THE INCOMPRESSIBLE PERFECT FLUID CYLINDER

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For a static incompressible perfect fluid cylinder, the physical radius R, and the parameter m of the exterior Levi-Civita solution, are numerically calculated in terms of the ratio of the central pressure p_0 and the mass density μ_0.

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1. *Introduction*

The exact solution for a static cylindrically symmetric incompressible perfect fluid in General Relativity is not known although the field equations can be reduced to a very symmetric first-order system of ordinary differential equations for two real functions, see equations (1) below. The results of a numerical study of that system, for constant mass density $\mu = \mu_0$, are given in the present note.

We assume that the solutions are regular at the symmetry axis and can be matched to the Levi-Civita vacuum metric at a finite radius, where the pressure p vanishes.

The qualitative behaviour of the incompressible perfect fluid is similar to that of the perfect fluid with the equation of state $\mu = \mu_0+5p$. The latter case has been solved analytically by Evans [1]. This exact solution served as a test-bed for our numerical calculations and is therefore discussed in Sect. 5.

2. *The equations to be solved*

Einstein's field equations for static cylindrically symmetric perfect fluids can be reduced to the first-order system [2]

$$\dot{y} = (1-yz)(Fy-2), \qquad \dot{z} = (1-yz)(Fz-2) \tag{1}$$

for the two real functions $y = y(x)$ and $z = z(x)$ (where a dot denotes derivative with respect to x). For a prescribed equation of state $\mu = \mu(p)$, the function F in (1) is deter-

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mined by

$$F = F(x) = \frac{1}{2}\frac{\mu+3p}{p}, \qquad \dot{p}+(\mu+p) = 0. \tag{2}$$

Once a solution to (1) and (2) is given, one gets the space-time metric

$$ds^2 = \frac{yz-1}{\kappa_0 p}dx^2+e^{-2x}(e^{2k}d\xi^2+e^{2h}d\varphi^2)-e^{2x}dt^2 \tag{3}$$

(κ_0 being Einstein's gravitational constant), simply by integrating from

$$y = \dot{k}, \qquad z = \dot{h}. \tag{4}$$

The independent coordinate x, which is in fact the gravitational potential, can be gauged so that the axis, and the surface of vanishing pressure, are given by $x = 0$, and $x = x_1 > 0$, respectively.

From (3) one finds the expression

$$R = \int_0^{x_1}\left(\frac{yz-1}{\kappa_0 p}\right)^{1/2}dx \tag{5}$$

for the physical radius R of the perfect fluid cylinder.

3. *The behaviour at the axis and at the boundary*

Introducing a radial coordinate, say r, which gives the physical distance from the axis, the leading terms in the expansions of the metric functions k and h in (3), and x, near the axis $r = 0$, should behave like

$$h \sim \ln r, \qquad k \sim r^2, \qquad x \sim r^2 \tag{6}$$

which leads to

$$z \sim r^{-2}, \qquad y \sim 1. \tag{7}$$

Since z becomes infinite at the axis, it is convenient to introduce the reciprocal function $w \equiv z^{-1}$. The system (1) then takes the form

$$\dot{y} = \left(1-\frac{y}{w}\right)(Fy-2), \qquad \dot{w} = (y-w)(F-2w). \tag{8}$$

From (7) and (8) one concludes

$$w_0 = 0, \qquad y_0 = \frac{2}{F_0}, \qquad \dot{w}_0 = 2, \qquad \dot{y}_0 = -\frac{\dot{F}_0}{F_0^2} \tag{9}$$

for the initial values of y and w, and their first derivatives, at $x = 0$ (the subscript 0 refers to $x = 0$). The expression for $\dot{y}_0$ can be derived from (8) by means of the Bernoulli–l'Hospital rule.

At the zero-pressure surface $x = x_1$, the two functions y and w must coincide and have equal but opposite derivatives,

$$y_1 = w_1, \quad \dot{y}_1 = -\dot{w}_1 \tag{10}$$

(note that $F_1 \equiv F(x_1) = \infty$).

4. *The Levi–Civita solution*

The general cylindrically symmetric vacuum solution [3]

$$ds^2 = \varrho^{-2m}[\varrho^{2m^2}(d\varrho^2+d\zeta^2)+\varrho^2 d\varphi^2]-\varrho^{2m}dt^2 \tag{11}$$

contains the real parameter m. The metric (11) can be cast into the form (3) by the substitutions

$$x = m \ln \varrho, \quad k = m^2 \ln \varrho, \quad h = \ln \varrho, \quad w = y = m. \tag{12}$$

Matching an interior solution to (11), one infers from the continuity of the metric, and its first x-derivative,

$$w_1 = y_1 = m \tag{13}$$

as the only boundary condition.

The value $m = \frac{1}{2}$ is distinguished because in that case the Levi–Civita solution is of Petrov type D.

5. *The Evans solution*

In our notation, the Evans solution [1] for the equation of state $\mu = \mu_0 + 5p$ is given by

$$y = \frac{a^2-4e^{3x}}{2(a^2-e^{3x})}, \quad w = \frac{1}{z} = \frac{2(1-e^{3x})}{1-4e^{3x}}, \tag{14}$$

$$p = \frac{\mu_0}{6}\left(\frac{a^2}{4}e^{-6x}-1\right), \quad F = \frac{a^2-e^{6x}}{\frac{a^2}{4}-e^{6x}}. \tag{15}$$

The parameter a is related to x_1 by

$$e^{3x_1} = \frac{a}{2}, \quad a \geqslant 2. \tag{16}$$

Remarkably, the function w does not depend on a.

The functions y and w as given in (14), and their first derivatives, take the following values:

At the axis ($x = 0$)

$$w_0 = 0, \quad y_0 = \frac{a^2-4}{2(a^2-1)}, \quad \dot{w}_0 = 2, \quad \dot{y}_0 = -\frac{9a^2}{2(a^2-1)^2} \tag{17}$$

at the boundary ($x = x_1$)

$$y_1 = w_1 = \frac{a-2}{2a-1} = m,$$

$$\dot{y}_1 = -\dot{w}_1 = -\frac{9a}{(2a-1)^2} \tag{18}$$

in accordance with (9) and (10).

The physical distance R from the axis to the surface is given by

$$R = \sqrt{\frac{2}{\kappa_0\mu_0}} \int_1^{\frac{a}{2}} \frac{d\zeta}{\sqrt{(a^2-\zeta)(\zeta-1)}} = \sqrt{\frac{2}{\kappa_0\mu_0}}\left[\frac{\pi}{2} - \arcsin\left(\frac{a^2-a+1}{a^2-1}\right)\right]. \tag{19}$$

R has its maximum at $a = a_m \equiv 2+\sqrt{3}$.

Fig. 1 illustrates the typical behaviour of y and w.

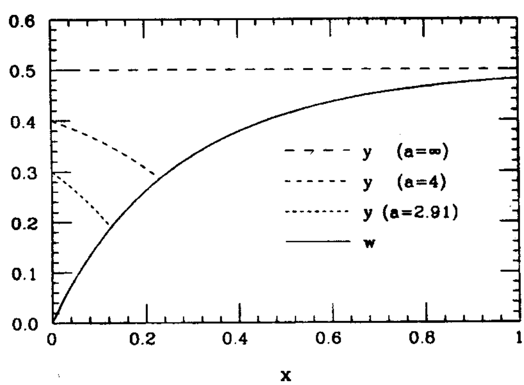

Fig. 1. Graphics of the functions $y(x)$ and $w(x)$ of Evans' solution, for different values of the parameter a. The function w tends asymptotically to $y = 0.5$

6. *The incompressible perfect fluid*

For the equation of state $\mu = \mu_0$ it follows from (2)

$$p = \mu_0(e^{x_1-x}-1), \quad F = \frac{\frac{3}{2}e^{x_1-x}-1}{e^{x_1-x}-1}. \tag{20}$$

The pressure is positive for $0 \leqslant x < x_1$ and takes its maximal value $p_0 = \mu_0(e^{x_1} - 1)$ at the axis $x = 0$.

The numerical integration starts at $x = 0$ with the initial conditions

$$w_0 = 0, \quad y_0 = 4\,\frac{e^{x_1} - 1}{3e^{x_1} - 2}\,. \tag{21}$$

The expression for $\dot{y}_0$

$$\dot{y}_0 = -\tfrac{1}{2}\,e^{x_1}(\tfrac{3}{2}\,e^{x_1} - 1)^{-2} \tag{22}$$

(see equations (9) and (20)), has to be put by hand into the numerical code in order to start the numerical integration of the system (8).

The energy condition $0 \leqslant p/\mu_0 \leqslant 1$ imposes the restriction

$$0 \leqslant x_1 \leqslant \ln 2, \quad 0 \leqslant y_0 \leqslant 1. \tag{23}$$

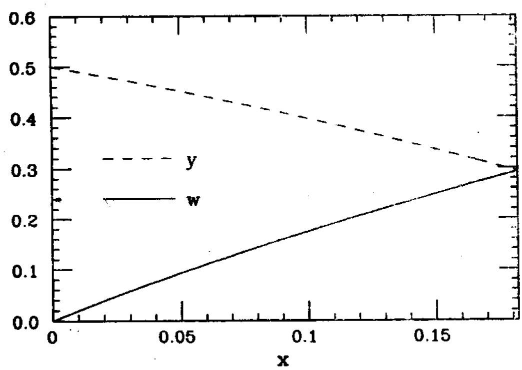

Fig. 2. Curves corresponding to $y(x)$ and $w(x)$ in the case of incompressible perfect fluid, for the initial value $y_0 = 0.5$. For this value, $y = w$ at $x = x_1 = \ln(6/5)$

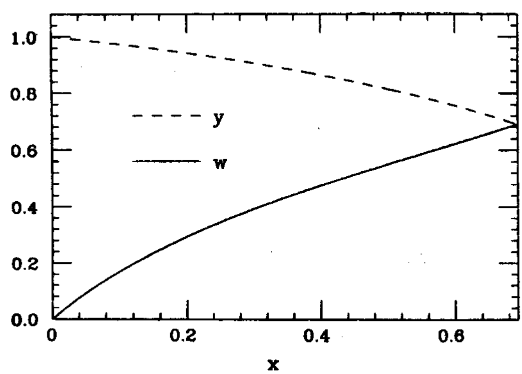

Fig. 3. The curves $y(x)$ and $w(x)$ are plotted for the initial value $y_0 = 1$, which is the maximum allowed for the energy condition. When $y_0 = 1$, $y = w$ at $x = x_1 = \ln(2)$

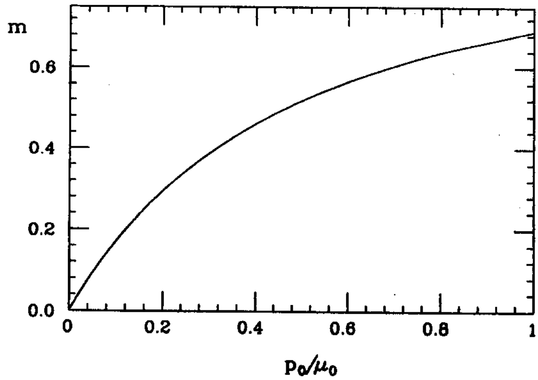

Fig. 4. The parameter m of the exterior Levi–Civita metric is represented as a function of the ratio p_0/μ_0.

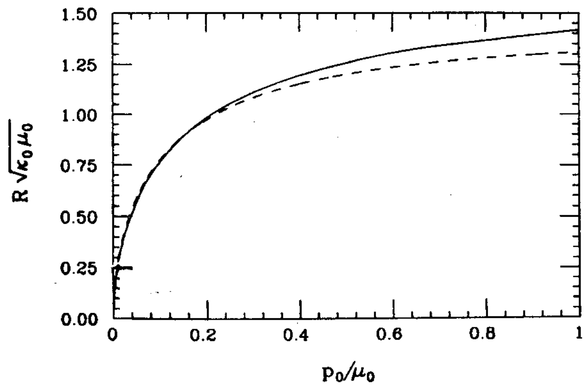

Fig. 5. Representations of the physical radius R (coveniently normalized) of an incompressible perfect fluid cylinder as functions of the ratio p_0/μ_0. The solid line represents the values obtained for numerical solutions of the functions $y(x)$ and $w(x)$, and the dashed one is for their linear approximation

In Figs 2 and 3, the curves $y = y(x)$ and $w = w(x)$ are plotted for the two initial values $y_0 = 0.5$ and $y_0 = 1$. The graphics for the functions $y(x)$ and $w(x)$ as seen in figures 2 and 3 suggest to try a linear approximation for the aforementioned functions y and w. According to (21) and (22), such a linear approximation would then be given by:

$$y \approx y_0 + \dot{y}_0 x = y_0 - \tfrac{1}{2}(y_0 - 2)(\tfrac{3}{2}y_0 - 2)x,$$

$$w \approx w_0 + \dot{w}_0 x = 2x. \tag{24}$$

Note that the smaller the value of the ratio p_0/μ_0 is the better the above approximation works, as Fig. 5 shows.

The numerical integration of the system (8), with F given in (20), assigns to each value of x_1 the corresponding parameter $m = y_1 = w_1$ of the Levi–Civita metric which describes the exterior gravitational field of the incompressible perfect fluid cylinder. Fig. 4 represents the parameter m as a function of the ratio of the central pressure p_0 and the constant mass density μ_0. The Levi-Civita parameter m which takes for the Evans solution the maximal value $m = 1/2$ can exceed $m = 1/2$ in the incompressible perfect fluid case (see Fig. 4) if one only imposes the condition $0 \leqslant p \leqslant \mu_0$. Other considerations (e.g., stability) might lead to stronger restrictions on m.

Finally, Fig. 5 shows how the physical radius R defined in (5) depends on that ratio.

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Editorial note. This article was proofread by the editors only, not by the authors.

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