BIRKHOFF'S THEOREM IN THE GENERALIZED FIELD THEORY

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It is shown that, contrary to previous expectations, Birkhoff's theorem is valid in the Generalized Field Theory.

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1. Introduction

It has been suggested several years ago (Ref. [1]) that there may exist genuinely time-dependent, spherically symmetric solutions of the system of field equations usually referred to (e.g., Ref. [2]) as the nonsymmetric Generalized Field Theory (GFT in the sequel). In an attempt to discover a solution of this type, it will be shown now that this result was wrong and, therefore, that Birkhoff's theorem holds rigorously. It seems that the reason for the previous mistake was insufficient attention apid to the so-called metric hypothesis

$$\tilde{\Gamma}^{\lambda}_{(\mu\nu)} = \left\{ {\lambda \atop \mu\nu} \right\}_a, \tag{1}$$

essential to the theory since it determines the metric $a_{\mu\nu}$ of the background Riemannian manifold — the space-time in which we actually live. It is now known (Ref. [2]) that, as long as very weak topological conditions are satisfied and which express the way in which we view the macrophysical continuum, the latter must be hyperbolic Riemannian. Hence, in the time-dependent, spherically symmetric case, its metric is of the form

$$ds^2 = c^2dt^2 + 2h\, dt\, dr - a^2dr^2 - b^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where a, b, c and h are functions of t and r only. However, if we now introduce a transformation of coordinates

$$\tau = \tau(t, r), \ \ \varrho = r \tag{2}$$

such that

$$h \frac{\partial \tau}{\partial t} - c^2 \frac{\partial \tau}{\partial r} = 0, \tag{3}$$

and this normal (Ref. [3]) partial differential equation is, in general, soluble (i.e., the transformation (2) exists), the cross term $d\tau d\varrho$ cancels out and we can put

$$\varrho = b$$

as well. Therefore, we can assume the background metric to have the much simpler form

$$ds^2 = c^2(t,r)dt^2 - a^2(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{4}$$

The corresponding Christoffel brackets are

$$\begin{Bmatrix}0\\00\end{Bmatrix} = \frac{c_0}{c}, \quad \begin{Bmatrix}0\\01\end{Bmatrix} = \frac{c_1}{c}, \quad \begin{Bmatrix}0\\11\end{Bmatrix} = \frac{aa_0}{c^2},$$

$$\begin{Bmatrix}1\\00\end{Bmatrix} = \frac{cc_1}{a^2}, \quad \begin{Bmatrix}1\\01\end{Bmatrix} = \frac{a_0}{a}, \quad \begin{Bmatrix}1\\11\end{Bmatrix} = \frac{a_1}{a}, \quad \begin{Bmatrix}1\\22\end{Bmatrix} = \begin{Bmatrix}1\\33\end{Bmatrix} \operatorname{cosec}^2\theta = -\frac{r}{a^2},$$

$$\begin{Bmatrix}2\\12\end{Bmatrix} = \begin{Bmatrix}3\\13\end{Bmatrix} = \frac{1}{r}, \quad \begin{Bmatrix}2\\33\end{Bmatrix} = -\sin\theta\cos\theta, \quad \begin{Bmatrix}3\\23\end{Bmatrix} = \cot\theta, \tag{5}$$

where $c_0 = \frac{\partial c}{\partial t}$, $c_1 = \frac{\partial c}{\partial r}$, etc., as in General Relativity.

2. *The affine connection*

The nonsymmetric affine connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ (to distinguish it from the "physical" connection $\Gamma^{\lambda}_{\mu\nu}$) for which

$$\tilde{\Gamma}_{\mu} \equiv \tilde{\Gamma}^{\sigma}_{[\mu\sigma]} \equiv 0 \tag{6}$$

(we use square brackets about the indices to denote the skew symmetric, and round ones, the symmetric part of an indexed quantity), is given by the equation

$$g_{\mu\nu,\lambda} - \tilde{\Gamma}^{\sigma}_{\mu\lambda} g_{\sigma\nu} - \tilde{\Gamma}^{\sigma}_{\lambda\nu} g_{\mu\sigma} = 0. \tag{7}$$

Its symmetric part is, by the metric hypothesis, given by the equations (5). Let us write

$$g_{\mu\nu} = h_{\mu\nu} + k_{\mu\nu} = h_{\nu\mu} - k_{\nu\mu} \tag{8}$$

and

$$h_{00} = \gamma, \quad h_{01} = \eta, \quad h_{11} = -\alpha, \quad h_{22} = h_{33}\operatorname{cosec}^2\theta = -\beta,$$

$$k_{01} = -v, \quad k_{23} = u\sin\theta \tag{9}$$

where α, β, γ, η, u and v are functions of t and r only. This is the most general form of the time-dependent, spherically symmetric field

$$g_{\mu\nu},$$

which is also independent of the expression (4) chosen for the metric

$$a_{\mu\nu}.$$

It is now easy to show that the only significant ones of Eq. (7) are

$$\gamma_0 - 2\frac{c_0}{c}\gamma - 2\frac{cc_1}{a^2}\eta = 0, \quad \gamma_1 - 2\frac{c_1}{c}\gamma - 2\frac{a_0}{a}\eta = 2\tilde{\Gamma}^1_{[01]}v,$$

$$\eta_0 - \left(\frac{c_0}{c} + \frac{a_0}{a}\right)\eta + cc_1\left(\frac{\alpha}{a^2} - \frac{\gamma}{c^2}\right) = -\tilde{\Gamma}^1_{[01]}v,$$

$$\eta_1 - \left(\frac{c_1}{c} + \frac{a_1}{a}\right)\eta + aa_0\left(\frac{\alpha}{a^2} - \frac{\gamma}{c^2}\right) = 0,$$

$$\frac{r\eta}{a^2} = -\tilde{\Gamma}^3_{[02]}u \sin\theta, \quad 0 = \tilde{\Gamma}^1_{[23]}v - \tilde{\Gamma}^3_{[03]}u \sin\theta,$$

$$\alpha_0 - 2\frac{a_0}{a}\alpha + 2\frac{c_1}{c}\eta = 0 = \alpha_1 - 2\frac{a_1}{a}\alpha + 2\frac{aa_0}{c^2}\eta,$$

$$r\left(\frac{\beta}{r^2} - \frac{\alpha}{a^2}\right) = -\tilde{\Gamma}^3_{[12]}u \sin\theta, \quad 0 = \tilde{\Gamma}^0_{[23]}v,$$

$$\beta_0 = -2\tilde{\Gamma}^3_{[02]}u \sin\theta, \quad \beta_1 - \frac{2}{r}\beta = -2\tilde{\Gamma}^3_{[12]}u \sin\theta,$$

$$v_0 - \left(\frac{c_0}{c} + \frac{a_0}{a}\right)v = -\tilde{\Gamma}^1_{[01]}\eta, \quad v_1 - \left(\frac{c_1}{c} + \frac{a_1}{a}\right)v = \tilde{\Gamma}^1_{[01]}\alpha,$$

$$\frac{rv}{a^2} = \tilde{\Gamma}^2_{[02]}\beta, \quad 0 = \tilde{\Gamma}^2_{[03]}\beta + \tilde{\Gamma}^0_{[23]}\gamma + \tilde{\Gamma}^1_{\{23\}}\eta,$$

$$\frac{u}{r}\sin\theta = -\tilde{\Gamma}^2_{[13]}\beta - \tilde{\Gamma}^0_{[23]}\eta + \tilde{\Gamma}^1_{[23]}\alpha,$$

$$u_0 = -2\tilde{\Gamma}^2_{[03]}\beta \operatorname{cosec}\theta, \quad u_1 - \frac{2}{r}u = -2\tilde{\Gamma}^2_{[13]}\beta \operatorname{cosec}\theta, \tag{10}$$

as well as

$$\tilde{\Gamma}^2_{[03]} = -\tilde{\Gamma}^3_{[02]}\sin^2\theta, \quad \tilde{\Gamma}^2_{[13]} = -\tilde{\Gamma}^3_{[12]}\sin^2\theta,$$

and

$$\tilde{\Gamma}^3_{[03]} = \tilde{\Gamma}^2_{[02]} = -1/2\tilde{\Gamma}^1_{[01]}.$$

We immediately see that either $\tilde{\Gamma}^0_{[23]} = 0$ or $v = 0$ but more about this in a moment. We also have

$$uu_0 + \beta\beta_0 = 0, \quad uu_1 + \beta\beta_1 = r\frac{2}{r}(u^2 + \beta^2),$$

whence

$$u^2 + \beta^2 = k^2 r^4, \tag{11}$$

k^2 being a constant. Since equation (7) (together with the metric hypothesis (1)) implies that

$$\det(g_{\mu\nu}) \propto \det(a_{\mu\nu})$$

we can write, without loss of generality.

$$k^2(\alpha\gamma + \eta^2 - v^2) = c^2 a^2. \tag{12}$$

If now $v \neq 0$ (so that $\tilde{\Gamma}^0_{[23]} = 0$) then we have (since $\tilde{\Gamma}^2_{[03]} = -\tilde{\Gamma}^3_{[02]} \sin^2\theta$)

$$\tilde{\Gamma}^3_{\{02\}} = \frac{ru\eta}{\beta^2 a^2} \operatorname{cosec}\theta = -\frac{r\eta}{a^2 u} \operatorname{cosec}\theta.$$

whence

$$\eta = 0 \quad (\tilde{\Gamma}^3_{[02]} = 0 = \tilde{\Gamma}^2_{[03]} \rightarrow u_0 \Rightarrow \beta_0 = 0),$$

so that

$$aa_0\left(\frac{\alpha}{a^2} - \frac{\gamma}{c^2}\right) = 0.$$

If a is independent of t ($a_0 = 0$) then so is α and it is easily shown that c is, in this case, at most a product of a function of r and a function of t only. Then, however, a simple transformation on t alone eliminates the time dependence as in General Relativity. Hence Birkhoff's theorem follows without any further investigation of the field equations.
If, on the other hand,

$$\frac{\alpha}{a^2} = \frac{\gamma}{c^2}$$

then

$$\tilde{\Gamma}^1_{[01]} = \tilde{\Gamma}^2_{[02]} = \tilde{\Gamma}^3_{[03]} = 0 \tag{13}$$

and therefore

$$v = 0. \tag{14}$$

In other words, if a time-dependent solution exists, it is necessary to assume (13), when

$$k^2(\alpha\gamma+\eta^2) = a^2c^2, \tag{15}$$

and equations (13) hold again.

Let us now put

$$\gamma = fc^2, \quad \alpha = ga^2, \quad \eta = hac, \quad \beta = r^2B \quad \text{and} \quad u = r^2U. \tag{16}$$

Equation (15) then becomes

$$fg+h^2 = 1/k^2 \tag{15'}$$

and Eqs (10) give immediately

$$\begin{aligned}
\tilde{\Gamma}^3_{[02]} &= -\frac{ch}{raU}\operatorname{cosec}\theta, \quad \tilde{\Gamma}^2_{[03]} = \frac{ch}{raU}\sin\theta,\\
\tilde{\Gamma}^3_{[12]} &= -\frac{B-g}{rU}\operatorname{cosec}\theta, \quad \tilde{\Gamma}^2_{[13]} = \frac{B-g}{rU}\sin\theta,\\
\tilde{\Gamma}^0_{[23]} &= -\frac{k^4rh}{acU}\sin\theta, \quad \tilde{\Gamma}^1_{[23]} = \frac{r(k^4f-B)}{a^2U}\sin\theta,
\end{aligned} \tag{17}$$

$$(U^2+B^2 = k^2)$$

and

$$\begin{aligned}
f_0-2\frac{c_1}{a}h &= 0, \quad f_1-2\frac{a_0}{c}h = 0,\\
g_0+2\frac{c_1}{a}h &= 0, \quad g_1+2\frac{a_0}{c}h = 0,\\
h_0+\frac{c_1}{a}(g-f) &= 0, \quad h_1+\frac{a_0}{c}(g-f) = 0.
\end{aligned} \tag{18}$$

The first four of Eqs (18) now give a further relation:

$$f+g = l^2, \tag{19}$$

a constant, with the last two becoming

$$h_0+\frac{c_1}{a}(l^2-2f) = 0, \quad h_1+\frac{a_0}{c}(l^2-2f) = 0. \tag{20}$$

Let us now consider the integrability condition

$$h_{01} = h_{10}$$

of Eqs (20). An easy calculation shows that this is

$$\frac{a_{00}}{c} - \frac{a_0 c_0}{c^2} - \frac{c_{11}}{a} + \frac{c_1 a_1}{a^2} = 0, \tag{21}$$

since the case $f = g = l^2/2$ is equivalent to $\alpha/a^2 = \gamma/c^2$ and has already been excluded.

3. *The field equations*

The remaining field equations of GFT are

$$R_{(\mu\nu)} = 0, \qquad R_{[[\mu\nu],\lambda]} = 0 \tag{22}$$

where $R_{\mu\nu}$ is the Ricci tensor

$$R_{\mu\nu} = -\tilde{\Gamma}^{\sigma}_{\mu\nu,\sigma} + \tilde{\Gamma}^{\sigma}_{(\mu\sigma),\nu} + \tilde{\Gamma}^{\sigma}_{\mu\varrho}\tilde{\Gamma}^{\varrho}_{\sigma\nu} - \tilde{\Gamma}^{\sigma}_{\mu\nu}\tilde{\Gamma}^{\varrho}_{(\sigma\varrho)}, \tag{23}$$

constructed from the "geometrical" connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$. (As we shall presently see, there will be no need, for the purpose of proving Birkhoff's theorem, to consider the equation

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu} = 0, \quad \mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}.)$$

With the Christoffel brackets given by (5) and the skew components of the connection by (17), the symmetric components of the Ricci tensor which are not identically zero are

$$R_{00} = \frac{c}{a}\left[-\frac{c_{11}}{a} + \frac{c_1 a_1}{a^2} + \frac{a_{00}}{c} - \frac{a_0 c_0}{c^2}\right] - \frac{2c^2}{ra^2}\left[\frac{c_1}{c} - \frac{h^2}{rU^2}\right],$$

$$R_{01} = -\frac{2c}{ra}\left[\frac{a_0}{c} - \frac{h(B-g)}{rU^2}\right],$$

$$R_{11} = \frac{a}{c}\left[-\frac{a_{00}}{c} + \frac{a_0 c_0}{c^2} + \frac{c_{11}}{a} - \frac{c_1 a_1}{a^2}\right] - \frac{2}{r}\left[\frac{a_1}{a} - \frac{(B-g)^2}{rU^2}\right],$$

$$R_{22}(= R_{33}\operatorname{cosec}^2\theta) = \frac{1}{a^2}\left[1 - a^2 + r\left[\frac{c_1}{c} - \frac{a_1}{a}\right] + 2\frac{k^4 h^2}{U^2} - 2\frac{(B-g)(k^4 f - B)}{U^2}\right]. \tag{24}$$

Equating the above to zero and recalling the integrability condition (21) as well as the integrals (15′) and (19) now requires the following set of equations to be satisfied:

$$g_0 + 2\frac{c_1}{a}h = 0 = g_1 + 2\frac{a_0}{c}h;$$

$$B_0 = 2\frac{ch}{ra}, \quad B_1 = \frac{2}{r}(B-g);$$

$$\frac{c_1}{c} = \frac{h^2}{r(k^2-B^2)}, \quad \frac{a_0}{c} = \frac{h(B-g)}{r(k^2-B^2)}, \quad \frac{a_1}{a} = \frac{(B-g)^2}{r(k^2-B^2)}, \tag{25}$$

and

$$3k^2+\frac{1}{k^2}-l^2g-2k^4(l^2-g)B-a^2(k^2-B^2) = 0.$$

(The last of these equations results from elimination of derivatives from

$$R_{22} = 0.)$$

The integrability conditions

$$a_{01} = a_{10} \quad \text{and} \quad B_{01} = B_{10},$$

both lead to the same condition

$$r\frac{h_1}{h} = 3+\frac{h^2+(B-g)^2}{kk^2-B^2}. \tag{26}$$

However, (19) and (15′) combined give

$$h^2 = \frac{1}{k^2}-(l^2-g)g \tag{27}$$

whence

$$2hh_1 = (-l^2+2g)g_1 = 2\frac{a_0}{c}h(l^2-2g) = 2\frac{h^2(B-g)}{r(k^2-B^2)}(l^2-2g),$$

so that

$$r\frac{h_1}{h} = \frac{(B-g)(l^2-2g)}{k^2-B^2}. \tag{28}$$

Equating (26) and (28) now results in

$$3k^2+\frac{1}{k^2}-l_B^2-2B^2 = 0,$$

so that

$$B = g = \text{constant} \quad \text{and} \quad h = 0.$$

Since $h = 0$ implies that $\eta = 0$, we have already shown that Birkhoff's theorem follows (in fact, we obtain a flat space-time as the only solution possible: it is, of course, not the solution when time dependence is not assumed).

4. *Conclusions*

The fact that Birkhoff's theorem holds has far-reaching consequences for GFT. As we know (Ref. [2]), the theory predicts, in the static, spherically symmetric case, a definite model of the universe. In view of the theorem, the static solution becomes unique. Therefore, the corresponding cosmological model is a deciding test of the theory, in so far as the cosmological evidence is capable of discriminating conclusion and, of course, providing spherical symmetry is a global rather than a local possibility. This remains to be investigated. It is, however, difficult to see how to carry out such an investigation without *a priori* invoking some kind of extraneous cosmological principle. Needles to say, such a principle would disturb the inner coherence of the theory quite as much as it does so in General Relativity.

Editorial note. This article was proofread by the editors only, not by the author.

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