G_{00}(dx^0)^2 + 2G_{\kappa 0}dx^0 dx^\kappa + (G_{\lambda 0}dx^\lambda)^2 \\ &+ (G_{\mu\nu} - G_{\mu 0}G_{\nu 0})dx^\mu dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu + (dx^0 + A_\mu dx^\mu)^2, \end{aligned} \tag{4.7}$$

where we have put $g_{\mu\nu} \equiv G_{\mu\nu} - A_\mu A_\nu$, $A_\mu \equiv G_{\mu 0} = G_{0\mu}$ and $G_{00} \equiv 1$.

Therefore, we can put in (4.2) $g_{\lambda\kappa}(x^\kappa, x^0) \equiv g_{\lambda\kappa}$ and $g_{00}(x^\kappa, x^0) \equiv 1$ with the definition $\delta x^0 \equiv dx^0 + A_\mu dx^\mu$. That is to say, x^0 is compared with the fifth coordinate of the Kaluza--Klein theory and N^0_μ is regarded as the electromagnetic potential A_μ. Under these assumptions, the torsion tensor defined by $R^0_{\lambda\mu} \equiv \dfrac{\delta N^0_\lambda}{\delta x^\mu} - \dfrac{\delta N^0_\mu}{\partial x^\lambda}$ may be identified with the (Finslerian) electromagnetic field tensor. In this case, if it is assumed that $F^\kappa_{\lambda\mu} = F^\kappa_{\mu\lambda}$ and $N^0_\lambda = N^0_\lambda(x^\kappa)$, then the Maxwell equations of the first class, i.e., $R^0_{\lambda\mu|\kappa} + R^0_{\mu\kappa|\lambda} + R^0_{\kappa\lambda|\mu} = 0$ can be obtained through a kind of Bianchi identity (cf. [12]). Therefore, the new meaning of $R^0_{\lambda\mu}$ may be somewhat effective from a physical viewpoint. These facts have not been noticed within the Finslerian or Lagrangian theory of electromagnetism (cf. [11]).

In the Lagrangian theory of electromagnetism [11], the (Finslerian) electromagnetic field tensor $F_{\lambda\kappa}$ is defined by

$$\begin{aligned} F_{\lambda\kappa}(x, y) &\equiv \tfrac{1}{2}(D_{\lambda\kappa} - D_{\kappa\lambda}); \\ D_{\lambda\kappa} &= g_{\lambda\nu}D^\nu_\kappa, \quad D^\nu_\kappa \equiv y^\nu{}_{|\kappa}, \end{aligned} \tag{4.8}$$

where D^ν_κ is called the deflection tensor. At this general stage, however, (4.8) cannot be given by the rotation of the (Finslerian) vector potential $A_\mu(x, y)$. So, in order to reduce (4.8) to the form

$$F_{\lambda\kappa} = A_{\lambda|\kappa} - A_{\kappa|\lambda}, \tag{4.9}$$

it is assumed in [11] that the metric tensor $g_{\lambda\kappa}(x, y)$ derived from the Lagrangian $\mathcal{L}(x, y)$ $\left(\text{i.e., } g_{\lambda\kappa} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^\lambda \partial y^\kappa}\right)$ becomes homogeneous of degree 0 in y. Then, $\mathcal{L}(x, y)$ is proved to be given by

$$\mathcal{L}(x, y) = g_{\lambda\kappa}(x, y)y^\kappa y^\lambda + A_\lambda(x)y^\lambda + U(x), \tag{4.10}$$

where $g_{\lambda\kappa}$, A_λ and U are physically regarded, from the standpoint of unified field theory, as the gravitational potential, the electromagnetic potential and the external potential respectively. By doing so, the general theory based on (4.8) becomes, for the first time, the electromagnetic theory by means of (4.9). Without (4.10), the Lagrangian theory of electromagnetism will lose all physical meaning.

In conclusion, our own theory of electromagnetism based on $R^0_{\lambda\mu}$ and (4.6) should be investigated in future; the theory is quite different from the Lagrangian theory.

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