Vol. B21 (1990) ACTA PHYSICA POLONICA No 10

ON THE DISTRIBUTION OF THE COMPOUND-NUCLEUS RESONANCES

BY N. ULLAH

Tata Institute of Fundamental Research, Homi Bhaba Road, Bombay 400005, India

(*Received February 5, 1990*)

The problem of distribution of the spacing of compound nucleus resonances is studied by using an identity which expresses a determinant in terms of the trace of log of the matrix. An explicit connection between the two-point correlation function and the fluctuation property of Gaussian Orthogonal Ensemble is shown.

PACS numbers: 24.60.Dr

1. *Introduction*

The compound nucleus reactions are one of the most interesting nuclear reactions which have been extensively studied for quite some time. In the beginning these reactions in which one measures the widths and positions of neutron resonance levels posed a challenging problem to theorists because they involved very large number of nuclear configurations. It was Wigner [1] who put forward the idea of random matrix to explain in a statistical way the widths and positions of these resonances. For the case of real symmetric Hamiltonians (Gaussian Orthogonal Ensemble (GOE)) Wigner gave a simple form of the distribution of the spacing of the eigenvalues of compound-nucleus Hamiltonian known as Wigner's surmise.

Recently there has been a new interest in the spacing distribution because it has been found [2] that quantum counter parts of classical chaotic systems show the same kind of spacing distribution as the one of GOE. This has necessiated the need to develop simple methods to derive spacing distribution.

Our aim here is to show that an identity which expresses a determinant in terms of trace of log of the matrix can be used to obtain in a simple way to study the problem of spacing distribution both for small values as well as for large values of the spacing.

We describe the formulation in the next Section. The concluding remarks are presented in Section 3.

2. Formulation

The pioneering work on the spacing distribution was carried out by Gaudin [3] who had studied this problem by first establishing an integral equation and then showing that its solutions are spheroidal eigen functions. Further in these studies it was found that it is much simpler if one first finds the probability of an interval which is free of any eigenvalue, the derivative of which gives exact distribution. It therefore suffices to derive expression for the probability distribution of the empty interval.

The starting point of our formulation is the expression [3] which gives the probability E of finding an interval s around origin free of any eigenvalues. It is given by

$$E = \det(1-n), \tag{1}$$

where 1 denotes unit matrix and the matrix elements of the matrix n are given by

$$n_{\mu\nu} = \int_{-s/2}^{s/2} \phi_{2\mu}(x)\phi_{2\nu}(x)dx, \tag{2}$$

$\phi_n(x)$ being the harmonic oscillator wave functions. They are given by [4]

$$\phi_n(x) = \pi^{-1/4}(2^n n!)^{-1/2} \exp\left(-\tfrac{1}{2}x^2\right)H_n(x), \tag{3}$$

H_n being the Hermite polynomial.

In Gaudin's method the probability E is expressed in terms of the eigenvalues of an integral equation, the solutions of which are spheroidal functions. As will be shown shortly the method which is developed here directly derives an expression for E by expressing the determinant in terms of trace of log of the matrix. A great advantage of the present method is that it does not need the properties of spheroidal functions to find an expression for E.

To proceed further we write the matrix element $n_{\mu\nu}$ as

$$n_{\mu\nu} = s \int_{-1/2}^{1/2} \phi_{2\mu}(sv)\phi_{2\nu}(sv)dv, \tag{4}$$

by putting $x = sv$ in expression (2).

We now use the identity which expresses determinant as trace of log of the matrix. E can then be written as

$$E = \exp(\mathrm{Tr}[\ln(1-n)]). \tag{5}$$

This identity was used earlier [5] to find a general expression for the Fourier transform of the single eigenvalue probability density for the three Gaussian ensembles defined by Dyson [6].

Expanding ln and writing the spacing s in terms of mean spacing $\pi/\sqrt{4m}$, where $N = 2m$ is the dimension of the real Hamiltonian matrix, we get from expression (5)

$$E = \exp\left[-\left(\frac{\pi t}{\sqrt{4m}}\right) \int_{-1/2}^{1/2} \sum_{\mu=0}^{m-1} \phi_{2\mu}^2\left(\frac{\pi t}{\sqrt{4m}}v\right) dv\right.$$

$$-\tfrac{1}{2}\left(\frac{\pi t}{\sqrt{4m}}\right)^2 \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dvdu \left[\sum_{\mu} \phi_{2\mu}\left(\frac{\pi t}{\sqrt{4m}}v\right)\phi_{2\mu}\left(\frac{\pi t}{\sqrt{4m}}u\right)\right]^2 + \ldots \Bigg], \tag{6}$$

where $t = s/(\pi/\sqrt{4m})$.

The limiting forms of the sums in expression (6) are already known [3, 7]. For $m \to \infty$, they can be expressed in terms of the following two functions

$$f(v) = 1 + \frac{\sin 2\pi t v}{2\pi t v}, \tag{7a}$$

$$g(v, u) = \frac{\sin \pi t(v-u)}{\pi t(v-u)} + \frac{\sin \pi t(v+u)}{\pi t(v+u)}. \tag{7b}$$

Using expressions (6), (7) we finally arrive at the following expression for E,

$$E = \exp\left[-\frac{t}{2}\int_{-1/2}^{1/2} dv f(v) - \tfrac{1}{2}\left(\frac{t}{2}\right)^2 \int_{-1/2}^{1/2} dv \int_{-1/2}^{1/2} du\right.$$

$$\left.(g(v,u))^2 - \tfrac{1}{3}\left(\frac{t}{2}\right)^3 \int_{-1/2}^{-1/2} dv \int_{-1/2}^{1/2} du \int_{-1/2}^{1/2} dw g(v,u)g(u,w)g(\omega,v) + \ldots\right]. \tag{8}$$

It is now a simple matter to expand sin functions in expression (7) and carry out the indicated integration in expression (8). This gives us the following expression for E for small t

$$E = \exp\left[-t - \frac{t^2}{2} - \tfrac{1}{3}t^3\left(1 - \frac{\pi^2}{12}\right) - \tfrac{1}{4}t^4\left(1 - \frac{\pi^2}{9}\right)\right.$$

$$\left. - t^5\left(\tfrac{1}{5} - \frac{\pi^2}{36} + \frac{\pi^4}{1200}\right)\ldots\right]. \tag{9}$$

If one wishes one could expand the exponential further and write E as

$$E = 1 - t + \frac{\pi^2}{36}t^3 - \frac{\pi^4}{1200}t^5 + \ldots \tag{10}$$

This is the form which was obtained earlier by Gaudin [3] by using the relation between spheroidal functions and Legendre polynomials for small values of t.

We next consider the expansion of the probability E for large values of t. The leading term in the expansion can easily be obtained by making diagonal approximation in expression (1) and making the substitution

$$x = \frac{s}{2} + u$$

in expression (2). This gives us

$$E(t) = \exp\left[\left(-\frac{\pi^2 t^2}{16}\right) + \ldots\right]. \tag{11}$$

This agrees with the first term of the asymptotic expansion of $E(t)$ given by Dyson [8] who had derived this expansion using Fredholm determinants and some exact results for inverse scattering problems.

3. Concluding remarks

We have shown that the identity which expresses a determinant in terms of Trace of log of the matrix can be used to find in a simple way the expansion of the probability of finding an interval t free of any eigenvalue. It gives the expansion both for small as well as large values of t. Looking at expression (9) we see that the coefiicients in the series in the exponential form decrease faster than the one given by expression (10) in which the exponential is further expanded.

If one also considers the Gaussian Unitary Ensemble (GUE) which is the ensemble of Hermitian Hamiltonians, then all one has to do is to derive similar expressions in which one uses odd Hermite polynomials also. This can be done very easily by putting a minus sign instead of plus sign in expression (7).

Our last remark is that since the function $g(v, u)$ given by expression (7b) is related to the two point correlation function, we find from expression (8) that unlike the average properties of matrix ensembles like single eigenvalue probability density function which is simply related to one point Green's function, the fluctuation property like E is related in quite an involved way with two-point Green's function of the Hamiltonian. This is the first time that such a relation has been explicitly shown.

REFERENCES

[1] E. P. Wigner, Gatlinberg Conference on Neutron Physics, Oak Ridge National Lab. Rep. no. ORNL 2309, p. 59.
[2] J. J. M. Verbaarschot, H. A. Weidenmüller, M. R. Zirnbauer, *Phys. Rep.*, **129**, 367 (1985).
[3] M. Gaudin, *Nucl. Phys.* **25**, 447 (1961).
[4] M. Abramowitz, I. Stegun, *Hanbook of Math. Functions*, New York, Dover 1965.
[5] N. Ullah, *Matrix Ensembles in the Many-Nucleon Problem*, Oxford Univ. Press, Oxford 1987.
[6] F. J. Dyson, *J. Math. Phys.* **3**, 1199 (1976).
[7] N. Ullah, Density and correlations in large one and three dimensional systems of fermions, *Fortschr. Phys.* **37**, 839 (1989).
[8] F. J. Dyson, *Comm. Math. Phys.* **47**, 171 (1976).