ON THE NEWTONIAN LIMIT OF GENERAL RELATIVITY

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The relations between the Newton-Cartan theory, Newton's classical gravitational theory and General Relativity are discussed. It is shown that the limit $c \to \infty$ of General Relativity becomes identical with the Newton-Cartan theory, provided the so-called time function — defined as proportional to the singular part of the covariant Einsteinian metric in the transition $c \to \infty$ — satisfies a certain boundary condition at spatial infinity. Using this limiting process, one obtains an asymptotic representation of Einsteinian fields "near" the Newtonian limit. This should allow us to identify post-Newtonian corrections for the Newton-Cartan theory.

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1. *Introduction*

Newton's gravitational theory governs the motions in the Solar system, post-Newtonian corrections are very small here. The theory has been applied also to larger systems, even to the whole Universe, establishing a Newtonian cosmology similar to Einstein's [1–5]. The question, to what extent we can trust Newtons's theory also in a cosmological context, — considering Einstein's as the correct one — is still open however [4–8]. A direct comparison between Newton's and Einstein's gravitational theories is difficult since the mathematical and physical framework of both theories is so different. Fortunately, there exists a space-time formulation of Newton's theory, dating back to Cartan [9] and Friedrichs [10], which has much similarity with the space-time formulation of Einstein's theory (see Ehlers [11] for a general review). Moreover, the Newton-Cartan theory can be obtained as that limiting case of Einstein's theory where Einstein's light cones open up to become the hypersurfaces of equal Newtonian time [10, 12]. For some Einsteinian fields an asymptotic representation becomes possible, where the lowest order terms just give the Newton-Cartan fields. We conjecture that a study of higher-order terms will help to clarify the relation between Newton's and Einstein's cosmology. The present article is a first contribution towards an answer to these questions. In Section 2 and 3 a short exposition of the Newton-Cartan theory is given, including some aspects of its physical interpretation. In Section 4 the transition from General Relativity to Newton-Cartan structures is considered.

2. *Newton-Cartan theory*

On a four-dimensional differentiable manifold V two geometrical objects are introduced, a degenerate semidefinite contravariant tensor field $h^{\alpha\beta}$ of rank 3, und a symmetric affine connection $\Gamma_{\alpha\beta}{}^{\mu}$. From the degeneracy of $h^{\alpha\beta}$ follows the existence of a covariant vector field t_α with

$$h^{\alpha\beta}t_\beta = 0. \tag{1}$$

$h^{\alpha\beta}$ and t_α are assumed to be covariantly constant with respect to the affine connection $\Gamma_{\alpha\beta}{}^{\mu}$:

$$h^{\alpha\beta}{}_{|\mu} = 0, \qquad t_{\alpha|\beta} = 0. \tag{2}$$

From (2) and the symmetry of the connection follows that t_α is the gradient of a scalar field $t(x^\alpha)$ on V, called "absolute Newtonian time", $t_\alpha = t_{,\alpha}$. The conditions (1), (2) do not fix the connection in terms of the fields $h^{\alpha\beta}$, t_α, some parts of $\Gamma_{\alpha\beta}{}^{\mu}$ serve as additional field components: With $\Gamma_{\alpha\beta}{}^{\mu}$ all connections generated by

$$\bar{\Gamma}_{\alpha\beta}{}^{\mu} = \Gamma_{\alpha\beta}{}^{\mu} + h^{\mu\sigma}K_{\sigma(\alpha}t_{\beta)} \tag{3}$$

also satisfy (1), (2), if $K_{\alpha\beta}$ is antisymmetric. This class of connections was called "Galilean" by Künzle [13]. Newton's theory corresponds to a restricted class of connections. The restrictions are formulated as conditions for the Riemann tensor formed with the connection, first given by Trautman [15]

$$h^{\alpha[\beta}R^{\tau]}{}_{(\mu\sigma)\alpha} = 0. \tag{4}$$

(Eq. (4) is only apparently weaker then condition IV in [15]. If [1, 2] are taken into account, both are equivalent. Note also; throughout this paper we follow the sign conventions of Misner, Thorne and Wheeler [14].) Galilean connections satisfying (4) were called "Newtonian" by Künzle [13]. As shown in Section 4, Newtonian connections are just those following from the limiting process $c \to \infty$ of General Relativity. Even then the resulting geometrical framework is more general than necessary for a spacetime formulation of Newton's theory. This generality can be removed, if also field equations are considered. But in this case boundary conditions at spatial infinity must be introduced, and this confines the theory to finite (localized) Newtonian matter distributions. Alternatively, one might restrict the geometry by a further local condition on the Riemann tensor, for instance [16]

$$h^{\alpha\beta}R^{\tau}{}_{\alpha\mu\sigma} = 0. \tag{5}$$

This condition does not follow from the $c \to \infty$ limit of General Relativity. We therefore drop (5) subsequently.

Field equations relate — as in General Relativity — geometrical quantities $h^{\alpha\beta}$, t_α $\Gamma_{\alpha\beta}{}^{\mu}$ to the matter distribution. A simple perfect fluid model with matter density ϱ will be assumed. The almost unique [16] choice of a field equation is similar to General Relativity:

$$R_{\alpha\beta} = 4\pi G \varrho t_\alpha t_\beta, \tag{6}$$

where $R_{\alpha\beta}$ is the Ricci tensor formed from the curvature tensor $R^\alpha{}_{\beta\mu\sigma}$. (6) reflects the fact, that only the matter density acts on the geometry directly, specific values of the momentum and stress of cosmic matter have no influence in Newtonian approximation. Let $v^\alpha = dx^\alpha/dt$ with $v^\alpha t_\alpha = 1$ be the Newtonian four velocity of a perfect fluid with pressure p, we have to postulate the equations of motion and conservation of mass independently of the field equations:

$$\varrho v^\alpha{}_{|\beta} v^\beta = -h^{\alpha\beta} p_{,\beta}, \tag{7}$$

$$v^\alpha \varrho_{,\alpha} + \varrho v^\alpha{}_{|\alpha} = 0. \tag{8}$$

Furthermore, one postulates that test particles follow a geodesic to $\Gamma_{\alpha\beta}{}^\mu$:

$$d^2x^\mu/dt^2 + \Gamma_{\alpha\beta}{}^\mu dx^\alpha/dt dx^\beta/dt = 0. \tag{9}$$

The equations of this Section — except of (5) — are the basic equations of the Newton -Cartan theory for a simple fluid.

3. Physical content of the Newton-Cartan theory

To study the physical content of the Newton-Cartan theory, it is useful to introduce adapted coordinates defined by the condition that x^0 equals Newton's time t. In these coordinates, $t_\alpha = t_{,\alpha} = \delta^0{}_\alpha$ and from (1), $h^{00} = h^{0i} = 0$ $(i = 1, 2, 3)$. The adopted coordinate system is fixed up to transformations

$$t' = t + \text{const}, \quad x^{k'} = f^k(x^l, t), \tag{10}$$

where f^k are three arbitrary functions with $\det f^k{}_{,l} \neq 0$. In adapted coordinates the relations (2) are equivalent to

$$\Gamma_{\alpha\beta}{}^0 = 0, \quad \Gamma_{00}{}^i = h^{ik}\Phi_k, \quad \Gamma_{0k}{}^i = h^{il}\dot{h}_{kl}/2 + h^{il}\omega_{kl},$$

$$\Gamma_{kl}{}^i = \left\{{}_{kl}^{\,i}\right\}, \tag{11}$$

(a dot denotes the time derivative). h_{kl} is the inverse matrix to h^{kl}, $\left\{{}_{kl}^{\,i}\right\}$ are the Christoffel symbols calculated with h_{kl} as 3-metric of the hypersurfaces $t = \text{const}$. The quantities Φ_k and $\omega_{kl} = -\omega_{lk}$ are fields remaining undetermined, i.e., they are not fixed by the fields $h^{\alpha\beta}$, t_α. Since Φ_k and ω_{kl} are parts of a connection, they do not simply transform as 3-tensors. We refer to Φ_k as acceleration force field and to ω_{kl} as Coriolis force field. Incidentally we note that the transformation (3) of the affine connection corresponds to the substitution $\Phi_k \to \Phi_k - K_{0k}$, $\omega_{kl} \to \omega_{kl} - K_{kl}/2$.

The Trautman restriction (4) for the Riemann tensor can now be written

$$\Phi_{i,k}-\Phi_{k,i} = -2\dot{\omega}_{ik}, \tag{12}$$

$$\omega_{ik,l}\varepsilon^{ikl} = 0, \tag{13}$$

and the field equations (6) split into

$$R_{kl}(h_{kl}) = 0, \tag{14}$$

$$h^{kl}(\dot{h}_{li||k}-\dot{h}_{lk||i})-h^{kl}\omega_{li||k} = 0, \tag{15}$$

$$h^{kl}\Phi_{k||l}+\omega_{kl}\omega^{kl}-\ddot{h}_{kl}h^{kl}/2-\dot{h}_{kl}\dot{h}^{kl}/4 = 4\pi G\varrho. \tag{16}$$

Here $||$ denotes the covariant derivative with respect to $\{{}_{kl}{}^{i}\}$. Since in three dimensions a vanishing Ricci tensor implies that the Riemann tensor is also zero, (14) states that the inner metric of the hypersurfaces of equal Newtonian time is flat. By means of the transformations (10) one can introduce a special class of adapted coordinates, so-called Galilean coordinates x^i, t, by the condition $h_{ik} = \delta_{ik}$. Galilean coordinates are preserved under transformations of the kinematical group

$$t' = t+\text{const}, \tag{17}$$

$$x^{i'} = a^{ik}(t)x^k+a^i(t) \tag{18}$$

with $a^{ik}a^{il} = \delta_{kl}$ (summation convention), corresponding to the transition to other arbitrarily accelerated and rotated Cartesian coordinate systems.

In Galilean coordinates, (15) becomes $h^{kl}\omega_{il,k} = 0$, which admits a representation $\omega_{ik} = \varepsilon_{ikl}f_{,l}$. From (13), $f_{,ll} = 0, f$ being a harmonic function. This allows to exclude the Coriolis field, if we assume $\omega_{ik} \to$ const (the constant may depend on t) at spatial infinity: $\omega_{ik} \to$ const implies $f \to f^i x_i+f_0$, and the function $\tilde{f} = f-f_i x^i-f_0$ tends to zero at spatial infinity. Since $\tilde{f} = 0$ is a solution, the uniqueness theorem implies $f = f_i x^i+f_0$, or $\omega_{ik} = \omega_{ik}(t)$. A saptially constant Coriolis field allows to introduce a special class of Galilean coordinates ("nonrotating" coordinates) by means of the kinematical group (17), (18) such that the Coriolis force field vanishes for all time in the new coordinates. Indeed, ω_{ik} transforms as

$$\omega'_{ik} = a^{ir}a^{ks}\omega_{rs}+a^{kl}\dot{a}^{il}. \tag{19}$$

(19) represents for $\omega'_{ik} = 0$ ordinary differential equations for the three independent components of a_{ik} (e.g. the Eulerian angles). $\omega_{ik} = 0$ is preserved under restricted kinematical transformations with a rotation matrix independent of t. However, there might be reasons not to impose boundary conditions. It is still an open question whether cosmological solutions require a nontrivial Coriolis field.

Because of (12), the acceleration force field derives from a potential $\Phi_k = \Phi_{,k}$ only if $\dot{\omega}_{ik} = 0$. The remaining equation (16) then simplifies to the Poisson equation for Φ.

$$\Delta\Phi = 4\pi G\varrho, \tag{20}$$

only if $\omega_{ik} = 0$.

For comparison with the $c \to \infty$ limit of General Relativity it is useful to define a covariant metric field $h_{\alpha\beta}$ besides the contravariant field $h^{\alpha\beta}$. Since $h^{\alpha\beta}$ is degenerate, $h_{\alpha\beta}$ cannot be defined uniquely. Let a contravariant vector field n^α with $n^\alpha t_\alpha = 1$ (i.e. a "time--like" or "null" unit vector) and a scalar H be given additionally. Then the relations

$$h_{\alpha\beta}h^{\beta\mu}+t_\alpha n^\mu = \delta^\mu_\alpha, \tag{21}$$

$$h_{\alpha\beta}n^\alpha n^\beta = H \tag{22}$$

uniquely determine a symmetric covariant tensor field $h_{\alpha\beta}$. Multiplying (21) with n^α, it also follows

$$h_{\alpha\beta}n^\beta = -Ht_\alpha. \tag{23}$$

Indeed, in adapted coordinates

$$h_{ik}h^{kl} = \delta^l_i, \quad h_{0i} = -h_{ik}n^k, \quad h_{00} = -H+h_{ik}n^i n^k, \tag{24}$$

so $h_{\alpha\beta}$ is determined by n^α and H. Usually, H is taken as zero [16, 17] and the vector field n^α is chosen such that [17]

$$n^\alpha{}_{|\beta}n^\beta = 0, \tag{25a}$$

$$h^{\alpha\beta}n^\mu{}_{|\beta}-h^{\mu\beta}n^\alpha{}_{|\beta} = 0. \tag{25b}$$

Here we proceed slightly differently. We try to define $h_{\alpha\beta}$ entirely in terms of the affine connection. In adapted coordinates we may introduce four functions Φ (related to a Newtonian potential) and h_{0k} (related to a vector potential), depending on all coordinates, by the differential equations

$$\dot{h}_{0k}+\Phi_{,k} = \Phi_k, \quad h_{0l,k}-h_{0k,l} = 2\omega_{kl}. \tag{26}$$

The integrability conditions for (26) are just (12) and (13). We may then consider the quantities

$$h_{\alpha\beta} = \begin{pmatrix} -2\Phi & h_{0k} \\ h_{0k} & h_{kl} \end{pmatrix} \tag{27}$$

as components of a covariant 4-tensor in adapted coordinates. Taking into account the proper transformation laws of Φ_k and ω_{kl} as parts of a connection, it can be checked that (27) indeed transforms as covariant tensor under (10). Defining a contravariant vector field t^α by means of

$$t^0 = 1, \quad t^i = -h^{ik}h_{0k}, \tag{28}$$

it is easily seen that (21)–(24) are satisfied for $t^\alpha = n^\alpha$. This also holds for (25b), but in general not for (25a), since (25a) requires $H_{,i} = 0$ in adapted coordinates. (25a) is in line with the assumption $H = 0$ made in [17], but not with our assumptions. The affine connection can now be written in general coordinates as

$$\Gamma^\mu_{\alpha\beta} = t^\mu t_{(\alpha,\beta)}+h^{\mu\sigma}(h_{\sigma\alpha,\beta}+h_{\sigma\beta,\alpha}-h_{\alpha\beta,\sigma})/2. \tag{29}$$

Notice that $h_{\alpha\beta}$ is in general not constant covariantly, for instance

$$h_{\alpha\beta|\mu}t^{\mu} = (H,_{\alpha}t_{\beta}+H,_{\beta}t_{\alpha})/2. \tag{30}$$

We further emphasize that t^{μ} and $h_{\alpha\beta}$ are not uniquely determined by $\Gamma_{\alpha\beta}{}^{\mu}$, t_{α} and $h^{\alpha\beta}$. It is seen from (26), that Φ and h_{0k} are fixed only up to the gauge

$$\bar{\Phi} = \Phi - F,_{0}, \quad \bar{h}_{0k} = h_{0k}+F,_{k}, \tag{31}$$

with an arbitrary function $F(x^i, t)$. h_{kl} is gauge-invariant, but t^i (see (28)) changes under a gauge. In general coordinates, $h_{\alpha\beta}$, t^{α} and H transform as

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta}+F,_{\alpha}t_{\beta}+F,_{\beta}t_{\alpha}, \tag{32}$$

$$\bar{t}^{\alpha} = t^{\alpha}-h^{\alpha\beta}F,_{\beta}, \tag{33}$$

$$\bar{H} = H+2f,_{\alpha}t^{\alpha}-h^{\alpha\beta}F,_{\alpha}F,_{\beta}. \tag{34}$$

The transformed quantities also satisfy all relations (21)–(24), (27)–(30). The affine connection is gauge-invariant. The gauge transformations (31) can be traced back to higher--order coordinate transformations in the $c \to \infty$ expansion of General Relativity (Section 4).

Sometimes the connection (29) is divided into components said to describe "inertial" and "gravitational" forces:

$$\Gamma_{\alpha\beta}{}^{\mu} = \mathring{\Gamma}_{\alpha\beta}{}^{\mu}+t_{\alpha}t_{\beta}h^{\mu\sigma}\psi_{,\sigma}, \tag{35}$$

where ψ is a suitable scalar function. (35) is a particular case of (3) with $K_{\gamma\beta} = \psi,_{\beta}t_{\alpha}-\psi,_{\alpha}t_{\beta}$, hence, if $\Gamma_{\alpha\beta}{}^{\mu}$ is a Galilean connection, so also $\mathring{\Gamma}_{\alpha\gamma}{}^{\mu}$, the only change is that the gradient part of Φ_k is different. The division (35) is clearly not unique if ψ is identified with the potential Φ, since it can be changed by the gauge (31). Instead, one may define ψ by dividing the gauge-independent quantitiy Φ_k into a rotation-free part $\psi,_{k}$ and a divergence-free part Φ_k, $\Phi_k = \psi,_{k}+\Phi_k$. Although this splitting is gauge-invariant, it lacks in general physical motivation, since $\mathring{\Gamma}_{\alpha\beta}{}^{\mu}$ still contains gravitational forces such as the Coriolis field, if the Riemann tensor calculated with $\mathring{\Gamma}_{\alpha\beta}{}^{\mu}$ does not vanish. It is to some degree a matter of tast whether the Coriolis field should be considered as "gravitational" or as "inertial" force (see also [18]). Although the field is sourceless, it cannot be transformed away globally if $\omega_{ik||l}$ is nonzero. It would be better to speak of a gravitational force of purely geometrical origin. Only if the Coriolis field is completely rejected, the Riemann tensor to $\mathring{\Gamma}_{\alpha\beta}{}^{\mu}$ vanishes, and the separation (35) makes sense physically (cf. Ehlers [11]).

To summarize, the physical content of the Newton-Cartan theory as defined in Section 2 appears to be slightly larger then that of Newton's classical theory. If a boundary condition at spatial infinity is imposed for the Coriolis force field, both theories coincide.

4. Newton-Cartan theory as degenerate case of General Relativity

We now turn to General Relativity and show, extending and improving an earlier derivation [12], that a certain degeneration of this theory leads asymptotically to the relations obtained in the previous Sections. Let $g^{\alpha\beta}(x^\mu, \varepsilon)$ be a family of contravariant metrics, depending smoothly on a parameter ε, and with a signature $(-+++)$ for every $\varepsilon \neq 0$. We further assume that $g^{\alpha\beta}$ satisfies the Einstein field equations for all $\varepsilon > 0$ and tends to a degenerate tensor $h^{\alpha\beta}$ of rank 3, signature $(0+++)$ for $\varepsilon \to 0$. $1/\varepsilon$ will later be taken as the square of the velocity of light, or $\varepsilon = 1/c^2$. A macroscopic matter tensor will be assumed, such that

$$\kappa(T_{\alpha\beta} - g_{\alpha\beta}T/2) = T^*_{\alpha\beta} \tag{36}$$

tends to a finite limit $\underset{0}{T^*_{\alpha\beta}}$ for $\varepsilon \to 0$ (note $\kappa = 8\pi G\varepsilon$). We shall show below that this limiting behaviour of the matter tensor is compatible with a reasonable Newtonian limit for a perfect fluid. We finally assume that for small ε an asymptotic representation

$$g^{\alpha\beta}(x^\mu, \varepsilon) = h^{\alpha\beta}(x^\mu) + \varepsilon \underset{1}{g^{\alpha\beta}}(x^\mu) + O(\varepsilon^2) \tag{37}$$

should be possible. Then the covariant metric admits a corresponding representation

$$g_{\alpha\beta}(x^\mu, \varepsilon) = \underset{-1}{g_{\alpha\beta}}(x^\mu)/\varepsilon + h_{\alpha\beta}(x^\mu) + \varepsilon \underset{1}{g_{\alpha\beta}}(x^\mu) + O(\varepsilon^2), \tag{38}$$

with a singular term $\underset{-1}{g_{\alpha\beta}}$. The relations between co- and contravariant components give to lowest orders

$$h^{\alpha\mu} \underset{-1}{g_{\beta\mu}} = 0, \tag{39}$$

$$\underset{1}{g^{\alpha\beta}} \underset{-1}{g_{\beta\mu}} + h^{\alpha\beta} h_{\beta\mu} = \delta^\alpha_\mu. \tag{40}$$

From (39) it is seen that $\underset{-1}{g_{\alpha\beta}}$ can be written

$$\underset{-1}{g_{\alpha\beta}} = -t_\alpha t_\beta. \tag{41}$$

With the minus sign $g_{\alpha\beta}$ has the correct signature. t_μ an eigen vector (to the eigenvalue zero) of $h^{\alpha\beta}$:

$$h^{\alpha\beta} t_\beta = 0. \tag{42}$$

We also have

$$-\underset{1}{g^{\alpha\beta}} t_\alpha t_\beta = 1 \tag{43}$$

from (40)–(42).

The Christoffel symbols admit a representation which follows from (37), (38). A short calculation gives

$$\Gamma_{\alpha\beta}{}^{\mu} = \varepsilon^{-1} \underset{-1}{\Gamma}{}_{\alpha\beta}{}^{\mu} + \underset{0}{\Gamma}{}_{\alpha\beta}{}^{\mu} + \varepsilon \underset{1}{\Gamma}{}_{\alpha\beta}{}^{\mu} + O(\varepsilon^2), \tag{44}$$

where

$$\underset{-1}{\Gamma}{}_{\alpha\beta}{}^{\mu} = h^{\mu\sigma}(t_{[\sigma,\alpha]}t_\beta + t_{[\sigma,\beta]}t_\alpha), \tag{45}$$

$$\underset{0}{\Gamma}{}_{\alpha\beta}{}^{\mu} = h^{\mu\sigma}(h_{\sigma\alpha,\beta} + h_{\sigma\beta,\alpha} - h_{\alpha\beta,\sigma})/2 + t^\mu t_{(\alpha,\beta)}$$

$$- \underset{1}{g}{}^{\mu\sigma}(t_\alpha t_{[\sigma,\beta]} + t_\beta t_{[\sigma,\alpha]}). \tag{46}$$

We have used the abbreviations

$$t^\alpha = -\underset{1}{g}{}^{\alpha\beta} t_\beta, \tag{47}$$

note $t^\alpha t_\alpha = 1$ from (43).

The behaviour of the terms in the expansion (37), (38) under coordinate transformations is easy to derive. Similar to the metric we expand the transformation functions in powers of ε:

$$\bar{x}^\alpha = \underset{0}{\bar{x}}{}^\alpha(x^\mu) + \xi^\alpha(x^\mu)\varepsilon + \zeta^\alpha(x^\mu)\varepsilon^2 + O(\varepsilon^3). \tag{48}$$

All expansion terms in (37), (38) and the singular components of the Christoffel symbols (45) transform as tensors under zero-order transformations $\bar{x}^\alpha = \underset{0}{\bar{x}}{}^\alpha(x^\mu)$, (46) transforms as affine connection. For the gauges

$$\bar{x}^\alpha = x^\alpha + \xi^\alpha(x^\mu)\varepsilon \tag{49}$$

we have

$$\bar{h}^{\alpha\beta} = h^{\alpha\beta}, \underset{1}{\bar{g}}{}^{\alpha\beta} = \underset{1}{g}{}^{\alpha\beta} + h^{\mu\beta}\xi^\alpha{}_{|\mu} + h^{\mu\alpha}\xi^\beta{}_{|\mu}, \tag{50}$$

$$\bar{t}_\alpha = t_\alpha, \quad \bar{h}_{\alpha\beta} = h_{\alpha\beta} + t_\beta(t_\mu\xi^\mu)_{|\alpha} + t_\alpha(t_\mu\xi^\mu)_{|\beta}, \tag{51}$$

where | denotes the covariant derivative with respect to the affine connection (46). With $F = t_\alpha\xi^\alpha$, (51) becomes (32). Thus the gauge transformation in the Newton-Cartan theory results from a higher-order coordinate transformation in the $c \to \infty$ limit of General Relativity.

The expansion of the Ricci tensor in powers of ε contains in general two singular terms

$$R_{\alpha\beta} = \varepsilon^{-2} \underset{-2}{R}{}_{\alpha\beta} + \varepsilon^{-1} \underset{-1}{R}{}_{\alpha\beta} + \underset{0}{R}{}_{\alpha\beta} + O(\varepsilon). \tag{52}$$

Since $T^{*}_{\alpha\beta}$ tends to a finite limit as $c \to \infty$, the singular terms in (52) must vanish:

$$\underset{-2}{R}_{\alpha\beta} = 0, \tag{53}$$

$$\underset{-1}{R}_{\alpha\beta} = 0. \tag{54}$$

(53) reduces to the simple relation

$$t_{[\alpha,\mu]}t_{[\beta,\sigma]}h^{\alpha\sigma}h^{\mu\beta} = 0. \tag{55}$$

In a local adapted coordinate system $t_\alpha = \delta^0_{\alpha,\mu}$, which can be introduced at any point of the manifold, it can be shown that (55) is algebraically equivalent to

$$t_{[\alpha,\beta]}t_\mu + t_{[\beta,\mu]}t_\alpha + t_{[\mu,\alpha]}t_\beta = 0 \tag{56}$$

(note that h^{ik} is a positive-definite metric). (56) shows that t_α can be written

$$t_\mu = ht_{,\mu}, \tag{57}$$

where t and h are two scalar functions. The field equation (54) reads

$$\underset{-1}{\Gamma}_{\alpha\beta}{}^{\mu}{}_{|\mu} = 0, \tag{58}$$

or with (45) and (57)

$$(h^{\alpha\beta}h_{,\beta}t_\mu t_\sigma/h)_{|\alpha} = 0. \tag{59}$$

A straightforward calculation yields

$$t_{\alpha|\beta} = -t_\beta(h_{,\alpha} - t_\alpha h_{,\mu}t^\mu)/h, \tag{60}$$

$$h^{\alpha\beta}{}_{|\mu} = h_{,\sigma}t_\mu(t^\alpha h^{\beta\sigma} + t^\beta h^{\alpha\sigma})/h, \tag{61}$$

and (59) becomes a differential equation for the function h:

$$h^{\alpha\beta}h_{,\alpha|\beta} = 0. \tag{62}$$

At first look (62) seems to be nonlinear, since h enters the affine connection (46). Actually however, (62) is linear in h: Using adapted coordinates with $x^0 = t$, the expression

$$\underset{0}{\Gamma}_{\alpha\beta}{}^{\mu}h^{\alpha\beta} = h^{\mu\sigma}h^{\alpha\beta}(h_{\sigma\alpha,\beta} + h_{\sigma\beta,\alpha} - h_{\alpha\beta,\sigma})/2$$

vanishes for $\sigma = 0$ and gives $\{{}_{ik}^{\,l}\}h^{ik}$ for $\sigma = 1$, hence (62) is a Laplace equation in the space of the definite three-dimensional Riemannian metric h_{ik},

$$h^{ik}h_{,i||k} = 0 \tag{63}$$

(notation as in Section 3). The dominant term of an expansion of the interval $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ is $ds^2 = -(cdt)^2h^2$, thus the proper time between two events separated by a Newtonian time interval dt is hdt, not dt. It is therefore justified to call h the time function. In the Newton-Cartan theory discussed in Sections 2 and 3 the time function is trivial, $h = 1$. This also follows from the differential equation (63) for h, if one imposes the boundary condition that h tends to a constant at the spatial infinity of every hypersurface $t = $ const, and invokes the uniqueness theorem. Then $h = h(t)$ everywhere. In a cosmological context more general solutions for h might exist. In our case, one introduces $\int h(t)dt$ as a new Newtonian time, subsequently also denoted by t. This is equivalent to put $h = 1$ in all previous equations. In particular, (60), (61) now become the basic relations (2) of the Newton-Cartan theory. The singular part (45) of the connection vanishes, and the expansion of the Ricci tensor starts with the regular term $\underset{0}{R}_{\alpha\beta}$ formed with the connection (46), which attends the form (29). Also the Trautman condition (4) follows (but not the Dixon condition (5)): (4) holds already in General Relativity with $h^{\alpha\beta}$ replaced by $g^{\alpha\beta}$. An expansion for the Riemann tensor similar to (44), but with regular terms only, gives (4) immediately. To discuss the further field equations in the expansion (44), a model for the matter tensor must be selected. We assume a simple perfect fluid

$$T_{\alpha\beta} = 8\pi G\varepsilon(\varrho+\varepsilon p)u_\alpha u_\beta + 8\pi G\varepsilon^2 p g_{\alpha\beta}. \tag{64}$$

The four velocity u^α is connected with the Newtonian velocity $v^\alpha = dx^\alpha/dt$ by the relation

$$u^\alpha = v^\alpha dt/ds, \tag{65}$$

and the expansion of the relativistic interval ds yields

$$ds/dt = (1-\varepsilon h_{\alpha\beta}v^\alpha v^\beta/2 + O(\varepsilon^2))/\sqrt{\varepsilon}. \tag{66}$$

This allows to calculate $T^*_{\alpha\beta}$ as defined by (36) as $4\pi G\varrho t_\alpha t_\beta$. Thus also (6) follows from the general-relativistic field equations. Similarly, an expansion of $T^{\alpha\beta}{}_{;\beta} = 0$ yields (7), (8). We have thus obtained the Newton-Cartan theory in the version given in Sections 2 and 3.

Let us shortly summarize the assumptions (for more exact definitions of the limiting procedure see [11, 19]). We consider a family of contravariant metric tensors $g^{\alpha\beta}$ labeled by a parameter ε, $g^{\alpha\beta}$ satisfies the Einstein field equations for all ε. For $\varepsilon \to 0$, $g^{\alpha\beta}$ is assumed to tend to a singular metric $h^{\alpha\beta}$ of rank 3, signature $(0+++)$, and $T^*_{\alpha\beta}$ as given by (36) should have a finite limit. For small ε, an asymptotic representation (37), (38) for the metric should be possible. Then the existence of a preferred set of spacelike hypersurfaces, called the hypersurfaces of equal Newtonian time, can be established, and one obtains a slighly extended version of the Newton-Cartan theory as degenerate limit $\varepsilon \to 0$ of General Relativity. To obtain the original Newton-Cartan theory, a further condition must be assumed. Define a time function $h(x^\alpha)$ as relating the relativistic interval ds to the Newtonian dt by $h(x^\alpha) = ds/dt$ in the limit $\varepsilon \to 0$. For h an elliptic partial differential equation holds. A boundary condition that h should tend to a constant at spatial infinity of the Newtonian hypersurfaces is sufficient to reduce the extended version to the usual Newton-Cartan theory.

5. *Concluding remark*

The method of the last Section may be used to derive higher-order corrections to the Newton-Cartan theory. The Newtonian approximation is given by the components of $h_{\alpha\beta}$, post-Newtonian term are contained in $\underset{1}{g}_{\alpha\beta}$. The situation is different for the contravariant components of the metric tensor. Here the Newtonian approximation is distributed over parts of $\underset{0}{g}^{\alpha\beta}$, $\underset{1}{g}^{\alpha\beta}$, $\underset{2}{g}^{\alpha\beta}$. Let us denote the space components of a covariant symmetric tensor $G_{\alpha\beta}$ by ${}_{SS}G$, by ${}_{ST}G$ the space-time components and by ${}_{TT}G$ the time component of $G_{\alpha\beta}$. In adapted coordinates, ${}_{SS}G = G_{ik}$, ${}_{ST}G = G_{0i}$, ${}_{TT}G = G_{00}$. For the contravariant tensor the notation is ${}^{SS}G$ ${}^{ST}G$, ${}^{TT}G$. Then the Newtonian approximation is given by ${}^{SS}\underset{0}{g}$, ${}^{ST}\underset{1}{g}$ and ${}^{TT}\underset{2}{g}$. In general, for all i $\geqslant 2$:

$$ {}^{TT}\underset{i}{g} = {}_{TT}\underset{i-2}{g}, \quad {}^{ST}\underset{i}{g} = {}_{ST}\underset{i-1}{g}, \quad {}^{SS}\underset{i}{g} = {}_{SS}\underset{i}{g}. $$

It is therefore not appropriate to use the four-dimensional representation of the Newton--Cartan theory also for its post-Newtonian corrections. A 3+1 split of the field equations is more appropriate. Post-Newtonian corrections to the Newton-Cartan theory will be discussed in a subsequent paper.

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