AMBIGUITIES AND PROBLEMS IN HIGHER ORDER CORRECTIONS*

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The formalism of optimization of perturbative higher order corrections with respect to the renormalization scales is extended to include the factorization scales. We consider the most general case with massless as well as massive particles. As a test of the optimization procedure, we apply our results to the recent $O(\alpha_s^3)$ calculation of the ratio $R_{e^+e^-} = \sigma_{tot}(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$, energy-energy correlation function and large P_t direct photon production in proton-antiproton inelastic collision. We conclude that the optimal scales give always a bit larger result and in some cases, the difference with respect to the physical scales is considerable.

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1. Introduction

In general there are two types of scales (free parameters) determining the perturbative higher order calculation: The renormalization and the factorization scales; e.g. in the theory of quantum chromodynamics (QCD) and up to the next-to-leading order, one has the scale μ entering the running coupling $\alpha_s(\mu)$ (related to the renormalization procedure) and the scales M and γ_1 entering the structure and/or the fragmentation function (related to the factorization procedure).

During the past few years, phenomenologists were rather uninterested in the ambi-

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guities related to the choice of these scales and took just the so called physical scales based on some naive physical arguments. However, this affects in general the result of the perturbative calculation. Recently and because of various motivations, interest has been revived to some extent. To bypass the choice problem of the scales and at least from the theoretical point of view, several procedures (called optimization) have been developped and proposed. The most appealing ones are the *Principle of Minimal Sensitivity* (PMS) [1, 2] and *the criterion of Fastest Apparent Convergence* (FAC) [3].

To give some theoretical motivations to the proposed optimization procedures; consider a physical quantity (e.g. the cross section for some physical processess) whose solution is presented as a power series in the coupling. One can stop at any order of the expansion. Then a feature of this approximation is that it depends on some parameters (scales) which are totally absent from the exact result. Now, the following question arises: given a calculation of the approximate solution to a particular order, how should one choose these unphysical parameters? One has to note that any arbitrary choice will yield automatically to an arbitrary result and no one can base his choice on just pure physical considerations. This is a very serious problem. In what sense is a perturbation prediction if it depends entirely on an arbitrary choice of the renormalization and/or the factorization scales (RS and/or FS)? To have a partial answer, Stevenson attacked the problem from a pure theoretical point of view [1]. He suggested that one has to take the approximate solutions which are stationary under small variations of the unphysical parameters and it is unjustified to assume from the begining that the same set of these free parameters must be used for every physical quantity. Before going one step further and giving other arguments in favour of Stevenson's idea, let us first mention that one has two kinds of information about the approximate calculation of any observable in question: (a) the first few terms of the approximation (perturbative expansion) in some RS and FS, (b) the full calculation (exact result) is RS and/or FS independent. If the chosen approximate answer was in a region of strong variations with respect to the unphysical parameters, it could not be acceptable because it will give completly a different answer. Then, it is necessary to minimize the sensitivity to the unphysical parameters. Moreover, by selecting the value at a stationary point, we are in a sense implementing a known property of the exact result (scheme independence). This does not mean that the optimization procedure yields an answer which is the best approximation to the exact result when it is compared to the experimental data. Hence, one cannot claim by any measure that this procedure is the right answer. However, in the absence of knowing the final answer, the optimization is the most plausible method of reducing at least the prescription ambiguity.

In Section 2 we develop a general formalism to treat the optimization procedure of physical quantities including structure and/or fragmentation functions. We consider both the massless and massive particles case together with the non-singlet and singlet parton distributions. In Section 3 we apply this formalism to various physical processess: the recent $O(\alpha_s^3)$ calculation of the ratio $\sigma_{e^+e^-} = \sigma_{tot}\,(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$, energy-energy correlation function and large P_t direct photon production in proton--antiproton inelastic reaction using the approximate corrections of the soft-gluon approach [4, 5]. Finally, in Section 4 we discuss our results and draw our conclusions.

2. Formalism

To a given order the scheme dependence of a perturbative calculation can be parametrized by a set of variables. If the calculation is repeated to one order higher, one new variable must be added for each renormalized physical quantity (observable). These unphysical variables parametrize essentially the freedom of finite renormalization (or factorization) of the observables. It is worth to mention that the restriction to the perturbation theory is very important. It expresses the fact that everything can be written as a power series in the coupling independently of RS and/or FS.

Let us consider a physical quantity like the cross section for a QCD processess containing structure and/or fragmentation functions. To be more specific, we start with the non-singlet partons distributions and consider the case of massless particles. We remind the reader that the cross section σ for a hadronic process of the form $A+B \to C+D$ has the following factorizable form [6–8]:

$$\sigma^{NS}(A+B \to C+D) = \sum_{\substack{a,b \\ c,d}} q^{NS}_{a/A} \otimes q^{NS}_{b/B} \otimes \hat{\sigma}(a+b \to c+d) \otimes D^{NS}_{C/c} \otimes D^{NS}_{D/d}, \tag{2.1}$$

where A, B, C, and D are hadrons, $q^{NS}_{a/A}$ and $q^{NS}_{b/B}$ ($D^{NS}_{C/c}$ and $D^{NS}_{D/d}$, respectively) are the non-singlet distribution (fragmentation) functions of the partons a and b (c and d, respectively) inside the hadrons A and B (C and D, respectively), $\hat{\sigma}(a+b \to c+d)$ is the parton cross section, $\otimes$ denotes the convolution product and NS the non-singlet part. In the following we restrict ourselves to the case with no partons fragmentation functions. Then Eq. (2.1) becomes

$$\sigma^{NS} = q^{NS} \otimes q'^{NS} \otimes \hat{\sigma}. \tag{2.2}$$

Note that we have omitted the sum and subscripts (q^{NS} for $q^{NS}_{a/A}$ and q'^{NS} for $q^{NS}_{b/B}$). Now in the perturbative expansion with respect to the coupling and up to the k^{th} order, one can write:

$$\hat{\sigma}^{(k)} = a^M[\hat{\sigma}_0 + a\hat{\sigma}_1 + \ldots + a^{(k)}\hat{\sigma}_k], \tag{2.3}$$

where $a \equiv \alpha_s/\pi$ (α_s is the QCD running coupling) is the coupling, $a^M\hat{\sigma}_0$ the Born term (of order a^M) and $a^{M+K}\hat{\sigma}_K$ the K^{th} term (of order a^{M+K}), the parton distribution q^{NS} (or q'^{NS}) satisfy the renormalization group equation [2, 9]:

$$M \frac{\partial q^{NS}}{\partial M} = \gamma^{NS} \tag{2.4}$$

where γ^{NS} is the non-singlet anomalous dimension with the following expansion [2, 9]:

$$\gamma^{NS} = \sum_{i=0} a^{i+1}\gamma^{NS}_i \tag{2.5}$$

and M is the factorization point. Usually Eq. (2.4) is written in the form [9]:

$$\frac{\partial q^{NS}}{\partial a} = \gamma^{NS}/\beta(a), \tag{2.6}$$

with $\beta(a)$ is the well known Callan-Symanzik beta function [9, 10]

$$\beta(a) \equiv \frac{\partial a}{\partial \tau} = -\sum_{i=0} a^{i+2} c_i \quad c_0 \equiv 1, \quad c_1 \equiv c, \tag{2.7}$$

where

$$\tau = b \ln(\mu/\Lambda), \tag{2.8}$$

$$b = (33-2N_f)/6, \tag{2.9}$$

$$c = (153-19N_f)/[2(33-2N_f)]. \tag{2.10}$$

N_f is the number of flavours, c_i's and γ_i^{NS}'s ($i > 1$) are RS and FS dependent coefficients and Λ is a QCD parameter [11]. Now, if one takes the Mellin transform of Eq. (2.2) he gets:

$$\tilde{\sigma}_n^{NS} = \tilde{q}_n^{NS} \tilde{q}_n'^{NS} \tilde{\delta}_n \tag{2.11}$$

("$\sim$" denotes the Mellin transforms of order n). The $\gamma_{n;0}^{NS}$ (Mellin transform of γ_0^{NS}) is given by the following expression [9]:

$$\gamma_{n;0}^{NS} = 8[-2/[n(n+1)] + 4 \sum_{j=2}^{n} 1/j]/3. \tag{2.12}$$

To have more simplification in our calculation, we take the renormalization point μ equals the factorization point M. Then, at the K^{th} order, $\tilde{\sigma}_n^{NS}$ depends just on μ, c_i and $\gamma_{n;i}^{NS}$ ($i = \overline{1, k}$) i.e.

$$\tilde{\sigma}_n^{NS(k)} = \tilde{q}_n^{NS(k)}(\mu, c_i, \gamma_{n;i}^{NS}) \tilde{q}_n'^{NS(k)}(\mu, c_i, \gamma_{n;i}^{NS}) \tilde{\delta}_n(\mu, c_i) \quad (i = \overline{1, k}). \tag{2.13}$$

Now, consider the perturbative approximations $\tilde{\sigma}_n^{NS(k)}$ and $\tilde{\sigma}_n'^{NS(k)}$ in two very close RS's and RF's. Since the exact result of the cross section can be written as:

$$\tilde{\sigma}_n^{NS} = \tilde{\sigma}_n^{NS(k)} + O(a^{M+k+1})$$

or

$$\tilde{\sigma}_n^{NS} = \tilde{\sigma}_n'^{NS(k)} + O(a^{M+k+1})$$

it follows that:

$$d\tilde{\sigma}_n^{NS(k)} = O(a^{M+k+1}). \tag{2.14}$$

Having in mind that τ, c_i's and $\gamma_{n;i}^{NS}$'s are independent parameters, Eq. (2.14) implies that:

$$\frac{\partial \tilde{\sigma}_n^{NS(k)}}{\partial(\tau, c_i, \gamma_{n;i}^{NS})} = O(a^{M+k+1}) \quad (i = \overline{1, k}). \tag{2.15}$$

We call Eq. (2.15) the self-consistent condition. Hence, imposing (2.15) yields (Appendix A):

$$\frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial \tau} = \sum_{l=0}^{j-1} \tilde{\hat{\sigma}}_{n;l} H^M_{j;l} \quad (j = \overline{1, k}), \tag{2.16}$$

$$\frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial c_i} = -\sum_{l=0}^{j-i} \sum_{p=0}^{j-i} \tilde{\hat{\sigma}}_{n;l} B^i_p H^M_{j-i-p+1;l}/(i-1) - \text{same terms } (j \to j-1) \; (j = \overline{i, k};\; i = \overline{2, k}) \tag{2.17}$$

and

$$\frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial \gamma^{NS}_{n;l}} = 2 \sum_{l=0}^{j-i} \tilde{\hat{\sigma}}_{n;l} F_{j-i-l}/(j-l) \quad (j = \overline{i, k};\quad i = \overline{2, k}). \tag{2.18}$$

Here $\tilde{\hat{\sigma}}_{n;j}$ denotes the Mellin transform of $\hat{\sigma}_j$ in the expession (2.3). The F's, B's and H's are defined such that:

$$1/\beta^{(k)}(a) = -\sum_{l=0}^{k} a^{l-2} F_l, \tag{2.19}$$

$$\frac{\partial a}{\partial c_i} = \sum_{l=0}^{k} a^{i+l+1} B^i_l/(i-1), \quad B^i_0 \equiv 1 \tag{2.20}$$

and

$$H^M_{i;l} = -2\gamma^{NS}_{n;i-l-1} + (M+l)c_{i-l-1}. \tag{2.21}$$

Now, combining Eqs. (2.7) and (2.20) we end up with the following recurrence formula:

$$B^i_j = -\sum_{l=1}^{k} (i+j-2l-1) c_l B^i_{j-1}/(i-j-1).$$

Now, the following important question arises: Knowing a result of a perturbative expansion at a given order in some RS and FS, how one can determine it at an other RS and FS? To answer this delicate question, one has to determine some quantities which remain invariant under any RS and FS. As we will see later, these scheme independent quantities play an important role in the optimization procedure especially for the FAC method [10]. One can show easily that the following quantities are RS and RF independent (Appendix B):

$$\eta_i = \sum_{l=1}^{5} S^l_i, \tag{2.22}$$

where

$$S^1_i = \tilde{\hat{\sigma}}_i + \sum_{l=0}^{i-2} \sum_{p=0}^{l} (M+p) \tilde{\hat{\sigma}}_p B^{i-l}_{i-p} c_{i-l}/(i-l-1),$$

$$S_i^2 = -\sum_{l=0}^{i-1} H_{i;l}^M V_l - \sum_{l=0}^{i-2}\sum_{t=0}^{l}\sum_{p=0}^{t-1} (M+t) B_{l-t}^{i-l} c_{i-l} H_{t;p}^M V_p/(i-l-1),$$

$$S_i^3 = 2\sum_{j=1}^{i}\sum_{l=0}^{i-j} F_{i-j-l} W_l^j/(i-l) + 2\sum_{j=1}^{i}\sum_{l=0}^{i-2}\sum_{p=0}^{t-j} (M+t) c_{i-l} F_{t-j-p} W_t^j/[(i-l-1)(t-p)],$$

$$S_i^4 = 2\sum_{j=1}^{i} N_{i-j-1}^j - 2\sum_{j=1}^{i}\sum_{l=0}^{i-1}\sum_{p=0}^{l-j} N_p^j H_{i;l}^M F_{l-j-p}/(l-p),$$

and

$$S_i^5 = 2\sum_{j=1}^{i}\sum_{l=0}^{i-2}\sum_{t=0}^{l} (M+t) c_{i-l} B_{l-t}^{i-l} N_{t-j-1}/(i-l-t)$$

$$-2\sum_{j=1}^{i}\sum_{l=0}^{i-2}\sum_{s=0}^{l}\sum_{p=0}^{s-1}\sum_{t=0}^{p-j} (M+s) c_{i-l} B_{l-s}^{i-l} N_t^j F_{p-j-t} H_{s;p}^M/[(i-l-1)(p-t)].$$

The functions V_l, W_l^j and $H_{i;l}^M$ are given respectively by the following expressions:

$$V_l = \int_0^{\tau} \tilde{\hat{\sigma}}_{n;l} d\tau,$$

$$W_l^j = \int_0^{\gamma_{n;j}^{NS}} \tilde{\hat{\sigma}}_{n;l} d\gamma_{n;j}^{NS},$$

and

$$N_t^j = \int_0^{\gamma_{n;j}^{NS}} \int_0^{\tau} d\gamma_{n;j}^{NS} d\tau \tilde{\hat{\sigma}}_{n;t}.$$

Once the values of $\eta_1, \eta_2, \ldots, \eta_k$ are known, one can determine exactly the form of $\tilde{\sigma}_n^{(k)}$ as a function of the RS and FS parameters $\mu, c_2, \ldots, c_k; \gamma_{n;1}^{NS}, \gamma_{n;2}^{NS}; \ldots, \gamma_{n;k}^{NS}$ (see Section 3). Now, to get the stationary (optimal) point, one has to apply the following condition:

$$\left.\frac{\partial \tilde{\sigma}_n^{NS(k)}}{\partial(\mu; c_2, \ldots, c_i; \gamma_{n;1}^{NS}, \ldots, \gamma_{n;i}^{NS})}\right|_{\text{optimal point}} = 0 \quad (i \leqslant k). \tag{2.23}$$

After a straightforward calculation, the condition (2.23) yields (Appendix C):

$$\sum_{j=0}^{k}\sum_{l=j}^{k} a^j \tilde{\hat{\sigma}}_{n;l} H_{j+k+1;l}^M = 0, \tag{2.24}$$

$$\sum_{j=0}^{k}\sum_{m=\alpha_1}^{\alpha_2} a^j \tilde{\hat{\sigma}}_{n;m} F_{k+j-i-m+1}/(k+j-i-m+1) = 0$$

$$\alpha_1 = \max(0, j+1-i), \quad \alpha_2 = \min(k, k+j+1-i) \tag{2.25}$$

and

$$\sum_{l=0}^{2k+i-1} \sum_{p=\alpha}^{\beta} \sum_{j=\alpha'}^{\beta'} a^l [2\gamma^{NS}_{n;k+l+1-i-p} - (M+j)c_{k+l+1-i-p}] \tilde{\hat{\sigma}}_{n;j} B^i_{p-j}/(i-1)$$

$$- \sum_{l=0}^{k} \sum_{j=l+1}^{k} c_{k+l+1-j} \frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial c_i} = 0, \tag{2.26}$$

where

$$\alpha = \text{mas}\,(0, l+1-i);\ \beta = \min(k+l+1-i, 2k);\ \alpha' = \max(0, p-k) \text{ and } \beta' = \min(k, p). \tag{2.26'}$$

It is worth to mention that an alternative way to the method of the principle of Minimal Sensitivity (PMS) is the Fastest Apparent Convergence procedure (FAC), which consists of absorbing all the perturbative higher order corrections in the Born term (see Section 3).

If one goes one step further and consider the singlet case with massless particles, then Eqs. (2.16), (2.17) and (2.18) will be generalized respectively as follows:

$$\frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial \tau} = \sum_{l=0}^{i-1} Y^{M+l;j-l-1}_{j-l-1;l}, \tag{2.27}$$

$$\frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial c_i} = - \sum_{l=0}^{j-1} \sum_{t=0}^{j-i} Y^{M+j;j-i-l-t}_{j-i-l-t;l} B^i_t/(i-1) - \text{same terms } (\alpha \to \alpha-1), \quad j = \overline{i, k} \tag{2.28}$$

and

$$\frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial \hat{\gamma}^{s\beta\delta}_{n;l}} = \sum_{t=0}^{j-i} Z^{\beta;\delta}_{j;i;t}, \tag{2.29}$$

where

$$Y^{l;j}_{i;t} = -[\hat{\gamma}^s_{n;l}, \tilde{\hat{\sigma}}_{n;t}]_+ + l c_j \tilde{\hat{\sigma}}_{n;t}$$

and

$$Z^{\beta,\delta}_{j;i;t} = [A(\beta, \delta), \tilde{\hat{\sigma}}_{n;t}]_+ F_{j-i-t}/(j-t).$$

The matrix elements $A(\beta, \delta)$ are zero except the one corresponding to the β^{th} row and δ^{th} column which equals to one. The notation $[,]_+$ means anticommutator. The matrix $\hat{\gamma}^s_{n;l}$ of the singlet anomalous dimensions has the form:

$$\hat{\gamma}^s_{n;l} = \begin{bmatrix} \gamma^{sqq}_{n;l} & 0 & N_f \gamma^{sqg}_{n;l} \\ 0 & \gamma^{sqq}_{n;l} & N_f \gamma^{sqg}_{n;l} \\ \gamma^{sgq}_{n;l} & \gamma^{sgq}_{n;l} & \gamma^{sgg}_{n;l} \end{bmatrix} \tag{2.30}$$

and

$$\hat{\delta}^{s}_{n;l} = \begin{bmatrix} \tilde{\sigma}^{qq}_{n;l} & \hat{\sigma}^{q\bar{q}}_{n;l} & \tilde{\sigma}^{qg}_{n;l} \\ \hat{\sigma}^{\bar{q}q}_{n;l} & \tilde{\sigma}^{\bar{q}\bar{q}}_{n;l} & \tilde{\sigma}^{\bar{q}g}_{n;l} \\ \hat{\sigma}^{gq}_{n;l} & \tilde{\sigma}^{g\bar{q}}_{n;l} & \tilde{\sigma}^{gg}_{n;l} \end{bmatrix}. \tag{2.31}$$

Now, applying the optimization condition, one ends up with the following equations:

$$\sum_{j=0}^{k} \sum_{l=j}^{k} a^j Y^{M+l;j+k-l}_{j+k-l;l} = 0, \tag{2.32}$$

$$\sum_{j=0}^{k} \sum_{t=\alpha_1}^{\alpha_2} a^j Z^{\beta_1,\delta_2}_{k+j+1;i;t} = 0, \tag{2.33}$$

$$\sum_{t=0}^{2k+i-1} \sum_{p=\alpha}^{\beta} \sum_{j=\alpha'}^{\beta'} a^t Y^{M+j;k+t-i+1}_{k+t-i-p+1;j} B^i_{p-j}/(i-1) + \sum_{t=i-1}^{k} \sum_{l=t+1}^{k} \frac{\partial \tilde{\delta}_{n;l}}{\partial c_i} c_{k+t-l+1} = 0, \tag{2.34}$$

where $i = \overline{2, k}$; $\beta_1, \delta_2 = \overline{1, 3}$, $\alpha_1 = \max(0, j-i+1)$; $\alpha_2 = \min(k, k+j-i+1)$, and α, β, α' and β' are defined in Eqs. (2.26′).

This type of analysis can be extended easily for the massive particles case to include the renormalized mass parameters. We remind the reader that the perturbation theory is not analytic with respect to the mass m, so we cannot expand any physical quantity (observable) in power series of m. Rather, we typically do the calculation order by order in the coupling a. This leads to the renormalization group equation of the form:

$$\frac{\partial m}{\partial \tau} = d_m m, \tag{2.35}$$

where d_m is the mass anomalous dimension which has the following perturbative expansion:

$$d_m = \sum_{i=1} a^{i+1} d_{m;i}. \tag{2.36}$$

The $d_{m;i}$'s ($i \geqslant 2$) are new RS parameters independent of the previous ones. Hence, implementing these new variables, the self-consistent conditions become:

$$\frac{\partial \tilde{\bar{\sigma}}^{(k)}_n}{\partial \tau} + \frac{\partial \tilde{\bar{\sigma}}^{(k)}_n}{\partial m} \frac{\partial m}{\partial \tau} = O(a^{M+k+1}), \tag{2.37}$$

$$\frac{\partial \tilde{\bar{\sigma}}^{(k)}_n}{\partial c_i} + \frac{\partial \tilde{\bar{\sigma}}^{(k)}_n}{\partial m} \frac{\partial m}{\partial c_i} = O(a^{M+k+1}), \tag{2.38}$$

$$\frac{\partial \tilde{\bar{\sigma}}^{(k)}_n}{\partial \hat{\gamma}^{\alpha\alpha\beta}_{n;i}} = O(a^{M+k+1}), \tag{2.39}$$

and

$$\frac{\partial \tilde{\hat{\sigma}}_n^{(k)}}{\partial m} \frac{\partial m}{\partial d_{m;i}} = O(a^{M+k+1}). \tag{2.40}$$

From the above equations, one can deduce that:

$$\frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial \tau} = \sum_{l=0}^{j-1} \left(Y_{j-l-1;l}^{M+l;j-l-1} - d_{m;j-l-1} m \frac{\partial \tilde{\hat{\sigma}}_{n;l}}{\partial m} \right) \tag{2.41}$$

$$\frac{\partial \tilde{\hat{\sigma}}_{n;\alpha}}{\partial c_i} = \sum_{j=0}^{\alpha-i} \sum_{l=0}^{\alpha-i} \left(- Y_{\alpha-i-l-j;j}^{M+j;\alpha-i-l-j} + d_{m;\alpha-i-j-l} m \frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial m} \right)$$

$$\times B_m^i/(i-1) - \text{same terms } (\alpha \to \alpha-1), \quad \alpha = \overline{i, k}; \quad i = \overline{2, k} \tag{2.42}$$

$$\frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial \hat{\gamma}_{n;i}^{s\beta\delta}} = \sum_{t=0}^{j-i} Z_{j;i;t}^{\beta,\delta}, \tag{2.43}$$

and

$$\frac{\partial \tilde{\hat{\sigma}}_{n;i}}{\partial d_{m;i}} = \sum_{l=0}^{j-i} \left[m \frac{\partial \tilde{\hat{\sigma}}_{n;l}}{\partial m} + \tilde{\hat{\sigma}}_{n;l} \right] F_{j-i-l}/(j-l), \tag{2.44}$$

where in the last two equations, $j = \overline{i, k}$; $i = \overline{1, k}$ and $\beta, \delta = \overline{1, 3}$. Eqs. (2.41), (2.42), (2.43) and (2.44) can be seen as a generalization of Eqs. (2.27), (2.28) and (2.29). In the same manner, one can get the stationary point by applying the optimization condition with respect to τ, c_i's, $\hat{\gamma}_{n;i}^{s\beta\delta}$'s and $d_{n;i}$'s. Doing this we end up with:

$$\sum_{j=0}^{k} \sum_{l=j}^{k} a^j \left(Y_{j+k-l;l}^{M+l;j+k-l} + d_{m;j+k-l} m \frac{\partial \tilde{\hat{\sigma}}_{n;l}}{\partial m} \right) = 0, \tag{2.45}$$

$$\sum_{l=0}^{2k+i-1} \sum_{p=\alpha}^{\beta} \sum_{j=\alpha'}^{\beta'} a^l \left(- Y_{k+l+1-i-p;j}^{M+j;k+j+1-l} + d_{m;j+k-l} m \frac{\partial \tilde{\hat{\sigma}}_{n;j}}{\partial m} \right) B_{p-j}^i/(i-1)$$

$$- \sum_{j=0}^{k} \sum_{l=j+1}^{k} \frac{\partial \tilde{\hat{\sigma}}_{n;l}}{\partial c_i} c_{k+j+1-l} = 0, \tag{2.46}$$

$$\sum_{j=0}^{k} \sum_{l=\alpha_1}^{\alpha_2} a^j Z_{k+j+1;i;l}^{\beta,\delta} = 0, \tag{2.47}$$

and

$$\sum_{j=0}^{k}\sum_{l=\alpha_1}^{\alpha_2} a^j\left[m\frac{\partial\tilde{\tilde{\sigma}}_{n;l}}{\partial m}+\tilde{\tilde{\sigma}}_{n;l}\right]F_{k+1+j-l-l}/(k+j+1-l)=p, \tag{2.48}$$

where

$$\alpha_1=\max(0,j-i+1) \quad \text{and} \quad \alpha_2=\min(k,k+j-i+1).$$

Similarly, Eqs. (2.45), (2.46), (2.47) and (2.48) are the generalization of Eqs. (2.32), (2.33) and (2.34). In this way, we have derived a general formalism for the application of the optimization procedure to all cases without any restrictions. To be more specific and as a test of the PMS and FAC methods we consider in Section 3, three phenomenologically important cases.

3. Applications

3.1. The ratio $R_{e^+e^-}=\sigma_{tot}(e^+e^-\to \text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$

Recently, a calculation of QCD corrections up to the $O(\alpha_s^3)$ (3-loop order) for the total cross section $\sigma_{tot}(e^+e^-\to\text{hadrons})$ has been completed [12], thus extending by one order of α_s the previous calculations (of 2-loop order) [13].

Now, consider the perturbation series expansion of the quantity [3, 14]:

$$R\equiv[R_{e^+e^-}/(3\Sigma Q_f^2)]-1=a(1+a\hat{\sigma}_1+a^2\hat{\sigma}_2\ldots), \tag{3.1.1}$$

where Q_f is the flavour charge. Notice that, in this case there are no structure and/or fragmentation functions. Now, up to the $O(a^2)$ and $O(a^3)$ respectively, Eq. (3.1.1) becomes:

$$R^{(1)}=a(1+a\hat{\sigma}_1), \tag{3.1.2}$$

$$R^{(2)}=a(1+a\hat{\sigma}_1+a^2\hat{\sigma}_2). \tag{3.1.3}$$

We remind the reader, that $R^{(2)}$ is a function of two renormalization points μ and c_2. However, $R^{(1)}$ depends just on one parameter μ. Now, by integrating Eq. (2.7) at the 2-loop order we obtain [10, 15]:

$$\tau^{(2)}=1/a+c\ln[ca/(1+ca)] \tag{3.1.4}$$

and at the 3-loop order the result depends on the relative magnitude of the quantities $4c_2$ and c^2. For $4c_2>c^2$ (case of our interest) and with $\Delta\equiv(4c_2-c^2)^{1/2}$:

$$\tau^{(3)}=1/a+c\ln(ca)-c\ln|1+ca+c_2a^2|+(2c_2-c^2)/\Delta$$
$$\times\{\arctan[(2c_2a+c)/\Delta]-\arctan[c/\Delta]\}. \tag{3.1.5}$$

For the application of PMS, it is supposed that the coeficients $\hat{\sigma}_1$ and $\hat{\sigma}_2$ have been given in a specific RS, e.g. the Minimal subtraction scheme (MS) [12], corresponding to the

scales μ and c_2. Now, at the 2-loop order; applying the basic conditions (2.15) and (2.23) lead to Eqs. (2.16) and (2.24) with $M = k = 1$ and give directly the following relation between the optimal $\hat{\sigma}_1^s \equiv \hat{\sigma}_1$ and $a^s \equiv a$:

$$\hat{\sigma}_1^s = -c/[2(1+ca^s)] \tag{3.1.6}$$

(here $\tilde{\sigma}_{n;i}$'s are $\hat{\sigma}_i$'s). Then the optimal physical quantity R becomes:

$$R_{\mathrm{PMS}}^{(1)} = a^s(1+ca^s/2)/(1+ca^s). \tag{3.1.7}$$

At the 3-loop order, the conditions (2.15) and (2.23) yield Eqs. (2.16), (2.17), (2.24) and (2.26) with $M = 1$ and $k = 2$. After straightforward simplifications on gets the optimal $\hat{\sigma}_1 \equiv \hat{\sigma}_1^s$ and $\hat{\sigma}_2 \equiv \hat{\sigma}_2^s$ in terms of the optimal $c_2 \equiv c_2^s$ and $a \equiv a^s$, i.e.

$$\hat{\sigma}_1^s = c_2^s a^s(2-ca^s)/[2(3+ca^s)] \tag{3.1.8}$$

and

$$\hat{\sigma}_2^s = c_2^s(-3+ca^s)/[3(3+c_2^s a^{s2})] \tag{3.1.9}$$

consequently, the value of the optimal physical quantity $R^{(2)}$ is:

$$R_{\mathrm{PMS}}^{(2)} = a^s(6-a^{s3}cc_2^s/(3+c_2^s a^{s2}))/6. \tag{3.1.10}$$

As it is mentioned in Section 2; at any order of a perturbative expansion, one can construct a set of RS and FS invariant quantities η_i given by Eq. (2.22). In the case of our interest (no structure and fragmentation functions) one obtains:

$$\eta_1 = \tau - \hat{\sigma}_1, \quad \eta_2 = c, \quad \eta_3 = c_2 + \hat{\sigma}_2 - c\hat{\sigma}_1 - \hat{\sigma}_1^2. \tag{3.1.11,a,b,c}$$

Now, at the 2-loop order Eqs. (3.1.11a), (3.1.4) and (3.1.6) imply the following equation determining the optimal a:

$$1/a + c\ln[ca/(1+ca)] + c/[2(1+ca)] = b\ln(\mu/\Lambda) - \hat{\sigma}_1. \tag{3.1.12}$$

Similarly, at the 3-loop order Eqs. (3.1.11), (3.1.5), (3.1.8) and (3.1.9) lead to:

$$1/a^s + c\ln(ca^s) - c\ln|1+ca^s+c_2^s a^{s2}| + (2c_2^s - \Delta^s)/\Delta^s$$
$$\times\{\arctan[(2c_2 a+c)/\Delta^s] - \arctan[c/\Delta^s]\} - \hat{\sigma}_1^s = b\ln(\mu/\Lambda) - \hat{\sigma}_1 \tag{3.1.13}$$

and

$$c_2^s + \hat{\sigma}_2^s - c\hat{\sigma}_1^s - \hat{\sigma}_1^{s2} = c_2 + \hat{\sigma}_2 - c\hat{\sigma}_1 - \hat{\sigma}_1^2 \tag{3.1.14}$$

here Δ^s stands for $(4c_2^s - c^2)^{1/2}$. The quantities $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are already given in Ref. [12] (in the MS scheme):

$$\hat{\sigma}_1^{\mathrm{MS}} = 7.539 - 0.441N_{\mathrm{f}},$$
$$\hat{\sigma}_2^{\mathrm{MS}} = 133.647 - 8.778N_{\mathrm{f}} + 0.176N_{\mathrm{f}} - 0.840(\Sigma Q_{\mathrm{f}})^2, \tag{3.1.15}$$

and in the $\overline{\text{MS}}$ scheme:

$$\hat{\sigma}_1^{\overline{\text{MS}}} = 1.986 - 0.115 N_f,$$

$$\hat{\sigma}_2^{\overline{\text{MS}}} = 70.985 - 1.2 N_f + 0.005 N_f - 0.840 (\Sigma Q_f)^2. \tag{3.1.16}$$

Where N_f and Q_f are the number of flavours and the flavour number respectively. To this purpose, we use the fact that in the MS and $\overline{\text{MS}}$ schemes [3, 12]:

$$c_2 = (2857 - 5033/9 N_f + 325/27 N_f^2)/64b. \tag{3.1.17}$$

Next, the FAC method is defined by requiring that:

$$\hat{\sigma}_1 \equiv \hat{\sigma}_1^F = 0, \quad \hat{\sigma}_2 \equiv \hat{\sigma}_2^F = 0. \tag{3.1.18}$$

At the 2-loop order, the quantity $\tau = \tau^F$ is determined from the invariant (3.1.11a):

$$\tau^F = b \ln (\mu/\Lambda) - \hat{\sigma}_1 \tag{3.1.19}$$

and the corresponding optimal value $a = a^F$ is determined from Eq. (3.1.4). At the 3-loop order, first we determine the optimal $c_2 \equiv c_2^F$ using the invariant (3.1.11c):

$$c_2^F = c_2 + \hat{\sigma}_2 - c\hat{\sigma}_1 - \hat{\sigma}_1^2. \tag{3.1.20}$$

Then with c_2^F known, one determines the optimal $a \equiv a^F$ using Eqs. (3.1.5) and (3.1.11a). We remind that, for this method and either for the 2-loop or 3-loop case the value of the optimal physical quantity R is:

$$R^{(i)} = a_i^F, \quad i = 1, 2. \tag{3.1.21}$$

Now, numerically, using $\check{\sigma}_1^{\text{MS}}$ and $\hat{\sigma}_2^{\text{MS}}$ we have carried the above optimizations for $\mu = \sqrt{s} = 34$ GeV ($\sqrt{s}$ is the e^-e^+ c-m energy) and flavour number $N_f = 5$. As a check of our calculation, we have repeated the procedures using $\hat{\sigma}_1^{\overline{\text{MS}}}$ and $\hat{\sigma}_2^{\overline{\text{MS}}}$; of course, the results are the same. From Ref. [12], we have used $\Lambda_{\overline{\text{MS}}} = 585$ Mev (and $\Lambda_{\text{MS}} = 0.377\, \Lambda_{\overline{\text{MS}}}$ [3]).

Table I presents the resulting $R^{(1)}$, $R^{(2)}$ and $[R^{(2)} - R^{(1)}]/R^{(1)}$, together with the corresponding quantities in MS and $\overline{\text{MS}}$. The same table presents also the running coupling a at the 2 and 3 loop order for the FAC, PMS as well as MS and $\overline{\text{MS}}$ schemes.

It is remarkable that $[R^{(2)} - R^{(1)}]/R^{(1)}$ is largest for the FAC procedure; and is next-

TABLE I

The physical quantities $R^{(1)}$, $R^{(2)}$, the fractional difference $(R^{(2)}-R^{(1)})/R^{(1)}$ and the running couplings a_2 (2-loop order) and a_3-(3loop order) in the considered schemes

	$R^{(1)} \times 10^2$	$R^{(2)} \times 10^2$	$\frac{R^{(2)} - R^{(1)}}{R^{(1)}}$	$a_2 \times 10^2$	$a_3 \times 10^2$
FAC	5.75	7.30	0.270	5.95	7.30
PMS	5.95	7.23	0.215	6.15	7.24
MS	5.36	6.13	0.144	4.33	4.34
$\overline{\text{MS}}$	5.64	6.52	0.156	5.23	5.26

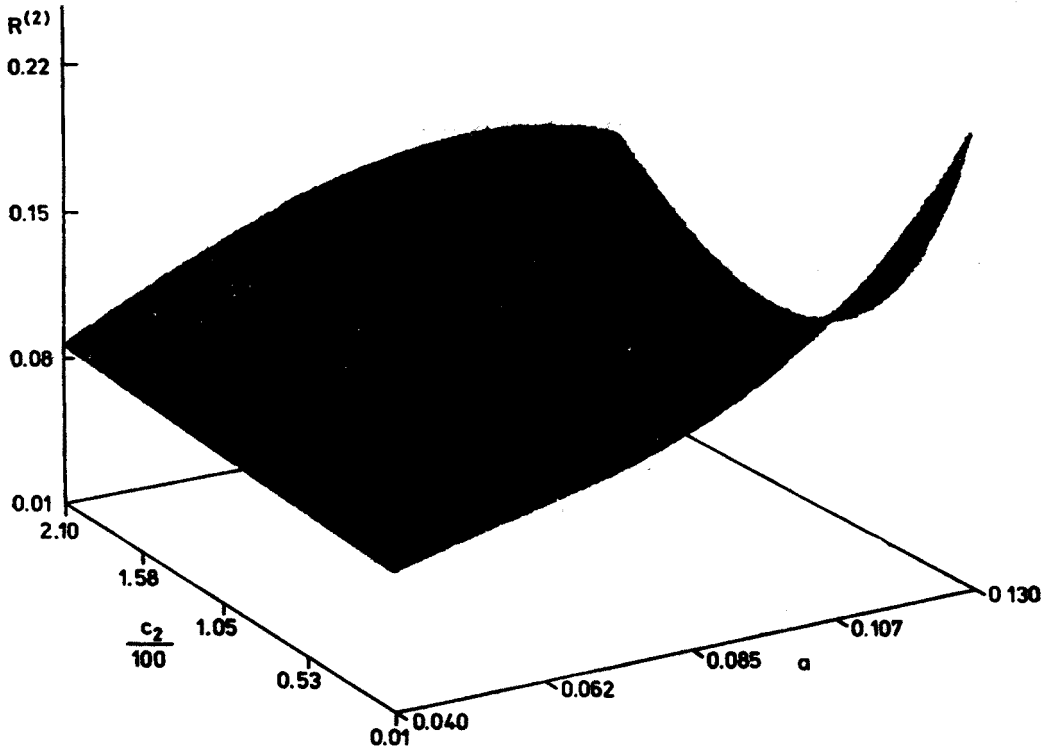

Fig. 1. The surface $R^{(2)} = R^{(2)}(c_2, a_3)$ (Eq. (3.1.3) determined at $\mu = \sqrt{s} = 34$ GeV

-to-largest for PMS. This can be taken to indicate that, regarding $R_{e^+e^-}$, these optimization procedures lead to a slower convergence than the usual MS and $\overline{\text{MS}}$ schemes.

Anyway, as a byproduct of our work, we also present in Fig. 1 the quantity $R^{(2)}$ as a function of c_2 and a. As anticipated [1], the optimal $c_2 = c_2^s$ and $a = a^s$ correspond to a saddle point of the surface $R^{(2)} = R^{(2)}(c_2, a)$.

3.2. Energy-energy correlation function

As our second application, we consider the energy-energy correlation function which has been proven to provide viable way of testing QCD and a powerful estimatior of the strong running coupling α_s [16]. The most recent calculation [17–20] gives results up to the 2-loop order, which can be written in the simple form:

$$R(\mu, \kappa) = a(\mu)\,[1 + a(\mu)\hat{\sigma}_1(\kappa)], \tag{3.2.1}$$

where κ is the relative angle between the two jet spectrometers. Now, in the $\overline{\text{MS}}$ scheme ($\mu = \sqrt{s}$; c.m. energy) and for $15° \leqslant \kappa \leqslant 165°$ and the number of flavours $N_f = 4$, $\hat{\sigma}_1(\kappa)$ has the following parametrized expression:

$$\hat{\sigma}_1(\kappa) = 5.454 + 2.819\omega^{1/2} + 1.013\omega^{-1/2} - 0.466\omega - 0.29\omega^{-1} + 0.03\omega^{3/2} + 0.017\omega^{-3/2}, \tag{3.2.2}$$

with

$$\omega = \cot^2(\kappa/2). \tag{3.2.3}$$

Notice that the expression (3.2.1) has the same form as Eq. (3.1.2), thus applying optimization, the PMS result is:

$$R_{\mathrm{PMS}} = a^s(1+ca^s/2)/(1+ca^s), \tag{3.2.4}$$

where a^s is given by Eq. (3.1.12). For the FAC method, it is trivial from the invariant (3.1.11a) that:

$$R_{\mathrm{FAC}} = a(\mu_{\mathrm{FAC}}), \tag{3.2.5}$$

where

$$\mu_{\mathrm{FAC}} = \sqrt{s}\exp(-\hat{\sigma}_1(\kappa)/b). \tag{3.2.6}$$

For $\sqrt{s} = 35$ GeV and with both $\Lambda = 0.2$ and 0.4 GeV, Figs. 2 and 3 display the optimal R_{PMS} and $R(\mu = \sqrt{s})$ ($\mu = \sqrt{s}$ = physical scale). They show that for $\Lambda = 0.2$ GeV, the difference between the two quantities compared to the Born contribution lies between 8% and 59% depending on the relative angle κ. For $\Lambda = 0.4$ GeV, the difference is even larger; it is between 19% and 100%. This means that using the optimal scale is equivalent to implementing a bit large correction with respect to the Born contribution, especially at low κ. These results suggest that one has to consider seriously this problem, notably from the phenomenological and experimental point of view, in order to have a precise measurements for the QCD running coupling α_s. Finally, it is worth to mention that the FAC method gives results similar to those of the PMS procedure (difference less than 3%). This is understandable from the expression (3.2.4). If the coupling a is small Eq. (3.2.4)

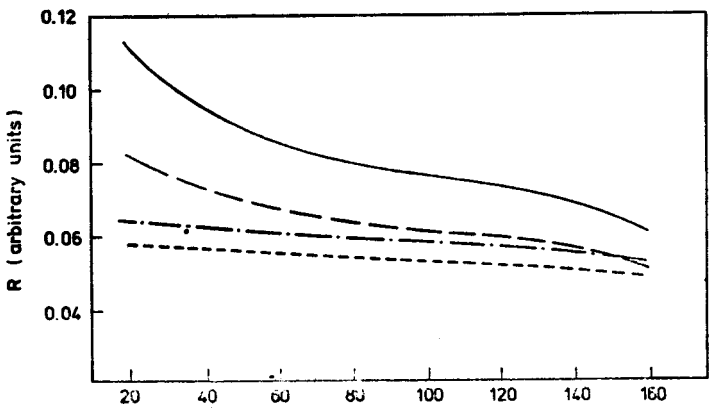

Fig. 2. The quantity $R(\mu, \kappa)$ (Eq. (3.2.1)) in arbitrary units as a function of the relative angle κ. Solid (long-dashed) line is the PMS result at c.m. energy $\sqrt{s} = 35$ GeV, and with $\Lambda = 0.4$ GeV ($\Lambda = 0.2$ GeV). Dashed-dotted (dotted) line is the result obtained with the physical scale $\mu = \sqrt{s} = 35$ GeV at $\Lambda = 0.4$ GeV ($\Lambda = 0.2$ GeV)

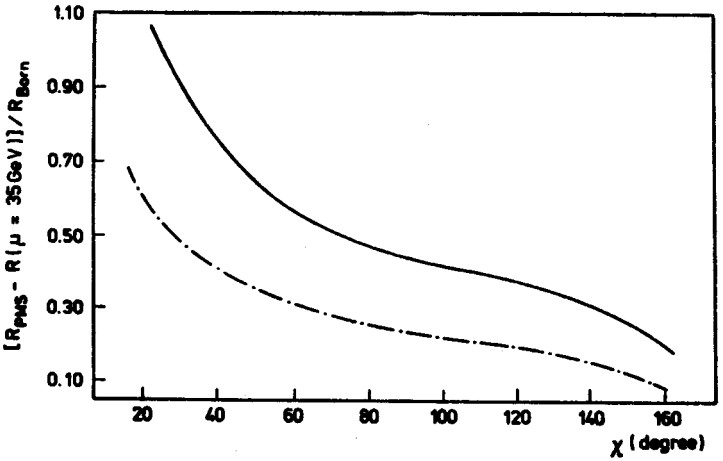

Fig. 3. The ratio $[R_{PM} - R(\mu = 35\ \text{GeV})]/R_{Born}$ for the energy-energy correlation function at c.m. energy $\sqrt{s} = 35$ GeV as a function of the relative angle κ. Solid (dash-dotted) line represents the result for $\Lambda = 0.4$ GeV ($\Lambda = 0.2$ GeV)

becomes

$$R_{PMS} \simeq a^s[1 - ca^s/2 + O(a^{s^2})] \simeq a^s. \tag{3.2.7}$$

This has the same form as that of the FAC method (see Eq. (3.2.5)).

3.3. Large-P_t direct photon production

As our final application, we consider the difference σ of the inclusive cross sections for $\bar{p}p \to \gamma X$ and $pp \to \gamma X$ (non-singlet case) at large transverse momentum P_t:

$$E\frac{d\sigma}{d^3p}(\bar{p}p \to \gamma X) - E\frac{d\sigma}{d^3p}(pp \to \gamma X) \equiv \sigma \tag{3.3.1}$$

we study this difference at fixed c.m. energy $\sqrt{s}$ and pseudo rapidity y. We remind the reader that this reaction is dominated by the subprocess $q_v\bar{q}_v \to \gamma g$ with $\bar{q}_v$ with q_v valence quarks and antiquarks parton distribution. Here is a case with non-singlet structure functions and massless particles. The Born and next-to-leading order contributions can be written as (see Eq. (2.11)):

$$\tilde{\sigma}_n^{NS} = \tilde{q}_n^{NS}\tilde{q}_n^{NS}\tilde{\hat{\sigma}}_n, \tag{3.3.2}$$

with

$$\tilde{\hat{\sigma}}_n = a[\tilde{\hat{\sigma}}_{n;0} + a\tilde{\hat{\sigma}}_{n;1}]. \tag{3.3.3}$$

As it is justified in our publications of Refs. [4, 21–23] $\tilde{\hat{\sigma}}_{n;1}$ can have as expression the approximate soft-gluon corrections (π^2-term) [4, 24]:

$$\tilde{\hat{\sigma}}_1 = C_F\pi^2\tilde{\hat{\sigma}}_{n;0}, \tag{3.3.4}$$

where $C_F = (N_c-1)/N_c$ in SU(N_c) color (N_c is the number of colors). Now, applying optimization, the expression (2.24) with $M = k = 1$ is reduced to

$$\tilde{\sigma}_{n;0}[2\gamma^{NS}_{n;1}-c]+2\tilde{\sigma}_{n;1}[\gamma^{NS}_{n;0}-1]+2a[\gamma^{NS}_{n;1}-c] = 0. \tag{3.3.5}$$

We have applied both conventions (PMS and FAC) using the soft-gluon approximation (3.3.4) at $P_t = 6$ GeV, c.m. energy $\sqrt{s} = 63$ GeV and pseudo rapidity $y = 0$. Also, we have used the quark valence distribution of Duke-Owens set 1 [25] which has $\Lambda = 0.2$ GeV and number of flavours $N_f = 4$. The FAC (PMS) solution is shown with a circle (a triangle, respectively). Notice that our optimal cross sections (FAC an PMS) differ very little from each other (the difference is less than 3%). Moreover, in spite of wide changes in the scales μ and M, the cross sections $\sigma(M, \mu)$ are relatively stable (see Fig. 4). In fact, the largest difference is less than 25%.

Finally, it is worth to mention that strictly speaking, one has to use structure functions calculated up to the first order in the anomalous dimensions $\gamma^{NS}_{n;1}$. However, and since there is no available parametrization for such structure functions, we restrict ourselves to the ones of Duke-Owens [25] which were calculated just to the zeroth order in the anomalous dimensions $\gamma^{NS}_{n;0}$.

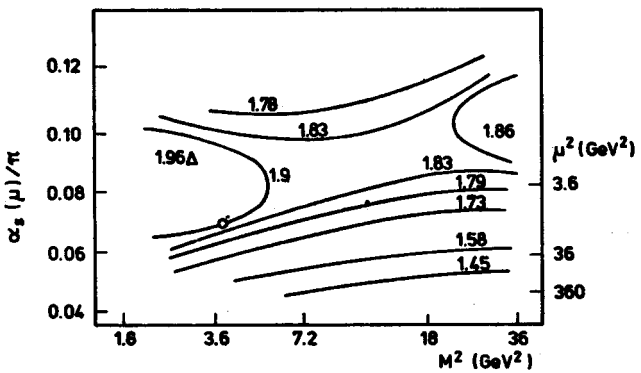

Fig. 4. Contour lines for $E\,d\sigma(\bar{p}p \to \gamma X)/d^3p - Ed\sigma(pp \to \gamma X)/d^3p \equiv \sigma$ at $\sqrt{s} = 63$ GeV and pseudo-rapidity $y = 0$, as functions of M^2 and μ^2 or $\alpha_s(\mu)/\pi$ at $P_t = 6$ GeV in units of 10^{-35} cm^2/GeV^2. The circle (triangle) denotes FAC (PMS) result

4. Conclusion

We have seen through the three previous applications and in all cases that the optimal scales give results larger than those of the so called physical scales. This is obvious, since, from our work of Ref. [4], and roughly speaking, the optimal scale amounts to exponentiating the corrections at the physical scale (true at the 1-loop order). Moreover, the stability of the results and the convergence of the perturbative expansion is a process dependent. In fact, in the energy-energy correlation function case the optimal scales depend on the relative angle κ, especially at smaller values where the two jets events dominate. As we have seen, the difference can be 100% with respect to the Born contribution with a K-factor

[(Born + corrections)/Born] of two. However, in the large P_t direct photon case and despite of the large interval of scales variations (one order of magnitude), the cross sections are stable (less than 20% difference). Similarly, in the e^+e^- case (ratio $R_{e^+e^-}$) and except the convergence problem of the perturbative expansion, the stability is fairly good.

We are indebted to Dr M. Dhina for useful discussions on the energy-energy correlation function. One of us (N.M.) would like to thank professor A. P. Contogouris for fruitfull discussions and encouragement during the time I spent at Mc. Gill university.

APPENDIX A

Let us consider the quantity (see Eq. (2.11)):

$$\tilde{\sigma}_n^{\mathrm{NS}} = \tilde{q}_n^{\mathrm{NS}} \tilde{q}_n'^{\mathrm{NS}} \tilde{\hat{\sigma}}_n \tag{A.1}$$

which can be written as

$$\tilde{\sigma}_n^{\mathrm{NS}} = \langle O \rangle \tilde{\hat{\sigma}}_n, \tag{A.2}$$

where $\langle O \rangle$ represents the long range contribution. Now, using the fact that:

$$\tilde{\hat{\sigma}}_n = \sum_{l=0}^{k} a^{M+l} \tilde{\hat{\sigma}}_{n;l}, \tag{A.3}$$

$$\frac{\partial a}{\partial \tau} = -\sum_{m=0}^{k} a^{m+2} c_m, \tag{A.4}$$

and

$$\frac{\partial \langle O \rangle}{\partial \tau} = 2\langle O \rangle \sum_{i=0}^{k} a^{i+1} \gamma_{n;i}^{\mathrm{NS}}, \tag{A.5}$$

and applying the self consistent condition (2.15) with respect to τ i.e.:

$$\frac{\partial \tilde{\hat{\sigma}}_n}{\partial \tau} = O(a^{M+k+l}). \tag{A.6}$$

We end up with the following expression:

$$\langle O \rangle \sum_{i=0}^{k} \sum_{l=0}^{k} a^{i+M+l+1} \tilde{\hat{\sigma}}_{n;l} [2\gamma_{n;i}^{\mathrm{NS}} - (M+1)c_i] + \langle O \rangle \sum_{l=0}^{k} a^{M+l} \frac{\partial \tilde{\hat{\sigma}}_{n;l}}{\partial \tau} = O(a^{M+k+1}). \tag{A.7}$$

After straigthforward simplifications we obtain:

$$\frac{\partial \sigma_{n;j}}{\partial \tau} = \sum_{l=0}^{j-1} \tilde{\hat{\sigma}}_{n;l} [-2\gamma_{n;j-l-1}^{\mathrm{NS}} + (M+1)c_{j-l-1}] \quad j = \overline{1, k} \tag{A.8}$$

which is exactly Eq. (2.16).

Now, applying the self-consistent condition with respect to c_i

$$\frac{\partial \sigma}{\partial c_i} = O(a^{M+k+1}), \tag{A.9}$$

or

$$\beta^{(k)} \frac{\partial \sigma}{\partial c_i} = O(a^{M+k+3}), \tag{A.10}$$

where $\beta^{(k)}$ is the k^{th} order Callan-Symanzik beta function. Since

$$\frac{\partial \langle O \rangle}{\partial c_i} = 2\langle O \rangle \sum_{m=0}^{k} \sum_{j=0}^{k} a^{j+i+m+2} \gamma_{n;j}^{\text{NS}} B_m^i / [\beta^{(k)}(i-1)] \tag{A.11}$$

and

$$\frac{\partial a}{\partial c_i} = \sum_{l=0}^{k} a^{i+l+1} B_l^i / (i-1). \tag{A.12}$$

We get

$$\sum_{j=0}^{k} \sum_{m=0}^{k} \sum_{l=0}^{k} a^{j+m+i+M+l+2} \tilde{\delta}_{n;j} B_m^i [\gamma_{n;l}^{\text{NS}} - (M+1)c_m/(i-1)]$$

$$- \sum_{l=0}^{k} \sum_{m=0}^{k} a^{m+M+l+2} c_m \frac{\partial \tilde{\delta}_{n;l}}{\partial c_i} = O(a^{M+k+3}). \tag{A.13}$$

After some simplifications and noticing that

$$\frac{\partial \tilde{\delta}_{n;l}}{\partial c_i} = 0 \quad \text{for} \quad 1 < i. \tag{A.14}$$

We obtain

$$\sum_{j=0}^{l-i} \sum_{m=0}^{l-i} \tilde{\delta}_{n;j} B_m^i [2\gamma_{n;l-i-j-m}^{\text{N}} - (M+j)c_{l-i-j-m}]/(i-1)$$

$$= \sum_{\alpha=0}^{l} c_{l-\alpha} \frac{\partial \tilde{\delta}_{n;\alpha}}{\partial c_i}, \quad l = \overline{i, k}, \quad i = \overline{2, k}. \tag{A.15}$$

Eq. (A.15) yields immediately the expression

$$\sum_{l=0}^{j-i}\sum_{m=0}^{j-l}\tilde{\delta}_{n;l}B_m^l[2\gamma_{n;j-i-l-m}^{NS}-(M+l)c_{j-i-l-m}]/(i-1)$$

$$=\frac{\partial\tilde{\delta}_{n;j}}{c_i}-\text{same terms}\,(j\to j-1),\quad j=\overline{i,k},\quad i=\overline{2,k}. \tag{A.16}$$

Then, we derive again Eq. (2.17).

Finally, the self-consistent condition with respect to $\gamma_{n;i}^{NS}$ i.e.:

$$\frac{\partial\tilde{\delta}_{n;\alpha}}{\partial\gamma_{n;i}^{NS}}=O(a^{M+k+1}) \tag{A.17}$$

and the relation

$$\frac{\partial\langle O\rangle}{\partial\gamma_{n;i}^{NS}}=-2\langle O\rangle\sum_{l=0}^{k}a^{l+i}F_l/(l+i) \tag{A.18}$$

together with Eq. (A.3), give:

$$\frac{\partial\tilde{\delta}_{n;j}}{\partial\gamma_{n;i}^{NS}}=2\sum_{l=0}^{k}\tilde{\delta}_{n;l}F_{j-i-l}/(j-l)\quad j=\overline{i,k},\quad i=\overline{2,k}. \tag{A.19}$$

We remind the reader that the B's and F's are defined in Eqs. (2.19)–(2.21). Notice that Eq. (A.19) has the same form as Eq. (2.18).

APPENDIX B

By construction and using Eqs. (2.16)–(2.18) one can see immediately that the invariant η_i has to have the following form:

$$\eta_i=S_i^1+f_i, \tag{B.1}$$

with

$$S_i^1=\tilde{\delta}_i+\sum_{l=0}^{i-2}\sum_{m=0}^{l}(M+m)\tilde{\delta}_m B_{l-m}^{i-l}c_{i-l}/(i-l-1), \tag{B.2}$$

where f_i is a function of τ and $\gamma_{n;j}^{NS}$'s ($j=\overline{1,k}$). By taking the partial derivative of (B.1) with respect to τ and using Eq. (2.16) we obtain:

$$f_i=S_i^2+L_i \tag{B.3}$$

with

$$S_i^2 = -\sum_{l=0}^{i-1}[-2\gamma_{n;i-l-1}^{NS}+(M+l)c_{i-l-1}]\int_0^\tau \tilde{\hat{o}}_{n;l}d\tau - \sum_{l=0}^{i-2}\sum_{m=0}^{l}\sum_{p=0}^{m-1}(M+m)$$

$$\times B_{l-m}^{i-l}c_{i-l}[-2\gamma_{n;m-p-1}^{NS}+(M+p)c_{m-p-1}]\int_0^\tau \tilde{\hat{o}}_{n;p}d\tau/(i-l-1). \tag{B.4}$$

L_i is a function of just the anomalous dimensions γ_j's ($j=\overline{1,k}$). Now, taking the total derivative of both (B.1) and (B.3) and using Eq. (2.18), we end up with:

$$L_i = S_i^3+S_i^4+S_i^5+\text{const}, \tag{B.5}$$

where

$$S_i^3 = 2\sum_{j=1}^{i}\sum_{l=0}^{i-j}F_{i-j-l}\int_0^{\gamma_{n;j}^{NS}}\tilde{\hat{o}}_{n;l}d\gamma_{n;j}^{NS}/(i-l)+2\sum_{j=1}^{i}\sum_{l=0}^{i-2}\sum_{p=0}^{m-j}(M+m)c_{i-l}$$

$$\times F_{m-j-p}\int_0^{\gamma_{n;j}^{NS}}\tilde{\hat{o}}_{n;m}d\gamma_{n;j}^{NS}/[(i-l-1)(m-p)], \tag{B.6}$$

$$S_i^4 = 2\sum_{j=1}^{i}\int_0^{\gamma_{n;j}^{NS}}\int_0^\tau \tilde{\hat{o}}_{n;i-j-1}d\gamma_{n;j}^{NS}d\tau+2\sum_{j=1}^{i}\sum_{l=0}^{i-1}\sum_{p=0}^{l-j}\int_0^{\gamma_{n;j}^{NS}}\int_0^\tau d\gamma_{n;j}^{NS}d\tau$$

$$\times[2\gamma_{n;i-l-1}^{NS}-(M+l)c_{i-l-1}]F_{l-j-p}\tilde{\hat{o}}_{n;p}/(l-p), \tag{B.7}$$

and

$$S_i^4 = 2\sum_{j=1}^{i}\sum_{l=0}^{i-2}\sum_{m=0}^{l}(M+m)c_{i-l}B_{l-m}^{i-l}\int_0^{\gamma_{n;j}^{NS}}\int_0^\tau \tilde{\hat{o}}_{n;m-j-1}d\gamma_{n;j}^{NS}d\tau+2\sum_{j=1}^{i}\sum_{l=0}^{i-2}$$

$$\times\sum_{m=0}^{l}\sum_{p=0}^{m-1}\sum_{t=0}^{p-j}(M+m)c_{i-l}B_{l-m}^{i-l}\int_0^{\gamma_{n;j}^{NS}}\int_0^\tau d\gamma_{n;j}^{NS}d\tau F_{p-j-t}\tilde{\hat{o}}_{n;t}$$

$$\times[2\gamma_{n;m-p-1}^{NS}-(M+p)c_{m-p-1}]/[(i-l-1)(p-t)]. \tag{B.8}$$

APPENDIX C

Using Eq. (A.8) which is a result of the self-consistent condition (A.6), we obtain from Eq. (A.7):

$$\sum_{m=0}^{k}\sum_{l=0}^{k}a^{m+M+l+1}\tilde{\hat{o}}_{n;l}[2\gamma_{n;m}^{NS}-(M+l)c_l] = -\frac{\partial\hat{o}_n^{NS}}{\partial\tau}. \tag{C.1}$$

Now, by setting $m+l=j+k$ and applying the optimization condition (2.23) with respect to τ we get:

$$\sum_{j=0}^{k}\sum_{l=j}^{k}a^j\tilde{\hat{o}}_{n;l}[2\gamma_{n;j+k-l}^{NS}-(M+l)c_{j+k-l}] = 0 \tag{C.2}$$

which is similar to Eq. (2.24).

For the optimization with respect to $\gamma^{NS}_{n;i}$ and after using Eq. (A.18) and the optimization condition (2.23), we obtain

$$\sum_{j=0}^{k} \sum_{m=\alpha_1}^{\alpha_2} a^j \tilde{\hat{\sigma}}_{n;m} F_{k+j-i-m+1}/(k+j-m+1) = 0,$$

$$\alpha_1 = \max(0, j+1-i), \quad \alpha_2 = \min(k, k+j+1-i) \tag{C.3}$$

again Eq. (C.3) is similar to Eq. (2.25). Finally, by imposing the relation (A.15) and applying the condition of optimization with respect to c_i, we end up with the following expression:

$$\sum_{l=0}^{2k+i-1} \sum_{p=\alpha}^{\beta} \sum_{j=\alpha'}^{\beta'} a^l [2\gamma^{NS}_{n;k+l+1-i-p} - (M+j) c_{k+l+1-i-p}]$$

$$\times \tilde{\hat{\sigma}}_{n;j} B^i_{p-j}/(i-1) - \sum_{l=0}^{k} \sum_{j=l+1}^{k} c_{k+l+1-j} \frac{\partial \hat{\sigma}_{n;j}}{\partial c_l}, \tag{C.4}$$

where α, β, α' and β' are given in Eq. (2.26').

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