A SEQUENCE OF CLIFFORD ALGEBRAS AND THREE REPLICAS OF DIRAC PARTICLE*

By W. Królikowski

III. Physikalisches Institut der RWTH Aachen, Theoretische Elementarteilchenphysik, Aachen, W. Germany

(Received April 23, 1990)

The embedding of Dirac algebra into a sequence $N = 1, 2, 3, \ldots$ of Clifford algebras is discussed, leading to Dirac equations with N-1 additional, electromagnetically "hidden" spins 1/2. It is shown that there are three and only three replicas $N = 1, 3, 5$ of Dirac particle if the theory of relativity together with the probability interpretation of wave function is applied both to the "visible" spin and "hidden" spins, and a new "hidden exclusion principle" is imposed on the wave function (then "hidden" spins add up to zero). It is appealing to explore this idea in order to explain the puzzle of three generations of leptons and quarks.

PACS numbers: 12.90.+b; 12.50.Ch

1. Introduction

As it was observed recently [1], the Dirac anticommutation relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu} \tag{1}$$

admit a remarkable sequence of representations having the form

$$\Gamma^\mu = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i^\mu, \tag{2}$$

$N = 1, 2, 3, \ldots$, where the matrices γ_i^μ, $i = 1, 2, \ldots, N$, are defined as basic elements of a sequence of Clifford algebras,

$$\{\gamma_i^\mu, \gamma_j^\nu\} = 2\delta_{ij} g^{\mu\nu}. \tag{3}$$

Except for $N = 1$ the representations (2) are reducible. In fact, they may be realized as

$$\Gamma^\mu = \gamma^\mu \otimes \underbrace{1 \otimes \ldots \otimes 1}_{N-1\,\mathrm{times}}, \tag{4}$$

* Supported by the Deutsche Forschungsgemeinschaft.

** On leave of absence from Institute of Theoretical Physics, Warsaw University, Warsaw, Poland.

where γ^μ and 1 are the usual Dirac 4×4 matrices. It is so, since for any $N > 1$ one can introduce, beside the linear combination (2), N-1 other Jacobi-type independent linear combinations $\Gamma_2^\mu, ..., \Gamma_N^\mu$ of γ_i^μ satisfying together with $\Gamma_1^\mu \equiv \Gamma^\mu$ the anticommutation relations of the form (3):

$$\{\Gamma_i^\mu, \Gamma_j^\nu\} = 2\delta_{ij}g^{\mu\nu} \tag{5}$$

that admit the representation (4). For instance, in the case of $N = 3$

$$\begin{aligned}\Gamma_1^\mu &= \frac{1}{\sqrt{3}}(\gamma_1^\mu+\gamma_2^\mu+\gamma_3^\mu),\\ \Gamma_2^\mu &= \frac{1}{\sqrt{2}}(\gamma_1^\mu-\gamma_2^\mu),\\ \Gamma_3^\mu &= \frac{1}{\sqrt{6}}(\gamma_1^\mu+\gamma_2^\mu-2\gamma_3^\mu)\end{aligned} \tag{6}$$

satisfy Eq. (5) and so may be represented as

$$\begin{aligned}\Gamma_1^\mu &= \gamma^\mu \otimes 1 \otimes 1,\\ \Gamma_2^\mu &= \gamma^5 \otimes i\gamma^5\gamma^\mu \otimes 1,\\ \Gamma_3^\mu &= \gamma^5 \otimes \gamma^5 \otimes \gamma^\mu,\end{aligned} \tag{7}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Thus, the Dirac equation, say, in an external electromagnetic field,

$$[\Gamma\cdot(p-eA)-m]\psi = 0, \tag{8}$$

may be rewritten as

$$[\gamma\cdot(p-eA)-m]\psi = 0, \tag{9}$$

when it is considered in the representations (2) realized in the form (4). Here, $\psi = (\psi_{\alpha_1\alpha_2...\alpha_N})$ carries N Dirac bispinor indices α_i, $i = 1, 2, ..., N$, of which only the first one is acted on by the usual Dirac matrices γ^μ and thus is "visible" in the electromagnetic field A^μ coupled to γ^μ. The rest of them are free (unless the mass m operates on them) and so are "hidden" in the electromagnetic field. In consequence, a particle, if described by Eq. (8), can display in the electromagnetic field only a "visible" spin 1/2, though it possesses also N-1 "hidden" spins 1/2.

In the case of $N = 1$ Eq. (9) reduces, of course, to the usual Dirac equation, while in the case of $N = 2$ it turns out to be equivalent [2] to the equation discovered in 1960 by Kähler [3, 4]. The latter equation linearizes the d'Alembertian differential operator in the space of antisymmetrical Lorentz tensors, realizing in this way the Dirac square-root procedure for total spin $0 \oplus 1$. Analogically, the general equation (9) realized this procedure for total spin $1/2 \oplus 3/2 \oplus ... \oplus 1/2\, N$ or $0 \oplus 1 \oplus ... \oplus 1/2\, N$ when $N = 1, 2, 3, 4, ...$ is odd or even, respectively.

Discussing the Dirac equation (8) in the representations (2) one may consider two natural interpretative options, where the physical Lorentz group of the theory of relativity is generated either by

$$J^{\mu\nu} = L^{\mu\nu}+\tfrac{1}{2}\sum_{i=1}^{N} \Sigma_i^{\mu\nu} \tag{10}$$

or by

$$J^{\mu\nu}_{\text{visible}} = L^{\mu\nu}+1/2\Sigma^{\mu\nu}, \tag{11}$$

where $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, $\Sigma_i^{\mu\nu} = \frac{i}{2}[\Gamma_i^\mu, \Gamma_i^\nu]$ and $\Sigma^{\mu\nu} = \frac{i}{2}[\Gamma^\mu, \Gamma^\nu] = \Sigma_1^{\mu\nu}$. The theory of relativity requires that in the case of the first option the mass m should commute with $J^{\mu\nu}$, while in the case of the second option it should commute with $J^{\mu\nu}_{\text{visible}}$ but not necessarily with

$$J^{\mu\nu}_{\text{hidden}} = J^{\mu\nu} - J^{\mu\nu}_{\text{visible}} = 1/2\sum_{i=2}^{N} \Sigma_i^{\mu\nu} \tag{12}$$

(note that m commutes with $J^{\mu\nu}_{\text{visible}}$ automatically, since in the Dirac equation (8) it must commute with Γ^μ).

In the present paper we choose the first option, where all matrices Γ_i^μ, $i = 1, 2, ..., N$, (not only $\Gamma_1^\mu \equiv \Gamma^\mu$) are connected with the physical spacetime governed by the theory of relativity. Then, the matrices $\Gamma_1^\mu \equiv \Gamma^\mu$ and Γ_i^μ, $i = 2, ..., N$, (cf. Eq. (6) for the case of $N = 3$) may be interpreted as the (relativistic) velocity of the particle's "centre of mass" and the relative (relativistic) velocities of the particles' "intrinsic constituents", although our particle is actually pointlike in the physical spacetime. Thus, we have here to do with an act of algebraic abstraction from the picture of a spatially extended composite particle. It may be compared with the famous act of algebraic abstraction from the picture of spatially extended rotator, that has led to the Dirac's concept of spinning particle.

2. *Probability conservation*

From the Dirac equation (8) and its Hermitian conjugate we readily deduce the local conservation equations

$$\frac{\partial}{\partial x^\mu}(\psi^+\Gamma_1^0\Gamma_1^\mu\psi) = 0 \tag{13}$$

and

$$\frac{\partial}{\partial x^\mu}(\psi^+\Gamma_1^0\Gamma_2^0 \dots \Gamma_N^0\Gamma_1^\mu\psi) = 0, \tag{14}$$

where the second conclusion is valid only when N is odd, $N = 1, 3, 5, ...$, and the mass m commutes with the matrix $\Gamma_2^0 \dots \Gamma_N^0$ (m commutes with Γ_1^μ automatically). The conserved current appearing in Eq. (13) is not covariant under the full Lorentz group generated by $J^{\mu\nu}$ defined in Eq. (10) (though it is covariant under the visible Lorentz group generated by J^{mn}_{visible} given in Eq. (11)). In contrast, the current in Eq. (14) is covariant under the full Lorentz group. Thus, in the case of our first option, the relativistic probability current can be described by

$$j^{\mu} = \eta_N \psi^{+} \Gamma_1^0 \Gamma_2^0 \dots \Gamma_N^0 \Gamma_1^{\mu} \psi, \tag{15}$$

where N must be odd, $N = 1, 3, 5, \dots$, while the mass m must commute with $\Gamma_2^0 \dots \Gamma_N^0$. Here, η_N is a phase factor making the matrix

$$P_{\text{hidden}} = \eta_N \Gamma_2^0 \dots \Gamma_N^0. \tag{16}$$

Hermitian (it can be chosen as $\eta_N = i^{1/2(N-1)(N-2)}$). Note that P_{hidden} describes the hidden internal parity. Since, due to Eq. (14), P_{hidden} is a constant of motion, one can consistently impose on ψ the constraint

$$\eta_N \Gamma_2^0 \dots \Gamma_N^0 \psi = \psi \tag{17}$$

in order to guarantee that the probability density be really positive:

$$j^0 = \eta_N \psi^{+} \Gamma_2^0 \dots \Gamma_N^0 \psi > 0. \tag{18}$$

It can be seen, however, that the constraint (17) is covariant under the full Lorentz group generated by $J^{\mu\nu}$ if and only if the wave function ψ is a scalar under hidden boosts generated by

$$J^{0l}_{\text{hidden}} = 1/2i \sum_{i=2}^{N} A_i^l, \tag{19}$$

$l = 1, 2, 3$, where $A_i^l = \Gamma_i^0 \Gamma_i^l$. Thus, the theory of relativity together with the constraint (17) requires that (in the case of our first option) the wave function ψ should be a hidden Lorentz scalar. Notice that the constraint (17) is always covariant under hidden spatial rotations generated by

$$J^{kl}_{\text{hidden}} = 1/2\varepsilon^{klm} \sum_{i=2}^{N} \Sigma_i^m, \tag{20}$$

$k, l, m = 1, 2, 3$, where $\Sigma_i^m = \Gamma_i^5 \Gamma_i^0 \Gamma_i^m$ and $\Gamma_i^5 = i\Gamma_i^0 \Gamma_i^1 \Gamma_i^2 \Gamma_i^3$ with $[\Sigma_i^m, \Sigma_j^n] = 0$ for $i \neq j$ and $[\Sigma_i^m, \Gamma_j^5] = 0$, $[\Gamma_i^5, \Gamma_j^5] = 0$. For instance, in the case of $N = 3$

$$\begin{aligned} \Sigma_1^m &= \sigma^m \otimes 1 \otimes 1, & \Gamma_1^5 &= \gamma^5 \otimes 1 \otimes 1, \\ \Sigma_2^m &= 1 \otimes \sigma^m \otimes 1, & \Gamma_2^5 &= 1 \otimes \gamma^5 \otimes 1, \\ \Sigma_3^m &= 1 \otimes 1 \otimes \sigma^m, & \Gamma_3^5 &= 1 \otimes 1 \otimes \gamma^5, \end{aligned} \tag{21}$$

where $\sigma^m = \gamma^5 \gamma^0 \gamma^m$. Here, the representation (7) is used.

In the chiral representation where γ^5 is diagonal with eigenvalues ± 1, the Dirac bispinor indices $\alpha_i = 1, 2, 3, 4$, $i = 1, 2, \ldots, N$, of ψ are defined by four pairs of eigenvalues ± 1 of Σ_i^3 and Γ_i^5, $i = 1, 2, \ldots, N$.

3. *Sector* $N = 3$

In the case of $N = 3$ the constraint (17) takes the form

$$i\Gamma_2^0\Gamma_3^0\psi = \psi. \tag{22}$$

Here, in the chiral version of representation (7) and (21)

$$i\Gamma_2^0\Gamma_3^0 = 1 \otimes \begin{pmatrix} 0 & 1_P \\ 1_P & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1_P \\ 1_P & 0 \end{pmatrix}, \quad \Gamma_2^5\Gamma_3^5 = 1 \otimes \begin{pmatrix} 1_P & 0 \\ 0 & -1_P \end{pmatrix} \otimes \begin{pmatrix} 1_P & 0 \\ 0 & -1_P \end{pmatrix}, \tag{23}$$

where 1_p is the Pauli 2×2 unit matrix. Thus, using the matrix notation $\psi = (\psi_{\alpha_2\alpha_3})$ where the bispinor index α_1 is suppressed, the constraint (22) implies in the chiral representation that

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{13} & \psi_{14} & \psi_{11} & \psi_{12} \\ \psi_{23} & \psi_{24} & \psi_{21} & \psi_{22} \end{pmatrix}. \tag{24}$$

Then, the eigenstates of hidden chirality $\Gamma_2^5\Gamma_3^5$ (which commutes with the hidden internal parity $i\Gamma_2^0\Gamma_3^0$) are given by

$$\psi^{(+)} = \begin{pmatrix} \psi_{11} & \psi_{12} & 0 & 0 \\ \psi_{21} & \psi_{22} & 0 & 0 \\ 0 & 0 & \psi_{11} & \psi_{12} \\ 0 & 0 & \psi_{21} & \psi_{22} \end{pmatrix}, \quad \psi^{(-)} = \begin{pmatrix} 0 & 0 & \psi_{13} & \psi_{14} \\ 0 & 0 & \psi_{23} & \psi_{24} \\ \psi_{13} & \psi_{14} & 0 & 0 \\ \psi_{23} & \psi_{24} & 0 & 0 \end{pmatrix}, \tag{25}$$

where

$$\Gamma_2^5\Gamma_3^5\psi^{(\pm)} = \pm\psi^{(\pm)}. \tag{26}$$

Making use of the usual charge conjugation matrix C in the chiral representation,

$$C = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = C^{-1}, \tag{27}$$

one can construct from the wave function (24) the following five candidates for hidden Lorentz covariants:

(*i*) hidden pseudoscalar (with hidden chirality $+1$):

$$C^{-1}_{\alpha_2\alpha_3}\psi_{\alpha_2\alpha_3} = 0, \tag{28}$$

(*ii*) hidden scalar (with the hidden chirality $+1$):

$$(C^{-1}\gamma^5)_{\alpha_2\alpha_3}\psi_{\alpha_2\alpha_3} = -2i(\psi_{12}-\psi_{21}), \tag{29}$$

(*iii*) hidden axial vector (with hidden chirality -1):

$$(C^{-1}\gamma^{\mu})_{\alpha_2\alpha_3}\psi_{\alpha_2\alpha_3} = \begin{cases} 0, & \mu = 0, \\ 2i(\psi_{13}-\psi_{24}), & \mu = 1, \\ -(\psi_{13}+\psi_{24}), & \mu = 2, \\ -2i(\psi_{14}+\psi_{41}), & \mu = 3, \end{cases} \tag{30}$$

(*iv*) hidden vector (with hidden chirality -1):

$$(C^{-1}\gamma^{5}\gamma^{\mu})_{\alpha_2\alpha_3}\psi_{\alpha_2\alpha_3} = \begin{cases} -2i(\psi_{14}-\psi_{41}), & \mu = 0, \\ 0, & \mu = 1, \\ 0, & \mu = 2, \\ 0, & \mu = 3, \end{cases} \tag{31}$$

(*v*) hidden pseudotensor (with hidden chirality $+1$):

$$\left(C^{-1}\frac{i}{2}[\gamma^0, \gamma^l]\right)_{\alpha_2\alpha_3}\psi_{\alpha_2\alpha_3} = \begin{cases} 2(\psi_{11}-\psi_{22}), & l = 1, \\ 2i(\psi_{11}+\psi_{22}), & l = 2, \\ -2(\psi_{12}+\psi_{21}), & l = 3, \end{cases}$$

$$\left(C^{-1}\frac{i}{2}[\gamma^k, \gamma^l]\right)_{\alpha_2\alpha_3}\psi_{\alpha_2\alpha_3} = \begin{cases} 0, & k = 1, \quad l = 2, \\ 0, & k = 2, \quad l = 3, \\ 0, & k = 3, \quad l = 1. \end{cases} \tag{32}$$

It is seen that, due to the constraint (22) spoiling generally the Lorentz covariance, Eqs. (30), (31) and (32) can be Lorentz covariant if and only if all of them give zero. This requires that the matrix $\psi = (\psi_{\alpha_2\alpha_3})$ should be antisymmetrical,

$$\psi_{\alpha_2\alpha_3} = -\psi_{\alpha_3\alpha_2}, \tag{33}$$

and in addition there should be $\psi_{14} = 0$. Since then, in particular, $\psi_{13} = \psi_{24} = \psi_{23} = \psi_{14} = 0$ (cf. Eq. (24)), one gets from Eq. (25) $\psi^{(-)} = 0$ and hence

$$\Gamma_2^5\Gamma_3^5\psi = \psi. \tag{34}$$

So, $\Gamma_2^5\Gamma_3^5$ must be a constant of motion, what implies that it ought to commute with the mass m in the Dirac equation (8) with $N = 3$ (m commutes with the visible chirality Γ_1^5 automatically).

Thus, in the sector $N = 3$, the theory of relativity and the constraint (22) enforce (in the case of the first interpretative option) the antisymmetry (33) and the additional constraint (34). Then, in the sector $N = 3$ there is one and only one Dirac particle, namely that corresponding to the hidden scalar (29):

$$A\frac{i}{4}(C^{-1}\gamma^5)_{\alpha_2\alpha_3}\psi_{\alpha_1\alpha_2\alpha_3} = A\psi_{\alpha_1 12}, \tag{35}$$

where A is a normalization constant. In this case, only four components $\psi_{\alpha_2\alpha_3}$ of ψ (with α_1 suppressed) are nonzero: $\psi_{12} = -\psi_{21} = \psi_{34} = -\psi_{43} \neq 0$.

4. *Hidden exclusion principle*

It is tempting to generalize the antisymmetry (33) to all sectors $N = 1, 3, 5, \ldots$ by conjecturing that all sector wave functions $\psi = (\psi_{\alpha_1\alpha_2\ldots\alpha_N})$ must be antisymmetrical with respect to the interchange of any pair of hidden bispinor indices $\alpha_2, \ldots, \alpha_N$. Such a conjecture, that may be called the "hidden exclusion principle" (or "Pauli hidden principle"), treats the hidden indices $\alpha_2, \ldots, \alpha_N$ as describing some identical Fermi degrees of freedom (they differ, of course, from the visible index α_1 which only couples to the particle's momentum and the electromagnetic field in Eq. (8)).

The hidden exclusion principle evidently restricts the sequence $N = 1, 3, 5, \ldots$ to its first three terms: $N = 1, 3, 5$. Moreover, it implies that in the sector $N = 5$ there is one and only one Dirac particle, namely that corresponding to the hidden scalar

$$A \tfrac{1}{24} \varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5} \psi_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = A\psi_{\alpha_1 1234}, \tag{36}$$

where A is a normalization constant. In this case only 24 components $\psi_{\alpha_2\alpha_3\alpha_4\alpha_5}$ of ψ (with α_1 suppressed) are nonzero: ψ_{1234} and its permutations (all equal up to sign of $\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}$).

It is interesting to note that for $N = 5$ the constraint (17),

$$i^2 \Gamma_2^0 \Gamma_3^0 \Gamma_4^0 \Gamma_5^0 \psi = \psi, \tag{37}$$

and the analogue of the additional constraint (34),

$$\Gamma_2^5 \Gamma_3^5 \Gamma_4^5 \Gamma_5^5 \psi = \psi, \tag{38}$$

are satisfied by

$$\psi_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = \varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5} \psi_{\alpha_1 1234} \tag{39}$$

automatically. In fact, in the chiral representation

$$i^2 \Gamma_2^0 \Gamma_3^0 \Gamma_4^0 \Gamma_5^0 = 1 \otimes \begin{pmatrix} 0 & 1_P \\ 1_P & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1_P \\ 1_P & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1_P \\ 1_P & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1_P \\ 1_P & 0 \end{pmatrix} \tag{40}$$

and

$$\Gamma_2^5 \Gamma_3^5 \Gamma_4^5 \Gamma_5^5 = 1 \otimes \begin{pmatrix} 1_P & 0 \\ 0 & -1_P \end{pmatrix} \otimes \begin{pmatrix} 1_P & 0 \\ 0 & -1_P \end{pmatrix} \otimes \begin{pmatrix} 1_P & 0 \\ 0 & -1_P \end{pmatrix} \otimes \begin{pmatrix} 1_P & 0 \\ 0 & -1_P \end{pmatrix}, \tag{41}$$

what gives for $(\alpha_2\alpha_3\alpha_4\alpha_5) = (1234)$

$$(i^2 \Gamma_2^0 \Gamma_3^0 \Gamma_4^0 \Gamma_5^0 \psi)_{1234} = \psi_{3412} = \psi_{1234} \tag{42}$$

and

$$(\Gamma_2^5 \Gamma_3^5 \Gamma_4^5 \Gamma_5^5 \psi)_{1234} = (-1)^2 \psi_{1234} = \psi_{1234}. \tag{43}$$

The additional constraint (38) satisfied automatically by the nonzero wave function implies that $\Gamma_2^5 \Gamma_3^5 \Gamma_4^5 \Gamma_5^5$ must be a constant of motion and so it ought to commute with the mass m in the Dirac equation (8) with $N = 5$.

In summary, the overall wave function comprising three nonzero sectors $N = 1, 3, 5$ has the form

$$\Psi = \begin{pmatrix} \psi_{\alpha_1} \\ \psi_{\alpha_1\alpha_2\alpha_3} \\ \psi_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \end{pmatrix}, \tag{44}$$

where

$$\begin{aligned} \Psi^+\Psi &= \psi^*_{\alpha_1}\psi_{\alpha_1} + 4\psi^*_{\alpha_1 12}\psi_{\alpha_1 12} + 24\psi^*_{\alpha_1 1234}\psi_{\alpha_1 1234} \\ &= \frac{1}{A^2}(\psi^{(1)+}\psi^{(1)} + 4\psi^{(3)+}\psi^{(3)} + 24\psi^{(5)+}\psi^{(5)}), \end{aligned} \tag{45}$$

with $\psi^{(1)} = A(\psi_{\alpha_1})$, $\psi^{(3)} = A(\psi_{\alpha_1 12})$ and $\psi^{(5)} = A(\psi_{\alpha_1 1234})$ describing three nonzero replicas of Dirac particle. They satisfy Eq. (9) with some masses $m^{(1)}$, $m^{(3)}$ and $m^{(5)}$. Here, $A = \sqrt{29}$ is the overall normalization constant. The same formula works for the product $\Psi'^+\Psi$ of two different overall wave functions Ψ and Ψ'. Of course, the kinetic part of lagrangian density for the fields $\psi^{(N)}$, $N = 1, 3, 5$, is

$$\overline{\Psi}\hat{\varrho}(\gamma \cdot p - \hat{m})\Psi = \sum_N \overline{\psi}^{(N)}(\gamma \cdot p - m^{(N)})\psi^{(N)}, \tag{46}$$

where $\overline{\Psi} = \Psi^+\gamma^0$, $\gamma^\mu = (\gamma^\mu_{\alpha_1\beta_1})$ and

$$\hat{\varrho} = 29\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/24 \end{pmatrix}, \quad \hat{m} = \begin{pmatrix} m^{(1)} & 0 & 0 \\ 0 & m^{(3)} & 0 \\ 0 & 0 & m^{(5)} \end{pmatrix}. \tag{47}$$

Here, $\mathrm{Tr}\,\hat{\varrho}^{-1} = 1$. One can formally write $\hat{\varrho} = \varrho(\hat{N}) \equiv 29\,[\frac{1}{8}(17\hat{N}^2 - 56\hat{N} + 47)]^{-1}$, where

$$\hat{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{48}$$

is the diagonal matrix with eigenvalues $N = 1, 3, 5$.

5. Conclusions

In conclusion, if the theory of relativity is applied to all bispinor indices involved in Eqs. (2) and (3), the sequence of representations (2) of Dirac algebra allows for probability interpretation of solutions ψ to the Dirac equation (8) only for odd $N = 1, 3, 5$ (this excludes from our discussion the case of $N = 2$ corresponding to the Kähler equation). At the same time, the (consistent) constraint (17) for ψ appears. When the hidden exclusion principle is imposed on ψ, the sequence of N terminates at $5 : N = 1, 3, 5$. Then, in each of three sectors $N = 1, 3, 5$ there is one and only one Dirac particle if the additional (consistent) constraint (34) for ψ with $N = 3$ is invoked. This constraint is enforced by the theory of relativity and the constraint (17).

Thus, making use of the representations (2) of Dirac algebra, the theory of relativity, the probability interpretation of ψ and the antisymmetry of ψ in hidden bispinor indices $\alpha_2, \ldots, \alpha_N$, taken jointly, are sufficient to provide the existence of three and only three replicas for any Dirac particle with definite electroweak and strong interactions coupled to its visible bispinor index α_1. The possible interpretation of these three replicas as three lepton and quark generations seems to be appealing and worth a deeper exploration.

I am indebted to Stefan Pokorski, Rudolf Rodenberg, Dieter Rein and Lalit Sehgal for helpful discussions.

Editorial note. This article was proofread by the editors only, not by the authors.

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