STATIC, CYLINDRICALLY SYMMETRIC SOLUTION OF THE GENERALIZED FIELD EQUATIONS

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The problem of finding cylindrically symmetric static solutions of the Generalized Field Theory is completely solved. The electromagnetic fields are shown to vanish faster with the distance from the axis of symmetry than the corresponding curvature of the associated space-time. On the other hand, when the latter becomes a flat Minkowski manifold, the symmetric field is zero but the theory predicts exactly the Maxwellian electric and magnetic fields.

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1. Introduction

The static, cylindrically symmetric solution of the nonsymmetric unified field theory of electromagnetism and gravitation has been studied in a number of works by the present author and his collaborators (Refs. [1–3 and [4]). No general solution corresponding to the above symmetry has been found in spite of the fact that the results obtained (especially in Ref. [3]) proved to be very interesting and led to a complete reformulation of the nonsymmetric theory itself (Ref. [5]). However, no attempt was made in the earlier studies to consider the effect of the so-called metric hypothesis which characterizes the Generalized Field Theory as I call our reformulation of Einstein's work.

It turns out that the hypothesis, namely, definition of the (hyperbolic, Riemannian) space-time metric

$$a_{\mu\nu}$$

by the condition or equations

$$\tilde{\Gamma}^{\lambda}_{(\mu\nu)} = \tfrac{1}{2}\, a^{\lambda\sigma}(a_{\sigma\mu,\nu} + a_{\mu\sigma,\nu} - a_{\mu\nu,\sigma}) \equiv \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}_a, \tag{1}$$

allows a complete resolution of the hitherto intractable problem of cylindrical symmetry.

Here, $\tilde{\Gamma}^{\lambda}_{(\mu\nu)}$ denotes the symmetric part of what I have called the geometric affine connection. It is related to its "physical" counterpart by Schrödinger's equation, which is not solvable for the latter,

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}+\tfrac{2}{3}\,\delta^{\lambda}_{\mu}\Gamma_{\nu},$$

where

$$\Gamma_{\nu} \equiv \Gamma^{\sigma}_{[\nu\sigma]} = \tfrac{1}{2}\,(\Gamma^{\sigma}_{\nu\sigma}-\Gamma^{\sigma}_{\sigma\nu}).$$

(Of course, $\tilde{\Gamma}^{\sigma}_{[\mu\sigma]} \equiv 0$.)

In the Generalized Field Theory, the nonsymmetric field

$$g_{\mu\nu} = h_{\mu\nu}+k_{\mu\nu} = k_{\nu\mu}-k_{\nu\mu},$$

is related to the geometric connection by the equations of Einstein

$$g_{\mu\nu,\lambda}-\tilde{\Gamma}^{\sigma}_{\mu\lambda}g_{\sigma\nu}-\tilde{\Gamma}^{\sigma}_{\lambda\nu}g_{\mu\sigma} = 0, \tag{2}$$

or

$$h_{\mu\nu;\lambda} \equiv n_{\mu\nu,\lambda}-\begin{Bmatrix}\sigma\\ \mu\lambda\end{Bmatrix}_a h_{\sigma\nu}-\begin{Bmatrix}\sigma\\ \lambda\nu\end{Bmatrix}_a h_{\mu\sigma} = \tilde{\Gamma}^{\sigma}_{[\mu\lambda]}k_{\sigma\nu}+\tilde{\Gamma}^{\sigma}_{[\lambda\nu]}k_{\mu\sigma}, \tag{3}$$

$$k_{\mu\nu;\lambda} \equiv k_{\mu\nu,\lambda}-\begin{Bmatrix}\sigma\\ \mu\lambda\end{Bmatrix}_a k_{\sigma\nu}-\begin{Bmatrix}\sigma\\ \lambda\nu\end{Bmatrix}_a k_{\mu\sigma} = \tilde{\Gamma}^{\sigma}_{[\mu\lambda]}h_{\sigma\nu}+\tilde{\Gamma}^{\sigma}_{[\lambda\nu]}k_{\mu\sigma}. \tag{4}$$

For the particular symmetry we are considering, these equations are written out in full in the Appendix.

Once the skew components $\tilde{\Gamma}^{\lambda}_{[\mu\nu]}$ of the connection are algebraically determined, we solve the field equations

$$R_{(\mu\nu)}(\tilde{\Gamma}^{\alpha}_{\beta\gamma}) = R^{E}_{\mu\nu}+\tilde{\Gamma}^{\lambda}_{[\mu\sigma]}\tilde{\Gamma}^{\sigma}_{[\lambda\nu]} = 0, \tag{5}$$

where $R^{E}_{\mu\nu}$ is the general relativistic Ricci tensor formed from the metric Christoffel symbols

$$\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}.$$

Both the field tensor $g_{\mu\nu}$ and the metric tensor $a_{\mu\nu}$ are assumed to be nonsingular and then equations (2)–(4) imply that

$$g = K^{2}a, \tag{6}$$

where K is a constant while g and a respectively denote their determinants. An exception is the flat (Minkowski) space-time for which we can expect the field $h_{\mu\nu}$ to vanish when we must also have $K = 0$.

Equations (5) do not constitute the full set of the Generalized Field equations but the remaining ones:

$$R_{[\mu\nu]}(\tilde{\Gamma}^{\alpha}_{\beta\gamma}) = -\tilde{\Gamma}^{\sigma}_{[\mu\nu];\sigma}, \tag{7}$$

serve to define (up to a constant of proportionality) the skew field which happens also to be proportional to the curl of the vector

$$\Gamma_\lambda.$$

Thus the latter plays the role of the four-potential up to the usual U(1) gauge.

2. *Preliminary results*

We choose the coordinate system

$$(x^0, x^1, x^2, x^3) = (t, r, z, \theta)$$

(with the speed of light in vacuum equal to unity) in such a way that the nonzero components of the static, cylindrically symmetric field, which then become functions of r only, are

$$h_{00} = \gamma, \quad h_{11} = -\alpha, \quad h_{22} = -\alpha, \quad h_{33} = -\beta, \quad k_{03} = u, \quad k_{23} = v. \tag{8}$$

These are the so-called isothermal coordinates. However, there seems to be no a priori reason why the space-time metric should also be isothermal (even though it will usually turn out to be the case for our symmetry) so that we write

$$ds^2 = c^2dt^2 - a^2dr^2 - p^2dz^2 - b^2d\theta^2, \tag{9}$$

with c, a, p and b, likewise, functions of r alone. The nonzero Christoffel brackets then are

$$\begin{Bmatrix}0\\01\end{Bmatrix} = \frac{c_1}{c},$$

$$\begin{Bmatrix}1\\00\end{Bmatrix} = \frac{cc_1}{a^2},$$

$$\begin{Bmatrix}1\\11\end{Bmatrix} = \frac{a_1}{a},$$

$$\begin{Bmatrix}1\\22\end{Bmatrix} = -\frac{pp_1}{a^2},$$

$$\begin{Bmatrix}1\\33\end{Bmatrix} = -\frac{bb_1}{a},$$

$$\begin{Bmatrix}2\\12\end{Bmatrix} = \frac{p_1}{p},$$

$$\begin{Bmatrix}3\\13\end{Bmatrix} = \frac{b_1}{b}, \tag{10}$$

the suffix denoting differentiation with respect to r. Hence the nonvanishing components of the general relativistic Ricci tensor are

$$R^E_{00} = -\frac{cc_1}{a^2}\left(\frac{c_{11}}{c_1} - \frac{a_1}{a} + \frac{p_1}{p} + \frac{b_1}{b}\right),$$

$$R^E_{11} = \frac{c_{11}}{c} + \frac{p_{11}}{p} + \frac{b_{11}}{b} - \frac{a_1}{a}\left(\frac{c_1}{c} + \frac{p_1}{p} + \frac{b_1}{b}\right),$$

$$R^E_{22} = \frac{pp_1}{a^2}\left(\frac{p_{11}}{p_1} - \frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c}\right),$$

$$R^E_{33} = \frac{bb_1}{a^2}\left(\frac{b_{11}}{b_1} - \frac{a_1}{a} + \frac{p_1}{p} + \frac{c_1}{c}\right). \tag{11}$$

Also, equation (6) becomes

$$\alpha\gamma(\alpha\beta + v^2) - \alpha^2 u^2 = K^2 a^2 p^2 b^2 c^2. \tag{12}$$

It may seem strange that we have chosen to impose the "isothermal" condition on the field rather than on the metric. In fact, it does not matter very much which choice we make. What we have done appears to be reasonable because, in the Generalized Field Theory, the form of the field, rather than that of the metric, is assumed as given although both are finally determined by the full set of the field equations (this is the "weak principle of geometrization"). The question can also be raised whether the field tensor $h_{\mu\nu}$ and the metric tensor $a_{\mu\nu}$ can be simultaneously diagonalized. The metric condition alone does not permit this conclusion to be drawn immediately. What can be shown directly is that the components

$$a^{01}, \quad a^{21} \quad \text{and} \quad a^{31}$$

are necessarily zero. That the diagonalization of the metric follows is, then, a consequence of the argument recorded in Ref. [1], namely, that, under the condition of cylindrical symmetry, only the "1–2" offdiagonal component (of the metric tensor) is nonzero. These remarks justify our form of both the field and the metric.

Inspection of the equations recorded in the Appendix now shows that the only nonzero skew components of the geometrical connection are as follows:

(*i*) When $u \neq 0$, $v \neq 0$ (the "electromagnetic" case):

$$\tilde{\Gamma}^0_{[13]}, \tilde{\Gamma}^1_{[03]}, \tilde{\Gamma}^1_{[23]}, \tilde{\Gamma}^2_{[13]};$$

(*ii*) when $u \neq 0$, $v = 0$ (the "magnetic" case):

$$\tilde{\Gamma}^0_{[13]}, \tilde{\Gamma}^1_{[03]}, \tilde{\Gamma}^3_{[01]};$$

and
(*iii*) when $u = 0$, $v \neq 0$ (the "electric" case):

$$\tilde{\Gamma}^1_{[23]}, \tilde{\Gamma}^2_{[13]}, \tilde{\Gamma}^3_{[12]}.$$

It is clear that the three cases must be considered separately. We take them up in turn in the next three Sections.

3. *The mixed, electric and magnetic, case*

If we assume that neither u nor v vanishes, the following equations (2) survive:

$$\gamma_1 - 2\frac{c_1}{c}\gamma = 0,$$

$$cc_1\left(\frac{\alpha}{a^2} - \frac{\gamma}{c^2}\right) = 0,$$

$$\alpha_1 - 2\frac{a_1}{a}\alpha = 0,$$

$$\frac{p_1}{p}\left(1 - \frac{a^2}{p^2}\right) = 0,$$

$$bb_1\left(\frac{\beta}{b^2} - \frac{\alpha}{a^2}\right) = \tilde{\Gamma}^0_{[13]}u + \tilde{\Gamma}^2_{[13]}v,$$

$$\alpha_1 - 2\frac{p_1}{p}\alpha = 0,$$

$$\beta_1 - 2\frac{b_1}{b}\beta = 2bb_1\left(\frac{\beta}{b^2} - \frac{\alpha}{a^2}\right),$$

$$\frac{b_1}{b}u = \tilde{\Gamma}^1_{[03]}\alpha + \tilde{\Gamma}^0_{[13]}\gamma,$$

$$u_1 - 2\frac{b_1}{b}u = 0 = v_1 - 2\frac{b_1}{b}v,$$

$$\tilde{\Gamma}^2_{[13]}\alpha = \tilde{\Gamma}^1_{[23]}\alpha - \frac{b_1}{b}v,$$

$$\tilde{\Gamma}^1_{[03]} = \frac{u}{\alpha}\frac{c_1}{c},$$

$$\tilde{\Gamma}^1_{[23]} = \frac{v}{\alpha}\frac{p_1}{p}, \qquad (13)$$

as well as the equation (12). Hence

$$u = hb^2, \quad v = eb^2 \tag{14}$$

and either

$$\gamma = kc^2, \quad \alpha = ka^2 \quad \text{or} \quad c_1 = 0; \tag{15}$$

and

$$p^2 = a^2 \quad \text{or} \quad p_1(= \alpha_1 = a_1) = 0. \tag{16}$$

Also

$$\beta = kb^2 + mb^4. \tag{17}$$

Here k, m, e and h are constants.

It is easily verified that the equation

$$bb_1\left(\frac{\beta}{b^2} - \frac{\alpha}{a^2}\right) = \tilde{\Gamma}^0_{[13]}u + \tilde{\Gamma}^2_{[13]}v \tag{18}$$

is identically satisfied. It is also readily seen that when $c_1 = 0$, the solution is either impossible (when merely $p^2 = a^2$) or trivial (when $p_1 = 0$ since, then, also $b_1 = 0$, the space-time becomes flat and the electromagnetic field vanishes). When $c_1 \neq 0$ but $p_1 = 0$ the field equations are again incompatible and this finally leaves as the only option

$$\gamma = kc^2, \quad \alpha = ka^2, \quad p = a \quad (c_1 \neq 0, \ a_1 \neq 0) \tag{19}$$

proving incidentally the isothermal condition. Then

$$\tilde{\Gamma}^2_{[13]} = \frac{v}{\alpha}\left(\frac{a_1}{a} - \frac{b_1}{b}\right), \quad \tilde{\Gamma}^0_{[13]} = \frac{u}{\gamma}\left(\frac{b_1}{b} - \frac{c_1}{c}\right). \tag{20}$$

The field equations become

$$\frac{c_{11}}{c_1} + \frac{b_1}{b} = 0, \tag{21}$$

$$\frac{c_{11}}{c} + \frac{b_{11}}{b} + \frac{a_{11}}{a} - \frac{a_1}{a}\left(\frac{c_1}{c} + \frac{b_1}{b} + \frac{a_1}{a}\right) = 0, \tag{22}$$

$$\frac{a_{11}}{a_1} - \frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} = 0, \tag{23}$$

and

$$\frac{b_{11}}{b} + \frac{b_1}{b}\frac{c_1}{c} - \frac{2e^2}{k^2}\frac{b^2}{a^2}\left(\frac{a_1}{a} - \frac{b_1}{b}\right)\frac{a_1}{a} - \frac{2h^2}{k^2}\frac{b^2}{c^2}\left(\frac{b_1}{b} - \frac{c_1}{c}\right)\frac{c_1}{c} = 0. \tag{24}$$

Equations (21) and (23) can be immediately integrated to give

$$c_1 b = A, \quad \frac{a_1}{a} bc = B, \tag{25}$$

where A and B are constants, whence

$$b = \frac{A}{c_1}, \quad \frac{a_1}{a} = \frac{B}{A}\frac{c_1}{c} = \lambda\frac{c_1}{c} \tag{26}$$

or

$$a = c^{\lambda} \tag{27}$$

without loss of generality since any proportionality constant between a and c can be removed at this stage by re-scaling.

Equations (22) and (24) now become, respectively,

$$-\frac{c_{111}}{c_1} + 2\left(\frac{c_{11}}{c_1}\right)^2 + (1+2\lambda)\frac{c_{11}}{c} - 2\lambda\frac{c_1^2}{c^2} = 0, \tag{28}$$

and

$$-\frac{c_{111}}{c_1} + 2\left(\frac{c_{11}}{c_1}\right)^2 - \frac{c_{11}}{c} - \frac{2\lambda E^2}{c_1^2 c^{2\lambda}}\left(\lambda\frac{c_1}{c} + \frac{c_{11}}{c_1}\right)\frac{c_1}{c} + \frac{2H^2}{c_1^2 c^2}\left(\frac{c_1}{c} + \frac{c_{11}}{c_1}\right)\frac{c_1}{c} = 0, \tag{29}$$

where $E = \dfrac{eA}{k}$ and $H = \dfrac{hA}{k}$.

Subtracting equation (29) from (28) gives

$$[(1+\lambda)c_1^2 c^{2+2\lambda} + E^2 c^2 - H^2 c^{2\lambda}]\frac{c_{11}}{c} = [\lambda c_1^2 c^{2+2\lambda} - \lambda^2 E^2 c^2 + H^2 c^{2\lambda}]\frac{c_1^2}{c^2}. \tag{30}$$

However, differentiating equation (6) and using (21)

$$[kmc^{2+2\lambda} + e^2 c^2 - h^2 c^{2\lambda}]\frac{c_{11}}{c} = [h^2 c^{2\lambda} - \lambda e^2 c^2]\frac{c_1^2}{c^2}, \tag{31}$$

while eliminating b between the first of the equations (26) and the equation (6) yields

$$c_1^2 c^{2+2\lambda} = \frac{A^2}{L}(kmc^{2+2\lambda} + e^2 c^2 - h^2 c^{2\lambda}), \tag{32}$$

where

$$L = \frac{K^2}{k^2} - k^2 \neq 0.$$

Equations (30), (31) and (32) must lead to an identity. Hence

$$m = 0 \tag{33}$$

($\beta = kb^2$ and $h_{\mu\nu} = ka_{\mu\nu}$) and elimination of c_1 and c_{11} gives

$$\lambda(2+\lambda)A^2e^4c^4+(1+2\lambda)A^2h^4c^{4\lambda}+[\lambda(\lambda-1)LE^2h^2-(1+2\lambda)A^2e^2h^2 +(1-\lambda)LH^2e^2-\lambda(2+\lambda)A^2h^2e^2]c^{2+2\lambda} = 0. \tag{34}$$

Hence, the only possible solution is obtained if, in addition to (33),

$$\lambda = 1, \quad e^2 = h^2. \tag{35}$$

The solution then becomes

$$ds^2 = \frac{r^4}{a_0^4}(dt^2-dr^2-dz^2)-\frac{b_0^4}{r^2}\,d\theta^2, \tag{36}$$

(strictly speaking, $a = c = \left(\frac{r-r_0}{a_0}\right)^2$, $b = \frac{b_0^2}{r-r_0}$, but the constant r_0 can be put equal to zero without loss of generality, the axis of space symmetry being chosen as $r = 0$; the constants a_0 and b_0 are, of course, inserted for dimensional reasons). We now have

$$\tilde{\Gamma}^1_{[03]} = \frac{2h}{k}\frac{b_0^4a_0^4}{r^7}, \quad \tilde{\Gamma}^1_{[23]} = \frac{2e}{k}\frac{b_0^4a_0^4}{r^7}, \quad h = \pm e \tag{37}$$

and so the skew field, as calculated from equation (7), has nonzero components

$$f_{03} = \pm\frac{n}{r^8}$$

in the transverse (θ) direction and

$$f_{23} = \pm\frac{n}{r^8}$$

in the radial (r) direction, both, of course, normal to each other and in the plane norma to the axis of the space symmetry.

Without attempting to interpret the above solution for the moment, we now pass to what is perhaps the most significant result of this article. It reflects on the validity and coherence of the Generalized Field Theory. Thus, let us assume "without prejudice" (i.e. without trying to solve a priori the resulting equation) that the last two terms in the equation (24) cancel out. It other words, we effectively solve the empty field equations of General Relativity though we retain a meaning of the skew field. The former are

$$\frac{c_{11}}{c_1}+\frac{b_1}{b} = 0, \tag{38}$$

$$\frac{c_{11}}{c}+\frac{b_{11}}{b}+\frac{a_{11}}{a}-\frac{a_1}{a}\left(\frac{a_1}{a}+\frac{b_1}{b}+\frac{c_1}{c}\right) = 0, \tag{39}$$

$$\frac{a_{11}}{a_1} - \frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} = 0, \tag{40}$$

$$\frac{b_{11}}{b} + \frac{c_1}{c} = 0. \tag{41}$$

Consider a solution of the form

$$c = r^{\lambda}, \quad a = r^{\mu}, \quad b = r^{\nu}, \quad \lambda, \mu, \nu \text{ constant.}$$

Then

$$\lambda + \nu = 1$$

and

$$2\lambda(\lambda-1) + \mu(\mu-1) - \mu(\mu+1) = 0,$$

so that

$$\mu = \lambda(\lambda-1) \quad \text{and} \quad \nu = 1-\lambda. \tag{42}$$

Therefore

$$\tilde{\Gamma}^1_{[03]} = \frac{h}{k}\frac{b^2}{a^2}\frac{c_1}{c} = \frac{\lambda h}{k} r^{2\nu-2\mu-1} = \frac{\lambda h}{k} r^{1-2\lambda^2}, \tag{43}$$

$$\tilde{\Gamma}^1_{[23]} = \frac{\lambda(\lambda-1)e}{k} r^{1-2\lambda^2}. \tag{44}$$

This, of course, is our previous solution (37) with $\lambda = 2$ and $h = \pm e$. However, we now notice that as

$$\lambda \to 0,$$

the skew field, as defined by the equations (7), tends to a constant value in both, 03 and 23, directions providing that we assume that simultaneously

$$k \to 0$$

as well.

This is precisely the classical Maxwell solution for static cylindrical symmetry ($f_{03} = h$, $f_{23} = e$ correspond to the solution $\frac{h}{r}$ and $\frac{e}{r}$, a result which follows from the transformation laws of tensors). Moreover, the limit is

$$c = a = 1 \quad \text{and} \quad b = r$$

so that the background space-time is Minkowski. What this means is that the Generalized Field Theory collapses and, because $k = 0$, the symmetric field $h_{\mu\nu}$ vanishes. Nevertheless, the corresponding recovery of the Maxwellian solution from its structure is a clear indication that the theory itself is, so to say, "on the right track".

We should point out that the metric relation (6) now requires that, in the above limit, we should also have

$$K^2 = 0.$$

4. *The magnetic case $u \neq 0$, $v = 0$*

When $v = 0$ (i.e. the field is "magnetic" according to Einstein's convention which interchanges the more common designation of the electric and magnetic field vectors), the connection-determining equations are:

$$\begin{aligned}
\gamma_1 - 2\frac{c_1}{c}\gamma &= -2\tilde{\Gamma}^3_{[01]}u, \\
cc_1\left(\frac{\alpha}{a^2} - \frac{\gamma}{c^2}\right) &= \tilde{\Gamma}^3_{[01]}u, \\
\alpha_1 - 2\frac{a_1}{a}\alpha = 0 &= \alpha_1 - 2\frac{p_1}{p}\alpha, \\
\frac{p_1}{p}\left(1 - \frac{p^2}{a^2}\right) &= 0, \\
bb_1\left(\frac{\beta}{b^2} - \frac{\alpha}{a^2}\right) &= \tilde{\Gamma}^0_{[13]}u, \\
\beta_1 - 2\frac{b_1}{b}\beta &= 2\tilde{\Gamma}^0_{[13]}u, \\
\frac{b_1}{b}u &= \tilde{\Gamma}^1_{[03]}\alpha + \tilde{\Gamma}^0_{[13]}\gamma, \\
u_1 - \left(\frac{c_1}{c} + \frac{b_1}{b}\right)u &= -\tilde{\Gamma}^3_{[01]}\beta + \tilde{\Gamma}^0_{[13]}\gamma, \\
\frac{c_1}{c}u &= \tilde{\Gamma}^1_{[03]}\alpha - \tilde{\Gamma}^3_{[01]}\beta,
\end{aligned} \tag{44}$$

together with the metric relation

$$\alpha^2(\gamma\beta - u^2) = K^2a^2p^2b^2c^2. \tag{45}$$

We immediately obtain (if $\alpha_1, p_1, a_1 \neq 0$)

$$\beta = kb^2 + mb^4, \quad \gamma = kc^2 + nc^4 \quad \text{and} \quad \alpha = ka^2 = kp^2$$

so that

$$\gamma\beta - u^2 = \frac{K^2}{k^2}\, b^2 c^2.$$

Therefore

$$\tilde{\Gamma}^0_{[13]} = \frac{mb^3 b_1}{u}, \quad \tilde{\Gamma}^3_{[01]} = -\frac{nc^3 c_1}{u}. \tag{46}$$

Furthermore,

$$\tilde{\Gamma}^1_{[03]}\alpha = \frac{c_1}{c}\,u - \frac{nc^3 c_1}{u}\,\beta = \frac{b_1}{b}\,u - \frac{mb^3 b_1}{u}\,\gamma,$$

or

$$\frac{b_1}{b\left(k^2 n - \dfrac{K^2}{k^2} + kmnb^2\right)} = \frac{c_1}{c\left(k^2 m - \dfrac{K^2}{k^2} + kmnc^2\right)}.$$

Solution is possible only if

$$m = n = 0, \quad b \text{ proportional to } c, \tag{47}$$

when the only nonzero skew component of the connection is

$$\tilde{\Gamma}^1_{[03]} = \frac{c_1 u}{c\alpha} = \frac{h}{k}\,\frac{cc_1}{a^2}. \tag{48}$$

The field equations (which again correspond to the general relativistic empty field) give immediately

$$c = \sqrt{\frac{r-r_0}{c_0}}, \quad b = b_0\sqrt{\frac{r-r_0}{c_0}}, \quad a = \left(\frac{r-r_0}{a_0}\right)^{\lambda},$$

constants a_0, c_0 and b_0 having the dimensions of a length and where we can, if we wish to, put $r_0 = 0$. In addition, the equation

$$2\frac{c_{11}}{c} - 2\frac{a_1}{a}\frac{c_1}{c} + \frac{a_{11}}{a} - \frac{a_1^2}{a^2} = 0$$

must be satisfied, whence

$$\lambda = -\tfrac{1}{4},$$

so that the solution becomes

$$ds^2 = \frac{r}{c_0}\,dt^2 - \left(\frac{a_0}{r}\right)^{1/2}(dr^2 + dz^2) - \frac{b_0^2}{c_0}\,r d\theta^2. \tag{49}$$

Also

$$\tilde{\Gamma}^1_{[03]} = H\sqrt{r}$$

corresponding to a classical field in the θ direction, proportional to

$$r^{-3/2}.$$

5. The electric case $u = 0$, $v \neq 0$

This time the equations which determine $\tilde{\Gamma}^\lambda_{[\mu\nu]}$ are:

$$\gamma_1 - 2\frac{c_1}{c}\gamma = 0,$$

$$cc_1\left(\frac{\alpha}{a^2} - \frac{\gamma}{c^2}\right) = 0,$$

$$\alpha_1 - 2\frac{a_1}{a}\alpha = 0,$$

$$pp_1\left(\frac{\alpha}{a^2} - \frac{\alpha}{p^2}\right) = \tilde{\Gamma}^3_{[12]}v,$$

$$bb_1\left(\frac{\beta}{b^2} - \frac{\alpha}{p^2}\right) = \tilde{\Gamma}^2_{[13]}v,$$

$$\alpha_1 - 2\frac{p_1}{p}\alpha = -2\tilde{\Gamma}^3_{[12]}v,$$

$$\beta_1 - 2\frac{b_1}{b}\beta = 2\tilde{\Gamma}^2_{[13]}v,$$

$$\frac{b_1}{b}v = -\tilde{\Gamma}^2_{[13]}\alpha + \tilde{\Gamma}^1_{[23]}\alpha,$$

$$\frac{p_1}{p}v = \tilde{\Gamma}^3_{[12]}\beta + \tilde{\Gamma}^1_{[23]}\alpha,$$

$$v_1 - \left(\frac{p_1}{p} + \frac{b_1}{b}\right)v = \tilde{\Gamma}^3_{[12]}\beta - \tilde{\Gamma}^2_{[13]}\alpha, \tag{50}$$

so that either

$$\alpha = ka^2, \quad \gamma = kc^2 \quad (c_1 \neq 0),$$

or

$$c_1 = 0, \quad \alpha = ka^2,$$

the second case not being significantly different from the first. Also

$$\alpha = ka^2 = kp^2+np^4, \quad \beta = kb^2+mb^4,$$

and

$$\alpha\beta+v^2 = \frac{K^2}{k^2}p^2b^2. \tag{51}$$

Then, substituting from (50.4) and (50.5) for $\tilde{\Gamma}^3_{[12]}$ and $\tilde{\Gamma}^2_{[13]}$ into the equation

$$\tilde{\Gamma}^1_{[23]}\alpha = -\tilde{\Gamma}^3_{[12]}\beta+\frac{p_1}{p}v = \frac{v_1}{b}v+\tilde{\Gamma}^2_{[13]}\alpha,$$

we find

$$\frac{p_1}{p(L-knp^2)} = \frac{b_1}{b(L-kmb^2)}, \quad L = \frac{K^2}{k^2}-k^2. \tag{52}$$

Hence, omitting, for the moment, dimensional constants of proportionality and as before

$$n = m \quad p = b \tag{53}$$

when, incidentally,

$$\alpha = \beta.$$

The skew connection components become

$$\tilde{\Gamma}^2_{[13]} = \frac{nb^3b_1}{v}, \quad \tilde{\Gamma}^3_{[12]} = -\frac{nb^3b_1}{v} \quad \text{and} \quad \tilde{\Gamma}^1_{[23]} = \frac{bb_1}{v}\left(\frac{L-knb^2}{k+nb^2}\right). \tag{54}$$

The field equations are now

$$\frac{c_{11}}{c_1}-\frac{a_1}{a}+2\frac{b_1}{b} = 0, \tag{55}$$

$$\frac{c_{11}}{c}+2\frac{b_{11}}{b}-\frac{a_1}{a}\left(\frac{c_1}{c}+2\frac{b_1}{b}\right)+2\frac{n^2b^6b_1^2}{v^2} = 0, \tag{56}$$

$$\frac{b_{11}}{b_1}-\frac{a_1}{a}+\frac{b_1}{b}+\frac{c_1}{c}-2\frac{nb^5b_1}{v^2}(L-knb^2) = 0, \tag{57}$$

(the fourth equation being identical to (57)), where

$$a^2 = b^2+\frac{n}{k}b^4.$$

Conditions of compatibility again require

$$n = 0, \quad c = b^{\lambda}$$

when the solution becomes:

$$ds^2 = \frac{c_0^2}{r^2}\,dt^2 - \left(\frac{r}{a_0}\right)^4 (dr^2+dz^2) - \frac{r^4}{b_0^2}\,d\theta^2 \tag{58}$$

with

$$\tilde{\Gamma}^1_{[23]} \text{ proportional to } \frac{1}{r}.$$

Hence the radial component of the field is proportional to r^{-3}.

6. *Conclusions*

None of the solutions obtained above appears to correspond to what is known about classical electromagnetism, except perhaps for very unusual and artificial distributions of charges. This could be regarded as a serious setback for the Generalized Field Theory were it not for two facts.

First of these is the approximation found in Section 3. It shows that, after all, the theory does allow the classical solution. Moreover, the latter occurs necessarily when the background space-time is flat. This confirms again the conclusion, reached in Ref. [5], that locally (i.e. where the Riemannian manifold is flat) the theories of electromagnetism and gravitation bifurcate (except for the results recorded in Ref. [6], which suggest that the correct theory of the electromagnetic field is, unlike Maxwell's, nonlinear). Of course, the result of Section 3 is not local. It is valid when the space-time tends globally to the Minkowski case, when also, as should be expected, there is no place for gravitation. It can be easily verified that the corresponding static field carries no energy as calculated from the standard energy-stress-momentum tensor.

Secondly, if we calculate the components of the Riemann-Christoffel curvature tensor for the three manifolds found in the preceding Sections, we see that the corresponding "electromagnetic" fields tend to zero much faster with the increasing distance from the axis of symmetry. In other words, from a purely practical point of view, we should be able to detect the curvature of the space-time long before the electromagnetic effects could be felt. Hence, although the results of this article are firm predictions of the theory, they are not likely to be of much empirical value. The alternative perhaps would be to seek an extremely strong field but then it may prove too difficult to maintain static, cylindrical symmetry.

APPENDIX

The non-trivial equations (2) are:

$$h_{00,0} = 0,$$

$$h_{00,1} - 2\begin{Bmatrix}0\\01\end{Bmatrix} h_{00} = 2\tilde{\Gamma}^3_{[01]}k_{30},$$

$$
\begin{aligned}
0 &= 2\tilde{\Gamma}^{3}_{[02]}k_{30}, \\
0 &= 2\tilde{\Gamma}^{3}_{[03]}k_{30}, \\
-\begin{Bmatrix}0\\01\end{Bmatrix}h_{00}-\begin{Bmatrix}1\\00\end{Bmatrix}h_{11} &= \tilde{\Gamma}^{3}_{[01]}k_{03}, \\
0 &= \tilde{\Gamma}^{3}_{[21]}k_{03}, \\
0 &= \tilde{\Gamma}^{3}_{[31]}k_{03}, \\
0 &= \tilde{\Gamma}^{3}_{[02]}k_{03}, \\
0 &= \tilde{\Gamma}^{3}_{[01]}k_{32}+\tilde{\Gamma}^{3}_{[12]}k_{03}, \\
0 &= \tilde{\Gamma}^{3}_{[02]}k_{32}, \\
0 &= \tilde{\Gamma}^{3}_{[03]}k_{32}+\tilde{\Gamma}^{3}_{[32]}k_{03}, \\
0 &= \tilde{\Gamma}^{3}_{[03]}k_{03}, \\
0 &= \tilde{\Gamma}^{0}_{[01]}k_{03}+\tilde{\Gamma}^{2}_{[01]}k_{23}, \\
0 &= \tilde{\Gamma}^{0}_{[02]}k_{03}+\tilde{\Gamma}^{2}_{[02]}k_{23}+\tilde{\Gamma}^{3}_{[23]}k_{03}, \\
0 &= \tilde{\Gamma}^{0}_{[03]}k_{03}+\tilde{\Gamma}^{2}_{[03]}k_{23}, \\
h_{11,0} &= 0, \\
h_{11}-2\begin{Bmatrix}1\\11\end{Bmatrix}h_{11} &= 0, \\
\tilde{\Gamma}^{3}_{[10]}k_{32} &= 0, \\
-\begin{Bmatrix}1\\22\end{Bmatrix}h_{11}-\begin{Bmatrix}2\\12\end{Bmatrix}h_{22} &= \tilde{\Gamma}^{3}_{[12]}k_{32}, \\
0 &= \tilde{\Gamma}^{3}_{[13]}k_{32}, \\
0 &= \tilde{\Gamma}^{0}_{[10]}k_{03}+\tilde{\Gamma}^{2}_{[10]}k_{23}, \\
0 &= \tilde{\Gamma}^{0}_{[12]}k_{03}+\tilde{\Gamma}^{2}_{[12]}k_{23}, \\
-\begin{Bmatrix}3\\13\end{Bmatrix}h_{33}-\begin{Bmatrix}1\\33\end{Bmatrix}h_{11} &= \tilde{\Gamma}^{0}_{[13]}k_{03}+\tilde{\Gamma}^{2}_{[13]}k_{23}, \\
h_{22,0} &= 2\tilde{\Gamma}^{3}_{[20]}k_{32}, \\
h_{22,1}-2\begin{Bmatrix}2\\12\end{Bmatrix}h_{22} &= 2\tilde{\Gamma}^{3}_{[21]}k_{32},
\end{aligned}
$$

$$
\begin{aligned}
0 &= 2\tilde{\Gamma}^3_{[23]}k_{32},\\
0 &= \tilde{\Gamma}^0_{[20]}k_{03}+\tilde{\Gamma}^2_{[20]}k_{23}+\tilde{\Gamma}^3_{[03]}k_{23},\\
0 &= \tilde{\Gamma}^0_{[21]}k_{03}+\tilde{\Gamma}^2_{[21]}k_{23}+\tilde{\Gamma}^3_{[13]}k_{23},\\
0 &= \tilde{\Gamma}^3_{[23]}k_{23},\\
0 &= \tilde{\Gamma}^0_{[23]}k_{03}+\tilde{\Gamma}^2_{[23]}k_{23},\\
h_{33,0} &= 2\tilde{\Gamma}^0_{[30]}k_{03}+2\tilde{\Gamma}^2_{[30]}k_{23},\\
h_{33,1}-2\begin{Bmatrix}3\\13\end{Bmatrix}h_{33} &= 2\tilde{\Gamma}^0_{[31]}k_{03}+2\tilde{\Gamma}^2_{[31]}k_{23},\\
0 &= 2\tilde{\Gamma}^0_{[23]}k_{30}+2\tilde{\Gamma}^2_{[23]}k_{32},\\
0 &= \tilde{\Gamma}^0_{[01]}h_{00},\\
0 &= \tilde{\Gamma}^1_{[01]}h_{11},\\
0 &= \tilde{\Gamma}^1_{[02]}h_{11}+\tilde{\Gamma}^0_{[21]}h_{00},\\
-\begin{Bmatrix}3\\13\end{Bmatrix}k_{03} &= \tilde{\Gamma}^1_{[03]}h_{11}+\tilde{\Gamma}^0_{[31]}h_{00},\\
0 &= \tilde{\Gamma}^0_{[02]}h_{00},\\
0 &= \tilde{\Gamma}^2_{[01]}h_{22}+\tilde{\Gamma}^0_{[12]}h_{00},\\
0 &= \tilde{\Gamma}^2_{[02]}h_{22},\\
0 &= \tilde{\Gamma}^2_{[03]}h_{22}+\tilde{\Gamma}^0_{[32]}h_{00},\\
k_{03,0} &= \tilde{\Gamma}^0_{[03]}h_{00},\\
k_{03,1}-\left(\begin{Bmatrix}0\\01\end{Bmatrix}+\begin{Bmatrix}3\\13\end{Bmatrix}\right)k_{03} &= \tilde{\Gamma}^3_{[01]}h_{33}+\tilde{\Gamma}^0_{[13]}h_{00},\\
0 &= \tilde{\Gamma}^3_{[02]}h_{33}+\tilde{\Gamma}^0_{[23]}\ {}_{00},\\
0 &= \tilde{\Gamma}^3_{[03]}h_{33},\\
0 &= \tilde{\Gamma}^2_{[10]}h_{22}+\tilde{\Gamma}^1_{[02]}h_{11},\\
0 &= \tilde{\Gamma}^1_{[12]}\ {}_{11},\\
0 &= \tilde{\Gamma}^2_{[12]}h_{22},\\
-\begin{Bmatrix}3\\13\end{Bmatrix}k_{32} &= \tilde{\Gamma}^2_{[13]}h_{22}+\tilde{\Gamma}^1_{[32]}\ {}_{11},
\end{aligned}
$$

$$-\begin{Bmatrix}0\\10\end{Bmatrix}k_{03} = \tilde{\Gamma}^3_{[10]}h_{33}+\tilde{\Gamma}^1_{[03]}h_{33}+\tilde{\Gamma}^1_{[03]}h_{11},$$

$$0 = \tilde{\Gamma}^1_{[13]\ \ 11},$$

$$-\begin{Bmatrix}2\\12\end{Bmatrix}k_{23} = \tilde{\Gamma}^3_{[12]\ \ 33}+\tilde{\Gamma}^1_{[23]}h_{11},$$

$$0 = \tilde{\Gamma}^3_{[13]}h_{33},$$

$$k_{23,0} = \tilde{\Gamma}^3_{[20]}h_{33}+\tilde{\Gamma}^2_{[03]}h_{22},$$

$$k_{23,1}-\left(\begin{Bmatrix}2\\12\end{Bmatrix}+\begin{Bmatrix}3\\13\end{Bmatrix}\right)k_{23} = \tilde{\Gamma}^3_{[21]}h_{33}+\tilde{\Gamma}^2_{[13]}h_{22},$$

$$0 = \tilde{\Gamma}^2_{[23]}h_{22},$$

$$0 = \tilde{\Gamma}^3_{[23]}h_{33},$$

together with (of course)

$$\tilde{\Gamma}^1_{[01]}+\tilde{\Gamma}^2_{[02]}+\tilde{\Gamma}^3_{[03]} = \tilde{\Gamma}^0_{[10]}+\tilde{\Gamma}^2_{[12]}+\tilde{\Gamma}^3_{[13]} = \tilde{\Gamma}^0_{[20]}+\tilde{\Gamma}^1_{[21]}+\tilde{\Gamma}^3_{[23]}$$
$$= \tilde{\Gamma}^0_{[30]}+\tilde{\Gamma}^1_{[31]}+\tilde{\Gamma}^2_{[32]} = 0.$$

Editorial note. This article was proofread by the editors only, not by the authors.

REFERENCES

[1] A. H. Klotz, G. K. Russell, *Acta Phys. Pol.* **B2**, 623 (1971).
[2] A. H. Klotz, G. K. Russell, *Acta Phys. Pol.* **B2**, 637 (1971).
[3] A. H. Klotz, G. K. Russell, *Acta Phys. Pol.* **B3**, 413 (1972).
[4] L. J. Gregory, A. H. Klotz, *Gen. Relativ. & Gravitation* **13**, 205 (1981).
[5] A. H. Klotz, *Macrophysics and Geometry*, Cambridge 1982.