ARISTOTELEAN DYNAMICS OF STATIC CHARGE DISTRIBUTIONS

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This paper presents a dynamical formulation of electrostatics i.e. a dynamics of static charge distributions. The theory can be quantized and leads to the conclusion that the total electric charge is a multiple of the elementary charge e. Further simple assumptions allow us to derive the inequality $0 < e^2/\hbar c < \pi$.

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1. *Introduction*

In the previous paper [1] we derived the inequality

$$0 < e^2/\hbar c < \pi$$

for the fine structure constant $e^2/\hbar c$. The derivation given in [1] is independent of any special assumptions about the dynamics of charged matter. It is based, however, on the assumption that the infrared electromagnetic field is a q-number, an assumption which admittedly may be challenged [2]. For this reason we give in this paper an independent derivation based on what seems to be a minimum of assumptions; we derive the inequality in the framework of electrostatics which is the simplest theoretical structure in which the notion of electric charge appears.

We start, in the next Section, with a paradox about the electrostatic field; the paradox suggests that the electrostatic field is not really static. We describe next a field theory which is a version of electrostatics and derive a counterpart of equal time canonical commutation relation for the phase, which is a field variable in this theory. Finally, in the last Section, we make the simplest dynamical assumption about the phase and derive the inequality $0 < e^2/\hbar c < \pi$ as a necessary condition for positivity of norm of charged states.

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2. *The paradox: the electrostatic field is not static*

The electromagnetic potential A_μ is usually assumed to be a covariant vector field i.e. a geometrical quantity whose Lie derivative is equal to

$$\pounds_\xi A_\mu = \xi^\lambda \partial_\lambda A_\mu + A_\lambda \partial_\mu \xi^\lambda.$$

This may be written in the form

$$\pounds_\xi A_\mu = \xi^\lambda F_{\lambda\mu} + \partial_\mu(\xi^\lambda A_\lambda),$$

where

$$F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda.$$

Since the potential is determined only up to a gradient we can drop the gradient from the Lie derivative obtaining thus a gauge invariant expression for the Lie derivative of the potential:

$$\pounds_\xi A_\mu = \xi^\lambda F_{\lambda\mu}. \tag{1}$$

The point of view contained in (1) is supported by several authors [3]. Dr. Salié from Jena has kindly informed us that Sommerfeld was led to an equivalent point of view by thermodynamical considerations. However, accepting the gauge invariant expression (1) for the Lie derivative of the potential we arrive at the paradoxical conclusion that the electrostatic field, for example the Coulomb field of a charge at rest, is not static:

$$\pounds_\xi A_\mu = \xi^\lambda F_{\lambda\mu} \neq 0$$

if ξ is a time-like vector and the electric field does not vanish.

Paradox is a "seemingly absurd though perhaps well-founded statement". We accept what Eq. (1) says, namely we assume that the electrostatic field is really static only if it vanishes completely; if, however, there is a nonvanishing charge density, the physical situation is not really static but instead a certain motion takes place. The moving part is the phase which in [1] was defined as a scalar field S such that the vector $eA_\mu + \partial_\mu S$ is gauge invariant; e is the elementary charge, $\hbar = 1 = c$. The notion of phase allows us to represent electrostatics as a dynamics of phase; this is done in the next Section.

3. *Aristotelean dynamics of static charge distributions*

In the static case $A_1 = A_2 = A_3 = 0$ and the electromagnetic field is described by a single function A_0, $\partial A_0/\partial x^0 = 0$. We replace A_0 by the phase putting

$$\frac{\partial S}{\partial x^0} = -eA_0. \tag{2}$$

Under electrostatic gauge transformations $A_0 \to A_0' = A_0 + \text{const}\, S$ behaves like the phase of the wave function, which explains the name. Putting $A_0 = (-1/e)\partial S/\partial x^0$, $A_1 = A_2$

$= A_3 = 0$ into the electromagnetic action

$$-\frac{1}{16\pi}\int d^4x F_{\mu\nu}F^{\mu\nu}$$

we obtain the expression

$$\frac{1}{8\pi e^2}\int d^4x \partial_{0i}S\partial_{0i}S, \quad i = 1, 2, 3, \tag{3}$$

which can be treated as the total action for a closed system described by a single field variable S. The action (3) has neither Poincaré nor Galilean but only Aristotelean symmetry which is the natural symmetry of electrostatics.

The momentum canonically conjugated with the phase S is

$$p = -\frac{1}{4\pi e}\Delta\frac{\partial S}{\partial x^0} = -\frac{1}{e}\varrho,$$

where ϱ is the electric charge density. Therefore

$$[S(x), \Delta\dot{S}(y)]_{x^0=y^0} = -4\pi i e^2\delta(\boldsymbol{x}-\boldsymbol{y}).$$

The Laplace operator is uniquely invertible; hence

$$[S(x), \dot{S}(y)]_{x^0=y^0} = i\frac{e^2}{|\boldsymbol{x}-\boldsymbol{y}|}. \tag{4}$$

This is the counterpart of canonical commutation relation for the phase.

A simple consequence of this relation is that [1]

$$[Q, S] = ie, \tag{5}$$

where

$$Q = \int d^3x\varrho$$

is the total charge. Eq. (5) shows that to explain quantization of charge in units equal to the constant e we have to assume that the phase S is periodic with the period 2π, an assumption perfectly natural for a "phase".

4. Positivity of norm of charged states

To discuss positivity we need the vacuum and to have the vacuum we need the time evolution of the phase. The trouble is that the time evolution given by (3) does not allow us to define the vacuum. We solve the difficulty by making the simplest Poincaré invariant assumption, namely that the phase S fulfils the wave equation $\Box S = 0$. This assumption, obviously inconsistent with (3), can be defended in this way: Eq. (4) is purely kinematical and thus it is likely to have a larger range of validity than the dynamics based on (3) which is rigorously restricted to the electrostatic situation. For example: canonical commutation relations for nonrelativistic and relativistic particle are the same, the difference being in different laws of time evolution.

Summing up we can say that we consider a quantum field theory defined by the equal time commutation relation

$$[S(x), \dot{S}(y)]_{x^0=y^0} = i\frac{e^2}{|\boldsymbol{x}-\boldsymbol{y}|},$$

the equation of motion

$$\Box S = 0$$

and the usual definition of the vacuum.

A charged state i.e. an eigenstate of the total charge Q is defined as

$$|f\rangle = \int d^4x f(x) :e^{iS(x)}: |0\rangle.$$

The scalar product of two such states is

$$\langle f|g\rangle = \int d^4x \overline{f(x)} \int d^4y g(y) \cdot K(x-y), \tag{6}$$

where

$$K(x) = \exp\frac{e^2}{2}\left\{\frac{1}{\pi r}\left[(t-r)\ln\left|\frac{t-r}{\lambda}\right| - (t+r)\ln\left|\frac{t+r}{\lambda}\right|\right] + i\frac{|t-r|-|t+r|}{2r}\right\},$$

$$t = x^0, \quad r = [(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}.$$

λ is the infrared cutoff whose actual value is irrelevant because a change of λ can be interpreted as a multiplication of wave functions f and g by a constant factor.

Positivity of the scalar product (6) is a difficult question. In this paper we wish to indicate only that the previously established inequality [1] $0 < e^2 < \pi$ is likely to appear here as a necessary condition. Consider states f and g localized at a space-like straight line:

$$f(x) = f(x^0)\delta(\boldsymbol{x}-\boldsymbol{v}x^0),$$

$$g(y) = g(y^0)\delta(\boldsymbol{y}-\boldsymbol{v}y^0),$$

where $|\boldsymbol{v}| = v > 1$. We can use so strongly localized states because their norm is finite for well behaved $f(x^0)$ and $g(x^0)$, as is seen from the scalar product

$$\langle f|g\rangle = \int dx^0\overline{f(x^0)} \int dy^0 g(y^0) \cdot K(t, r),$$

$$t = x^0 - y^0, \quad r = v|x^0-y^0|.$$

This simplified scalar product will be positive definite if

$$K(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} K(t, v|t|)dt > 0.$$

This integral exists for $0 < e^2 < \pi$ and defines an analytic function of e^2 proportional to

$$\frac{\sin \frac{e^2}{2}\left[1+\frac{1}{v}\,\text{sign}\,(\omega\lambda)\right]}{\Gamma(e^2/\pi)\sin e^2};$$

the remaining factors are positive for all real e^2. This is to be positive for all $v > 1$ and for all real ω which will be the case only if

$$0 < e^2 < \pi. \tag{7}$$

5. Conclusions

The argument given in this paper is completely different from that given in [1]. In [1] it was made in a model with full Lorentz symmetry but without space-time translations. Here it is made in a model whose symmetry includes space-time translations and rotations but not Lorentz boosts. One has thus the impression that the inequality $0 < e^2 < \pi$ characterizes somehow the electric charge as such, not a special representation of it within a model.

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