# ON INSTANTON-INDUCED EFFECTIVE ACTION

By S. I. Kruglov

Byelorussian Institute of Agricultural Mechanization, Leninsky Prospect 99, Minsk, 220608, USSR

(*Received April 19, 1990*)

A four-quark model motivated by determinant 't Hooft interaction for the case of $SU(2)_f \otimes SU(2)_f$ symmetry is investigated. Using the method of functional integration, meson fields are introduced and the perturbation theory based on expansion in loops is considered. It is shown that a dynamic symmetry breaking takes place in the model; the role of Goldstone bosons is played by $\pi$-mesons. A constituent quark mass dependence on the quark condensate and a dimensional coupling constant (associated with the instanton density) dependence on the momentum-cutoff, whose inverse value gives the instanton size, have been obtained. At the momentum-cutoff of $\Lambda = 1$ GeV and current mass of quarks $m_0 = 5$ MeV experimental values of the $\pi$-mesons mass, quark condensate and dynamic quark mass have been reproduced. Completely effective action describing the interaction of mesons has been derived within the framework of the proposed model.

PACS numbers: 11.15.Tk

## 1. *Introduction*

One of the urgent problems of today is the investigation of nonperturbative effects in quantum chromodynamics (QCD). Among them are chiral symmetry breaking, confinement of quarks, etc. This area is difficult to study because of the impossibility of applying the perturbation theory using expansion in coupling constant $\alpha_s$. The understanding of these effects would provide the key to the calculation of the low-energy characteristics of hadrons-mass spectra, decay width, etc. This area of soft interactions of quarks and gluons is characterized by a complex structure of QCD vacuum and its nonperturbative fluctuations [1]. Long-wavelength fluctuations provide quark capture and are not considered in the present paper. This type of fluctuations can be taken into account, for example by using the bag model [2]. We shall consider only small-size fluctuations caused by instantons [3]. Thus we shall discuss the intermediate region between the asymptotic freedom and confinement of quarks. The model of vacuum as an instantonic liquid has been developed in [4, 9].

In [5] (see also [6]) it is shown that instantons generate a quark interaction of the form

$$\lambda \det \overline{\psi}_i(1+\gamma_5)\psi_j + \text{h.c.} \tag{1}$$

where $\psi_i$ is the ith-flavor quark field, h.c. is a hermitian conjugate, $\lambda$ is the constant which can be associated with the instanton density [7].

In [8–15] the role of interactions of the form of (1) for the low-energy physics of mesons was noted. In particular, it was shown [5, 8] that if we take into account the gluon field fluctuations contributing to (1), the U(1)-problem becomes solvable, spontaneous breaking of chiral invariance occurs [8–11]. All these points to the importance of further investigations of the results obtained by studying the interactions of (1).

Considering only the u, d-quarks in Eq. (1) and taking into account the free Lagrangian, we arrive at a Lagrangian with a four-quark interaction [13, 6]

$$\mathscr{L} = -\bar{\psi}(\gamma_\mu\partial_\mu+m_0)\psi+\frac{\lambda}{2}[(\bar{\psi}\psi)^2+(\bar{\psi}\gamma_5\psi)^2-(\bar{\psi}\tau^a\psi)^2-(\bar{\psi}\gamma_5\tau^a\psi)^2]. \tag{2}$$

Here $\tau^a$ are the Pauli matrices, $m_0 = \text{diag}\,(m_{01}, m_{02})$; $m_{01}, m_{02}$ are current quark masses. In (2) summation over color degrees of freedom of quarks $n = 1, 2, ..., N_c$ has been performed. We shall consider equal bare masses of quarks $m_{01} = m_{02} = m_0$. Then Lagrangian (2) is invariant under transformations of the group $SU(2)_f \otimes SU(2)_f$. Note that (2), even in the case of $m_0 = 0$, is not invariant under U (1)-chiral transformations, i.e. it brakes the U (1)-symmetry. In this respect it differs from the Lagrangian (2); it also differs from the previously considered chiral-symmetric Lagrangians of [17–19] proposed to describe mesons at low energies.

The aim of the present paper is to investigate the model based on postulated Lagrangian (2) motivated by the presence of instantons. Section 2 considers the perturbation theory as a mean-field approximation and the possibility of the appearance of condensates. In Section 3 the quadratic part of the effective action is calculated and the mass spectra of mesons are found. In Section 4 the Goldberger-Treiman relation is derived, the dependence of the quark dynamic mass on the momentum-cutoff is given and the value of the quark condensate is calculated. In Section 5 the complete effective action is calculated. In Conclusion the status of the model is discussed.

## 2. *Perturbation theory*

Let us consider the generating functional for the Green functions corresponding to the Lagrangian (2)

$$Z[\bar{\eta}, \eta] = N_0 \int \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp\,[i \int d^4x(\mathscr{L}+\bar{\psi}\eta+\bar{\eta}\psi)] \tag{3}$$

where $\eta$, $\bar{\eta}$ are external sources. Redefining the normalization factor $N_0$, multiplying by the constant

$$\int \mathscr{D}\phi_0\mathscr{D}\tilde{\phi}_0\mathscr{D}\phi_a\mathscr{D}\tilde{\phi}_a \exp\left\{-i\int d^4x\,\frac{\mu^2}{2}\left[\left(\phi_0-\frac{g_0\bar{\psi}\psi}{\mu^2}\right)^2\right.\right.$$

$$\left.\left.-\left(\tilde{\phi}_0+\frac{ig_0\bar{\psi}\gamma_5\psi}{\mu^2}\right)^2-\left(\phi_a+\frac{g_0\bar{\psi}\tau^a\psi}{\mu^2}\right)^2+\left(\tilde{\phi}_a-\frac{ig_0\bar{\psi}\gamma_5\tau^a\psi}{\mu}\right)^2\right]\right\},$$

we write (3) as

$$Z[\bar{\eta}, \eta] = N \int \mathscr{D}\bar{\psi}\mathscr{D}\psi\mathscr{D}\phi_0\mathscr{D}\tilde{\phi}_0\mathscr{D}\phi_a\mathscr{D}\tilde{\phi}_a \exp\left\{i\int d^4x\left[-\bar{\psi}(\gamma_\mu\partial_\mu + m_0 - g_0\phi_0 - ig_0\gamma_5\tilde{\phi}_0 - g_0\tau^a\phi_a - ig_0\gamma_5\tau^a\tilde{\phi}_a)\psi - \frac{\mu^2}{2}(\phi_0^2 - \tilde{\phi}_0^2 - \phi_a^2 + \tilde{\phi}_a^2) + \bar{\psi}\eta + \bar{\eta}\psi\right]\right\}. \quad (4)$$

Here $\lambda = g_0^2/\mu^2$, $g_0$ is a dimensionless coupling constant and a constant $\mu$ is mass-dimensional. Thus, meson fields representing coupled states of quark-antiquark pairs have been introduced into (4) (see, for example, [17–21]). The fields $\tilde{\phi}_a$ will be identified with the triplet of pseudoscalar $\pi_a$-mesons.

Calculations will use the momentum-cutoff $\Lambda$ which specifies the region of nonlocal interaction of quarks. This region is determined by the instanton size $\varrho = 1/\Lambda$ [5, 8] and is responsible for the quark pairing.

We can integrate Eq. (4) over the quark fields $\bar{\psi}$, $\psi$ and obtain

$$Z[\bar{\eta}, \eta] = N \int \mathscr{D}\phi_A \det(-\gamma_\mu\partial_\mu - m_0 + g_0\phi_A\Gamma_A) \times \exp\left\{i\int d^4x d^4y\left[-\frac{\mu^2}{2}\phi_A^2\varepsilon_A\delta(x-y) + \bar{\eta}(x)S_f(x, y)\eta(y)\right]\right\}, \quad (5)$$

where

$$\phi_A = (\phi_0, \tilde{\phi}_0, \phi_a, \tilde{\phi}_a), \quad \mathscr{D}\phi_A = \mathscr{D}\phi_0\mathscr{D}\tilde{\phi}_0\mathscr{D}\phi_a\mathscr{D}\tilde{\phi}_a,$$

$$\Gamma_A = (I_{(2)}, i\gamma_5, \tau^a, i\gamma_5\tau^a), \quad \varepsilon_A = (1, -1, -1, 1).$$

In Eq. (5) summation over repeated indexes is understood. The Green function of quarks in the external meson fields $S_f(x, y)$ obeys the equation

$$(\gamma_\mu\partial_\mu + m_0 - g_0\phi_A\Gamma_A)S_f(x, y) = \delta(x-y). \quad (6)$$

In modes with four-fermion scalar-scalar interaction, the symmetric vacuum is not stable [16–21]. More advantageous from the point of view of energy is the rearrangement of the physical vacuum and the appearance of the condensate, which leads to the dynamical breaking of the initial $SU(2)_f \otimes SU(2)_f$-symmetry. In order to take into account and to determine the condensate the fields should be "shifted" by the constants. Using admissible gauging for the group $SU(2)_f \otimes SU(2)_f$, it is enough to assume the following conditions

$$\langle\bar{\psi}\psi\rangle \neq 0, \quad \langle\bar{\psi}\gamma_5\psi\rangle \neq 0, \quad \langle\bar{\psi}\tau^3\psi\rangle \neq 0, \quad \langle\bar{\psi}\gamma_5\tau^3\psi\rangle \neq 0.$$

As a result, it is necessary to make the substitution in (5), (6)

$$\phi_0 = \phi_0' + \sigma_0, \quad \phi_3 = \phi_3' + \sigma_3, \quad \phi_i = \phi_i', \quad \tilde{\phi}_0 = \tilde{\phi}_0' + \tilde{\sigma}_0,$$
$$\tilde{\phi}_3 = \tilde{\phi}_3' + \tilde{\sigma}_3 \tilde{\phi}_i = \tilde{\phi}_i, \quad (7)$$

where $i = 1, 2$; $\sigma_0, \sigma_3, \tilde{\sigma}_0, \tilde{\sigma}_3$ are coordinate-independent constants. These constants determine the stationary point of the fields for the generating functional (5) and are $x$-independent by virtue of the Lorentz-invariance.

Below, the values of the constants $\sigma_0, \sigma_3, \tilde{\sigma}_0, \tilde{\sigma}_3$ will be determined from the minimum of the effective potential which determines the vacuum energy density.

To formulate the perturbation theory, let us use the saddle-point method. The fields $\phi'_A$ (7) represent quantum excitations over vacuum and are assumed small.

Using the equality $\det Q = \exp \operatorname{tr} \ln Q$, let us rewrite Eq. (5) taking into account (7)

$$Z[\bar{\eta}, \eta] = N \int \mathscr{D}\phi_A \exp \{i[S_{\rm eff} + \int d^4x d^4y \bar{\eta}(x) S_f(x, y) \eta(y)]\},$$

$$S_{\rm eff} = -\frac{\mu^2}{2} \int d^4x [(\phi'_0 + \sigma_0)^2 - (\tilde{\phi}'_0 + \tilde{\sigma}_0)^2 - \phi'^2_1 - \phi'^2_2 - (\phi'_3 + \sigma_3)^2$$

$$+ \tilde{\phi}'^2_1 + \tilde{\phi}'^2_2 + (\tilde{\phi}'^2_3 + \tilde{\sigma}_3)^2] - i \operatorname{tr} \ln(-\gamma_\mu \partial_\mu - m + i\tilde{m}\gamma_5 + g_0 \phi'_A \Gamma_A). \tag{8}$$

Here the following notation has been introduced $m = \operatorname{diag}(m_1, m_2)$, $m_1 = m_0 - g_0(\sigma_0 + \sigma_3)$, $m_2 = m_0 - g_0(\sigma_0 - \sigma_3)$, $\tilde{m} = \operatorname{diag}(\tilde{m}_1, \tilde{m}_2)$, $\tilde{m}_1 = g_0(\tilde{\sigma}_0 + \tilde{\sigma}_3)$, $\tilde{m}_2 = g_0(\tilde{\sigma}_0 - \tilde{\sigma}_3)$. Let us recall that we assume that the current masses of u, d quarks are equal, i.e., $m_{01} = m_{02} = m_0$. The operator, tr in (8) includes tracing in matrix and space-time variables.

Let us use the equality $\operatorname{tr} \ln(-\gamma_\mu \partial_\mu - m + i\tilde{m}\gamma_5 + g_0 \phi'_A \Gamma_A) = \operatorname{tr} \ln(-\gamma_\mu \partial_\mu - m + i\tilde{m}\gamma_5)$ $+ \operatorname{tr} \ln(1 - g_0 S_{0f}(x, y) \phi'_A \Gamma_A)$, where the "free" Green function of quarks $S_{0f}(x, y)$ satisfies the equation

$$(\gamma_\mu \partial_\mu + m - i\tilde{m}\gamma_5) S_{0f}(x, y) = \delta(x - y). \tag{9}$$

Taking this into account, in expansion of the logarithm in (8) in small fluctuations of $\phi'_A$, we obtain the expression for the effective action

$$S_{\rm eff} = -\frac{\mu^2}{2} \int d^4x \Bigg[ (\phi'_0 + \sigma_0)^2 - (\tilde{\phi}'_0 + \tilde{\sigma}_0)^2 - \phi'^2_1 - \phi'^2_2 - (\phi'_3 + \sigma_3)^2$$

$$+ \tilde{\phi}'^2_1 + \tilde{\phi}'^2_2 + (\tilde{\phi}'_3 + \tilde{\sigma}_3)^2 - i \operatorname{tr} \ln(-\gamma_\mu \partial_\mu - m + i\tilde{m}\gamma_5)$$

$$+ \sum_{n=1}^{\infty} \frac{i}{n} \operatorname{tr} (g_0 S_{0f} \phi'_A \Gamma_A)^n \Bigg], \tag{10}$$

where

$$\operatorname{tr} (g_0 S_{0f} \phi'_A \Gamma_A)^n = \operatorname{tr} [g_0^n \int d^4x_1 \ldots d^4x_n S_{0f}(x_n - x_1) \phi'_{A_1}(x_1) \Gamma_{A_1}$$

$$\times S_{0f}(x_1 - x_2) \phi'_{A_2}(x_2) \Gamma_{A_2} \ldots S_{0f}(x_{n-1} - x_n) \phi'_{A_n}(x_n) \Gamma_{A_n}]. \tag{11}$$

### 3. *Mass spectrum*

The values of the vacuum fields $\sigma_0, \sigma_3, \tilde{\sigma}_0, \tilde{\sigma}_3$ are obtained from the requirement of the absence of terms linear in fields $\phi'_A$ in the effective action (10). These terms correspond to the "tadpole"-type diagram given in Fig. 1. The stationary conditions for the action $S_{eff}$ are written as

$$\left.\frac{\delta S_{eff}}{\delta \phi'_A}\right|_{\phi_A'=0} = -\mu^2 \sigma_A \varepsilon_A + i\,\mathrm{tr}\,(g_0 \Gamma_A S_{0f}) = 0. \tag{12}$$

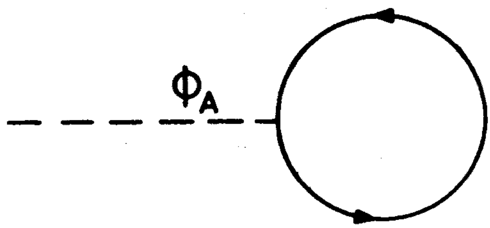


Fig. 1. The "tadpole"-type diagram participating in the condensate formation

To calculate the trace entering into (12), it is necessary to find the solution of equation (9). Passing into the momentum space, we find

$$S_{0f}(p) = \begin{pmatrix} \dfrac{-i\hat{p}+m_1+i\tilde{m}_1\gamma_5}{p^2+M_1^2} & 0 \\ 0 & \dfrac{-i\hat{p}+m_2+i\tilde{m}_2\gamma_5}{p^2+M_2^2} \end{pmatrix}. \tag{13}$$

Here $\hat{p} = p_\mu \gamma_\mu$, $M_1^2 = m_1^2 + \tilde{m}_1^2$, $M_2^2 = m_2^2 + \tilde{m}_2^2$. The poles of the Green functions (13) determine the constituent (dynamic) quark masses $M_1$, $M_2$. Thus, even at equal bare masses $m_0$ of u, d-quarks, as a result of vacuum reconstruction, the quarks acquire different masses $M_1 \neq M_2$. The components containing the matrix $\gamma_5$ in (13) violate CP-parity. It should be remembered that the $\theta$-term in the QCD Lagrangian caused by the presence of instantons gives a complicated topological vacuum structure and breakes PC-parity (see for example [22]). However, this breaking is assumed very small.

Substituting (13) into (12) and calculating the traces, we find the self-consistency condition.

$$\mu^2 g_0 \sigma_0 = m_1 I_1 + m_2 I_2, \quad \mu^2 g_0 \sigma_3 = -m_1 I_1 + m_2 I_2,$$

$$\mu^2 g_0 \tilde{\sigma}_0 = \tilde{m}_1 I_1 + \tilde{m}_2 I_2, \quad \mu^2 g_0 \tilde{\sigma}_3 = -\tilde{m}_1 I_1 + \tilde{m}_2 I_2,$$

$$I_j = \frac{i g_0^2 N_c}{4\pi^4} \int \frac{d^4 p\, \theta(p^2+\Lambda^2)}{p^2+M_j^2} \quad (j = 1, 2). \tag{14}$$

It is convenient to rewrite (14) as

$$\mu^2(m_0-m_1) = 2m_2I_2, \quad \mu^2(m_0-m_2) = 2m_1I_1, \tag{15a}$$

$$\mu^2\tilde{m}_1 = 2\tilde{m}_2I_2, \quad \mu^2\tilde{m}_2 = 2\tilde{m}_1I_1. \tag{15b}$$

These equations are gap equations [16]. Note that with cutoff regularization used here, equations (15a) have solutions: $m_1 > m_0$, $m_2 > m_0$ and equations (15b) have solutions provided that the constants $\tilde{m}_1$ and $\tilde{m}_2$ have different signs.

Let us now show that equations (15) follow from the condition of the effective potential minimum. Combining the constant terms in (10), we write

$$S_{\rm eff}^{\rm const} = -\frac{\mu^2}{2}\int d^4x(\sigma_0^2-\tilde{\sigma}_0^2-\sigma_3^2-\tilde{\sigma}_3^2)-i\,{\rm tr}\ln(-\gamma_\mu\partial_\mu-m+i\tilde{m}\gamma_5) \tag{16}$$

Taking into account that for the constant fields there is a relationship [23] $S_{\rm eff}^{\rm const} = -\int d^4xV_{\rm eff}$, we find from (16) the effective potential

$$V_{\rm eff} = \frac{\mu^2}{2}[(m_1-m_0)(m_2-m_0)-\tilde{m}_1\tilde{m}_2]+\frac{ig_0^2N_c}{8\pi^4}\int d^4p\ln(p^2+M_1^2)(p^2+M_2^2). \tag{17}$$

Equation (15) is obtained from the condition of the potential minimum (17):

$$\frac{\partial V_{\rm eff}}{\partial m_1} = \frac{\partial V_{\rm eff}}{\partial m_2} = \frac{\partial V_{\rm eff}}{\partial \tilde{m}_1} = \frac{\partial V_{\rm eff}}{\partial \tilde{m}_2} = 0.$$

Here we shall be interested in the case where there is no dynamic breaking of the CP-parity, i.e. where $\tilde{m}_1 = \tilde{m}_2 = 0$ ($\tilde{\sigma}_0 = \tilde{\sigma}_3 = 0$). Besides, we shall neglect the mass splitting of the u,d-quarks. For this purpose we assume $m_1 = m_2 = m$ or, which is the same, $\sigma_3 = 0$, As a result of the above limitation, only one of the two equations (15) "survives", which, after the calculation of the integral $I_1 = I_2 = I$, takes the form

$$\frac{\mu^2(m_0-m)}{m} = \frac{g_0^2N_c}{2\pi^2}\left[m^2\ln\left(\frac{\Lambda^2}{m^2}+1\right)-\Lambda^2\right]. \tag{18}$$

Equation (18) has nontrivial solutions non-analytical with respect to the constant $\lambda = g_0^2/\mu^2$ provided that $\alpha \equiv 2\pi^2/\lambda\Lambda^2N_c < 1$ [16]. From (18) the dynamic mass of the quarks $m$ is determined when the values of the constant $\lambda$ and of the momentum-cutoff $\Lambda$ are specified (provided that $\alpha < 1$).

Expansion of (10) in the number $n > 1$ of external legs is represented by graphs in Fig. 2. To obtain the mass spectrum of mesons, it is necessary to calculate the field $\phi'_A$-quadratic term in (10) which determines the propagation of mesons. From (10) we find

$$S_{\rm eff}^{(2)} = -\frac{\mu^2}{2}\int d^4x\phi_A'^2\varepsilon_A+\frac{i}{2}{\rm tr}(g_0S_{0f}\phi'_A\Gamma_A)^2$$

$$\equiv -\tfrac{1}{2}\int d^4xd^4y\phi'_A(x)\Delta_{AB}^{-1}(x,y)\phi'_B(y). \tag{19}$$

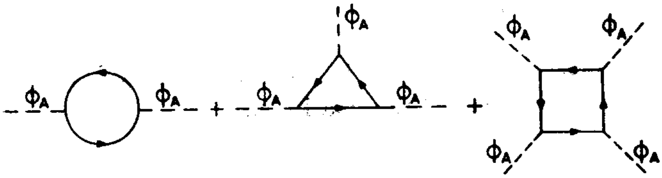


Fig. 2. Quark loops contributing to the complete effective action

The inverse propagator $\Delta_{AB}^{-1}$ can be written in momentum space as

$$\Delta_{AB}^{-1}(p) = \mu^2 \varepsilon_{AB}\delta_{AB} - ig_0^2 \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} S_{0f}(k)\Gamma_A S_{0f}(k-p)\Gamma_B. \tag{20}$$

Calculating (20) and taking into account (13), (18) (at $\tilde{m}_1 = \tilde{m}_2 = 0$, $m_1 = m_2 = m$) we find

$$\Delta_{00}^{-1}(p) = -\frac{2m_0 I}{m-m_0} + (p^2+4m^2)(Z_3^{-1} - J(g_0^2)),$$

$$\Delta_{\bar{0}\bar{0}}^{-1}(p) = 2I\frac{2m-m_0}{m-m_0} + p^2(Z_3^{-1} - J(g_0^2)),$$

$$\Delta_{11}^{-1}(p) = \Delta_{22}^{-1}(p) = \Delta_{33}^{-1}(p) = 2I\frac{2m-m_0}{m-m_0} + (p^2+4m^2)(Z_3^{-1} - J(g_0^2)),$$

$$\Delta_{\bar{1}\bar{1}}^{-1}(p) = \Delta_{\bar{2}\bar{2}}^{-1}(p) = \Delta_{\bar{3}\bar{3}}^{-1}(p) = -\frac{2m_0 I}{m-m_0} + p^2(Z_3^{-1} - J(g_0^2)), \tag{21}$$

where $J(g_0^2) = \frac{g_0^2 N_c}{4\pi^2}\int_0^1 dx \ln\left[1+\frac{p^2}{m^2}x(1-x)\right]$, $Z_3^{-1}$ is

$$Z_3^{-1} = -\frac{ig_0^2 N_c}{4\pi^4}\int \frac{d^4p\,\theta(p^2+\Lambda^2)}{(p^2+m^2)^2} = \frac{dI}{dm^2} = \frac{g_0^2 N_c}{4\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}+1\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right]. \tag{22}$$

Let us redefine the fields $\phi'_A Z_3^{-1/2} = \phi_A$ and the constant

$$g^2 = g_0^2 Z_3 = \frac{4\pi^2}{N_c}\left[\ln\left(\frac{\Lambda^2}{m^2}+1\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right]^{-1}.$$

It follows from this formula that the expansion in $g^2/4\pi^2$ in (10) corresponds to the $1/N_c$ expansion provided that $[\ln(\Lambda^2/m^2+1) - \Lambda^2/(\Lambda^2+m^2)]^{-1} < 1$. For the values of the parameters $\Lambda$, $m$ used below this condition will be fulfilled. It follows from (21) that the $\pi$-mesons are Goldstone bosons, since, if the bare mass of the quarks $m_0 = 0$, their masses tend to zero (see also [17, 13]). In this case, the scalar meson mass $m_\sigma = 2m$ is in agreement with [16, 17].

To calculate the masses from (21), it is necessary to specify the momentum-cutoff and calculate the consistent mass of the quarks $m$.

### 4. Determination of $\Lambda$, $m$, $\langle\bar{\psi}\psi\rangle$

To fix $\Lambda$ and calculate $m$ we find the relation of these quantities to the constant $F_\pi$ [17]. Let us construct an axial current corresponding to the $SU_A(2)$ transformations of the Lagrangian (2)

$$\psi' = \exp\left(i\varphi\gamma_5\frac{\vec{n}\vec{\tau}}{2}\right)\psi, \quad \bar{\psi}' = \bar{\psi}\exp\left(i\varphi\gamma_5\frac{\vec{n}\vec{\tau}}{2}\right), \tag{22}$$

where $\vec{n}$ is the unit vector, i.e., $\vec{n}^2 = 1$.

Transformations of the quark fields (22) generate infinitely small transformations of the meson fields

$$\phi_0+\sigma_0 \to \phi_0+\sigma_0+\varphi n_a\tilde{\phi}_a, \quad \tilde{\phi}_0 \to \tilde{\phi}_0-\varphi n_a\phi_a,$$
$$\tilde{\phi}_a \to \tilde{\phi}_a-\varphi n_a(\phi_0+\sigma_0), \quad \phi_a \to \phi_a+\varphi n_a\tilde{\phi}_0. \tag{23}$$

Let us take into account that by renormalization of the fields $\sigma_0 = (m_0-m)/g_0Z_3^{1/2} = (m_0-m)/g$. Now, using the Gell-Mann-Levy method [24], we find the expression for the axial current

$$A_\mu = \frac{\partial\delta L_{\text{eff}}}{\partial\partial_\mu\varphi(x)} = n_a\left(\phi_a\partial_\mu\tilde{\phi}_0-\tilde{\phi}_0\partial_\mu\phi_a\right.$$
$$\left.+\phi_0\partial_\mu\tilde{\phi}_a-\tilde{\phi}_a\partial_\mu\phi_0+\frac{(m_0-m)}{g}\cdot\partial_\mu\tilde{\phi}_a\right). \tag{24}$$

If we use (24) to describe the decay $\pi^\pm \to \mu^\pm\nu$ (see [17]), it is necessary to identify (see also [13, 17, 25])

$$\frac{m-m_0}{g} = F_\pi, \tag{25}$$

where the pion decay constant $F_\pi = 93$ MeV. Relation (25) is the Goldberger-Treiman identity. Substituting into (25) the value of $g$, we arrive at a transcendent equation relating $F_\pi$ and the momentum-cutoff $\Lambda$:

$$\left(\frac{F_\pi}{m-m_0}\right)^2 = \frac{N_c}{4\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2}+1\right)-\frac{\Lambda^2}{\Lambda^2+m^2}\right]. \tag{26}$$

Fig. 3 shows the graphical dependence of the quark dynamic mass $m$ on the momentum--cutoff at $N_c = 3$, $m_0 = 0$ (see also [13]). Choosing the value of $\Lambda = 1$ GeV, we obtain the value $m = 241$ MeV. From (18) the value of the constant $\lambda$ can be found from

$$\lambda = \left[\frac{N_c \Lambda^4}{2\pi^2(\Lambda^2+m^2)} - \frac{2F_\pi^2 m^2}{(m-m_0)^2}\right]^{-1}. \tag{27}$$

The value of the constant $\lambda$ vs the momentum-cutoff is graphically represented in Fig. 4. At $\Lambda = 1$ GeV we obtain the value $\lambda = 7.9\ (\text{GeV})^{-2}$. Finally, we calculate the value of the quark condensate. We have

$$\langle\bar{\psi}\psi\rangle = \langle\bar{u}u\rangle + \langle\bar{d}d\rangle = i \operatorname{tr} S_{0f}(x, x)$$

$$= \frac{iN_c m}{2\pi^4}\int \frac{d^4p\,\theta(p^2+\Lambda^2)}{p^2+m^2} = -\frac{N_c m}{2\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}+1\right)\right]. \tag{28}$$

Fig. 5 shows the condensate vs the momentum-cutoff. Taking the value of $\Lambda = 1$ GeV, we obtain a reasonable value

$$\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = (-248\ \text{MeV})^3. \tag{29}$$

It should be recalled that phenomenology yields the value $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = -(240$–$250\ \text{MeV})^3$.

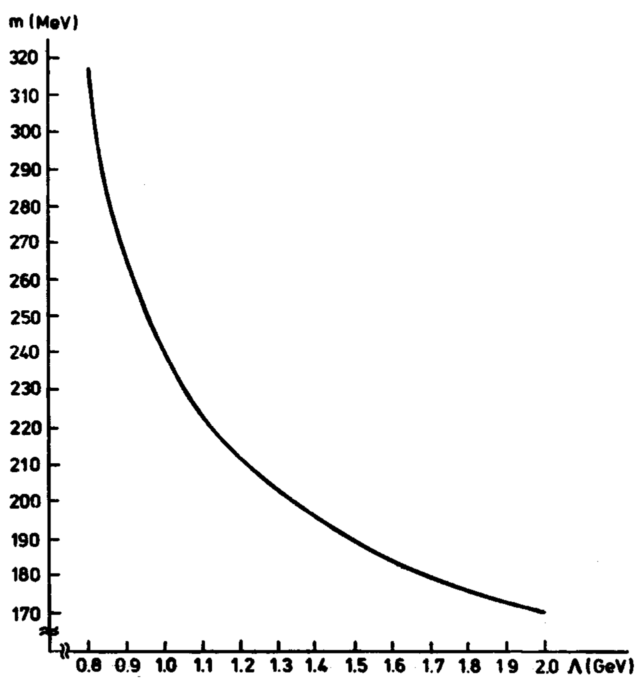


Fig. 3. The constituent (dynamic) mass of quarks vs the momentum-cutoff at $m_0 = 0$

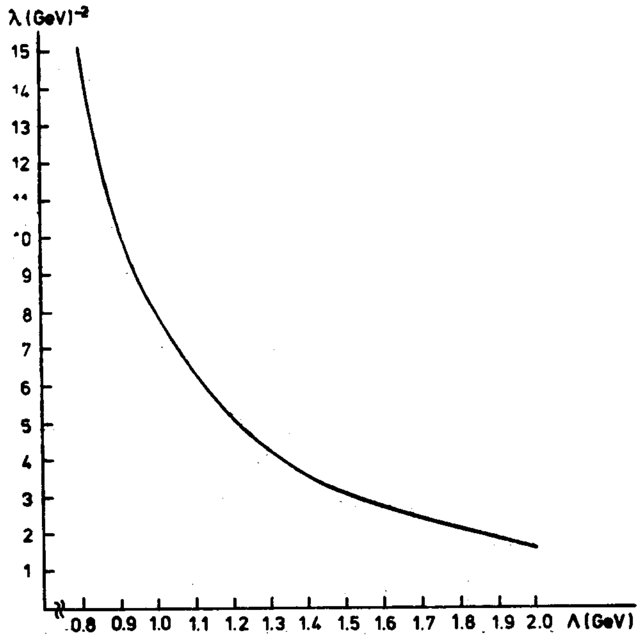


Fig. 4. The plot of the dependence of the coupling constant entering into the original Lagrangian on the momentum-cutoff ($m_0 = 0$)

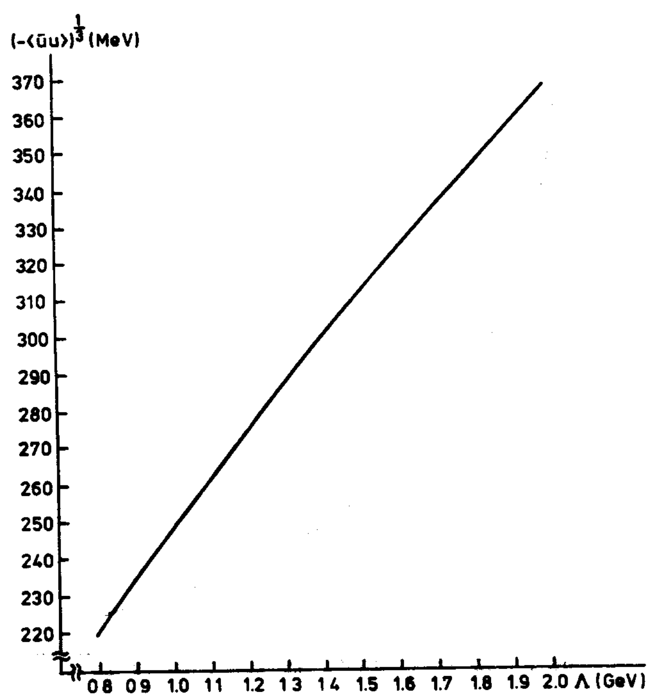


Fig. 5. Quark condensate vs the momentum-cutoff

We can easily see from (21), (25) and (28) that the following relation takes place (see also [13]):

$$F_\pi^2 m_\pi^2 \approx -m_0\langle\bar{\psi}\psi\rangle, \tag{30}$$

which can also be obtained in terms of current algebra [26, 27]. We performed a numerical calculation without approximations given by the equations

$$\Delta^{-1}_{\tilde{a}\tilde{a}}(p) = 0, \quad \Delta^{-1}_{00}(p) = 0 \tag{31}$$

at $\Lambda = 1$ GeV, $m = 0.24$ GeV, $m_0 = 5$ MeV.

We obtain

$$m_\pi = 140\text{ MeV}, \quad m_\sigma = 500\text{ MeV}. \tag{32}$$

The computation has shown that there are no solutions of equations

$$\Delta^{-1}_{\tilde{0}\tilde{0}}(p) = 0, \quad \Delta^{-1}_{aa}(p) = 0 \tag{33}$$

neither at $p^2 > 0$ nor at $p^2 < 0$.

Thus the propagators $\Delta_{\tilde{0}\tilde{0}}$, $\Delta_{aa}$ do not give particle propagation.

## 5. *Complete effective action*

Let us calculate the components in the sum (10) which are graphically represented in Fig. 2. Note that after the fields redistribution the expansion parameter of (10) is the quantity $g^2/4\pi^2 \approx 1/6$ (at $\Lambda = 1$ GeV). Therefore, the expansion in constant $g^2/4\pi^2$ of (10) is justified. Let us count the components of (10) whith $n = 3$ and $n = 4$. At $n > 4$ the quark loops will give expressions convergent at $\Lambda \to \infty$ which are infinitely small. For three- and four-point functions we have the expressions

$$\Gamma_{ABC}(k_1, k_2) = i\,\mathrm{tr}\left\{\int\frac{d^4p}{(2\pi)^4}\,[S_{0f}(p+k_1-k_2)\Gamma_A S_{0f}(p)\Gamma_B S_{0f}(p+k_1) + S_{0f}(p-k_1)\Gamma_B S_{0f}(p)S_{0f}(p+k_2-k_1)]\Gamma_C\right\}, \tag{34}$$

$$\begin{aligned}\Gamma_{ABCD}(k_1, k_2, k_3) = i\,\mathrm{tr}\Big\{\int\frac{d^4p}{(2\pi)^4}\,\{&S_{0f}(p)\Gamma_A S_{0f}(p+k_2)\,[\Gamma_C S_{0f}(p+k_2-k_3)\Gamma_B \\ &\times S_{0f}(p-k_1)\Gamma_A + \Gamma_C S_{0f}(p+k_2-k_3)\Gamma_A S_{0f}(p+k_1+k_2-k_3)\Gamma_B \\ &+\Gamma_B S_{0f}(p-k_1+k_3)\Gamma_C S_{0f}(p-k_1)\Gamma_A + \Gamma_A S_{0f}(p+k_1+k_2)\Gamma_C S_{0f}(p+k_1+k_2-k_3)\Gamma_B \\ &+\Gamma_A S_{0f}(p+k_1+k_2)\Gamma_B S_{0f}(p+k_3)\Gamma_C + \Gamma_B S_{0f}(p+k_3-k_1)\Gamma_A S_{0f}(p+k_3)\Gamma_C]\}\Big\}.\end{aligned} \tag{35}$$

Substituting (13) into (34), (35) at $\tilde{m}_1 = \tilde{m}_2 = 0, m_1 = m_2 = m$ and calculating the integral to an accuracy of $O(g^2)$, we find

$$S_{\text{eff}}^{\text{int}} = S_{\text{eff}}^{(3)} + S_{\text{eff}}^{(4)} = \int d^4x \Big\{ 2gm[\phi_0^3 + 3\phi_0\phi_a^2 + \phi_0(\tilde{\phi}_0^2 + \tilde{\phi}_a^2)$$

$$+ 2\tilde{\phi}_0(\phi_a\tilde{\phi}_a)] - \frac{g^2}{2}[\phi_0^4 + (\phi_a^2)^2 + 6\phi_0^2\phi_a^2 + \tilde{\phi}_0^4 + (\tilde{\phi}_a^2)^2 + 6\tilde{\phi}_0^2\tilde{\phi}_a^2$$

$$+ 6(\phi_0^2\tilde{\phi}_0^2 + \tilde{\phi}_0^2\phi_a^2 + \phi_0^2\tilde{\phi}_a^2) + 24\phi_0\tilde{\phi}_0(\phi_a\tilde{\phi}_a) + 2\phi_a^2\tilde{\phi}_b^2 + 4(\phi_a\tilde{\phi}_a)^2] \Big\}. \tag{36}$$

The effective action of the interacting meson fields of (36) can be rewritten in a more compact form by introducing the matrices

$$\phi = \phi_0 I_{(2)} + \phi_a\tau^a, \quad \tilde{\phi} = \tilde{\phi}_0 I_{(2)} + \tilde{\phi}_a\tau^a, \tag{37}$$

where $I_{(2)}$ is a unit $2\times2$ matrix. Taking into account the relations

$$\text{tr}\,\phi^3 = 2(\phi_0^3 + 3\phi_0\phi_a^2),$$

$$\text{tr}\,\phi\tilde{\phi}^2 = 2[\phi_0(\tilde{\phi}_0^2 + \tilde{\phi}_a^2) + 2\tilde{\phi}_0\phi_a\tilde{\phi}_a],$$

$$\text{tr}\,\phi^4 = 2[\phi_0^4 + (\phi_a^2)^2 + 6\phi_0^2\phi_a^2],$$

$$\text{tr}\,\phi^2{}_r{}^2 = 2(\phi_0^2 + \phi_a^2)(\tilde{\phi}_0^2 + \tilde{\phi}_a^2) + 8\phi_0\tilde{\phi}_0(\phi_a\tilde{\phi}_a),$$

$$\text{tr}\,\phi\ \ \phi\tilde{\phi} = 2\phi_0^2(\phi_a^2 + \tilde{\phi}_0^2) + 2\tilde{\phi}_0^2\phi_a^2 + 8\phi_0\tilde{\phi}_0(\phi_a\tilde{\phi}_a) + 4(\phi_a\tilde{\phi}_a)^2 - 2\phi_a^2\tilde{\phi}_b^2$$

formula (36) will be written as

$$S_{\text{eff}}^{\text{int}} = \int d^4x \Big\{ gm\,\text{tr}\,[\phi(\phi^2 + \tilde{\phi}^2)] - \frac{g^2}{4}\text{tr}\,[\phi^4 + \tilde{\phi}^4 + 4\phi^2\tilde{\phi}^2 + 2(\phi\tilde{\phi})^2] \Big\}. \tag{38}$$

Expressions (36), (38) can be used to find the decay widths and cross-sections of mesons scattering. Formula (38) is obviously invariant under transformations of the group $U(2)_f \otimes U(2)_f$.

## 6. *Discussion*

We have considered a model based on the determinant 't Hooft interaction in the case of $SU(2)_f \otimes SU(2)_f$ symmetry. It describes well the $\pi$-mesons. For Lagrangian (2) the complete effective action has been found. For correct quantitative estimation of the mass spectra of scalar and pseudoscalar mesons, we should, perhaps, not only generalize the model to the case of $SU(3)_f \otimes SU(3)_f$-symmetry (see [13–15]), but also take into account the contribution of the gluon, four-quark states [28] and, possibly, of the longwave fluctuations associated with the confinement.

The autor would like to thank M. I. Levchuk for help in the calculations. I have much benefited from discussions with D. I. Diakonov and V. Yu. Petrov.

**Editorial note.** This article was proofread by the editors only, not by the authors.

## REFERENCES

[1] V. A. Novikov et al., *Nucl. Phys.* **B165**, 67 (1980).
[2] P. N. Bogolubov, *Ann. Inst. Henri Poincare* **8**, 163 (1967).
[3] A. A. Belavin et al., *Phys. Let.* **B59**, 85 (1975).
[4] E. V. Shuryak, *Phys. Lett.* **B107**, 103 (1981); *Nucl. Phys.* **B203**, 93 (1982); **B214**, 237 (1983); *Phys. Rep.* **115**, 151 (1985).
[5] G. 't Hooft, *Phys. Lett.* **37**, 8 (1976); *Phys. Rev.* **D14**, 3432 (1976).
[6] D. I. Dyakonov, V. Yu. Petrov, In: *Hadron matter under extreme conditions*, Kiev 1986, part I, p. 192, in Russian.
[7] M. A. Shifman et al., *Nucl. Phys.* **B147**, 385 (1979); **B163**, 46 (1980).
[8] G. G. Callan, R. Dashen, D. J. Gross, *Phys. Rev.* **D16**, 2526 (1977); **D17**, 2717 (1978); **D19**, 1826; **D20**, 3279 (1979).
[9] D. I. Dyakonov, V. Yu. Petrov, *Zh. Eksp. Teor. Fiz.* **86**, 25 (1984); **89**, 361 (1985); *Nucl. Phys.* **B245**, 259 (1984); **B272**, 457 (1986).
[10] D. G. Caldi, *Phys. Rev. Lett.* **39**, 121 (1977).
[11] R. D. Garlitz, D. B. Creamer, *Ann. Phys.* (*USA*) **118**, 429 (1979).
[12] V. V. Anisovich, S. M. Gerasyuta, I. V. Keltuyala, *Yad. Fiz.* **46**, 1508 (1987).
[13] T. Hatsuda, T. Kunihiro, *Prog. Theor. Phys.* **74**, 765 (1985); T. Kunihiro, *Prog. Theor. Phys.* **80**, 34 (1988).
[14] V. Bernard, R. L. Jaffe, V. G. Meissner, *Nucl. Phys.* **B308**, 753 (1988); M. Takizawa et al., *Prog. Theor. Phys.* **82**, 481 (1989).
[15] H. Reinhardt, R. Alkofer, *Phys. Lett.* **B207**, 482 (1988); *Z. Phys. C* (*Particles and Fields*) **45**, 275 (1989).
[16] Y. Nambu, G. Jona-Lasinio *Phys. Rev.* **122**, 345 (1961); V. G. Vaks, A. I. Larkin, *Zh. Eksp. Teor. Fiz.* **40**, 282 (1961).
[17] M. K. Volkov, D. Ebert, *Yad. Fiz.* **36**, 1265 (1982); D. Ebert, M. K. Volkov, *Z. Phys. C* (*Particles and Fields*) **16**, 205 (1983); M. K. Volkov, *Elem. Chast. Atom. Yad.* **17**, 433 (1986); *Ann. Phys.* (*USA*) **157**, 282 (1984).
[18] A. Dhar, R. Shankar, S. R. Wadia, *Phys. Rev.* **D31**, 3256 (1985).
[19] D. Ebert, H. Reinhardt, *Nucl. Phys.* **B271**, (1986).
[20] V. P. Pervushin, H. Reinhardt, D. Ebert, *Elem. Chast. Atom. Yad.* **10**, 1114 (1979).
[21] S. I. Kruglov, *Izv. Vuz. Fizika* **11**, 71 (1982); **6**, 5 (1989); *Acta Phys. Pol.* **B15**, 725 (1984); **B20**, 729 (1989).
[22] K. Huang, *Quarks, leptons and gauge fields*, World Scientific, 1982.
[23] S. Coleman, E. Weinberg, *Phys. Rev.* **D7**, 1888 (1973).
[24] V. De Alfaro, S. Fubini, G. Furlan, S. Rossetti, *Currents in Hadron Physics*, New York 1973.
[25] V. Bernard, *Phys. Rev.* **D34**, 1601 (1986).
[26] J. Gasser, H. Leutwyler, *Phys. Rep.* **87**, 74 (1982).
[27] L. J. Reinders, H. R. Rubinstein, S. Yazaki, *Phys. Rep.* **127**, 1 (1985).
[28] N. N. Achasov, S. A. Devyanin, G. N. Shestakov, *Yad. Fiz.* **33**, 1337 (1981).