COMPUTER SIMULATIONS OF THE BOSONIC STRING*

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New extensive computer simulations of the bosonic string carried out in collaboration with D. Boulatov and V. Kazakov are presented.

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1. Introduction

The theory of triangulated random surfaces was suggested as a regularized theory of the bosonic string some years ago [1, 2, 3]. It is somewhat remarkable that the regularized theory can be solved analytically in a number of (unphysical) dimensions, namely $d = -2, 0, 1$. The critical exponent γ for the susceptibility can be calculated in these dimensions. Further critical exponents can be calculated for $d = 0$ by coupling the $2D$--gravity part of the string to spin fields [4].

A problem with the suggested regularization of the string and $2D$-gravity is that it has no naive weak coupling limit where it agrees with the formally defined continuum theory. In this respect the theory is different from e.g. lattice gauge theory, which has a weak coupling limit where it can be identified with the continuum theory. However, recently Knizhnik et al. have been able by analytical means to solve the minimal conformal theories coupled to $2D$-gravity. Agreement with the former calculations in the triangulated random surface model was observed. This gives confidence to both classes of calculations, and in particular it showed that the triangulated randoms urface model provided a good regularization of the bosonic string and $2D$-gravity.

An interesting problem with the calculation of Knizhnik et al. is the breakdown when the central charge c of the conformal theory is larger than one. In the application of the theory to bosonic strings the central charge is the dimension d of space-time in which the string is embedded. The formula for the susceptibility is

$$\gamma = \frac{d-1-\sqrt{(d-1)(d-25)}}{12} \tag{1}$$

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and it is seen that the susceptibility becomes complex when d is larger than one. In fact this is nothing but a paraphasing of the old result of Otto and Weigt [5] that the Liouville theory cannot be quantized in a way consistent with reparametrization invariance if $d > 1$.

Contrary to this result the randomly triangulated model is well defined in any dimension. It is therefore interesting to investigate what happens in the neighbourhood of $d = 1$. Is there a transition to a new phase where the metric of the worldsheet is singular (branched polymers [1, 7, 8, 9, 10]), or do we have a new theory which by some reason is inaccessible by the methods of conformal field theory. Unfortunately no analytical methods are presently available for solving the randomly triangulated models for $d > 1$. One has to rely on numerical Monte Carlo methods for simulating the surfaces. These numerical methods have been extensively used in the past with somewhat limited success [8, 9, 11, 12, 13, 14]. It is the purpose of the present article to describe the results of new extensive numerical simulations for low dimensions $d = 1 \div 6$. A new algorithm has been developed. It combines the advantages of the canonical and grand canonical algorithms used in the past and even allow for a reasonable vectorization. It is the purpose of this article to describe this algorithm and the results obtained by the Monte Carlo simulation.

2. *The model*

According to the prescription of Polyakov the loop Green function of string theory is given by

$$G(\gamma_1, \ldots, \gamma_n) = \int \mathscr{D} g_{ab} \int\limits_{\gamma_1 \cup \ldots \cup \gamma_n} \mathscr{D} x \exp(-\beta \int d^2\xi \sqrt{g}\, \partial_\alpha x \partial^\alpha x). \tag{2}$$

The integration in (2) is over all connected surfaces with $\gamma_1, \gamma_2 \ldots \gamma_n$ as fixed boundary loops. Further one has to integrate over all metrics. Interactions are introduced in a natural way by including surfaces of different topology. Here we will restrict ourself to surfaces with spherical topology.

In the triangulated random surfaces model the formal expression (2) is given a precise meaning by defining the analog of the Green loop function (2)

$$G_\beta(\gamma_1, \ldots, \gamma_n) = \sum_{T \in \mathscr{T}} \varrho(T) \int \prod_{i \in T/\partial T} dx_i \exp(-\beta \sum_{\langle ij \rangle} (x_i - x_j)^2). \tag{3}$$

Here T denotes a triangulation. The boundary loops $\gamma_1, \ldots, \gamma_n$ are approximated by polygonal loops which constitute the boundary ∂T of T. The integration over metrics is replaced by summation over triangulations $T \in \mathscr{T}$, where $\mathscr{T}$ is a suitable class of triangulations with trivial topology. If the integral (2) is extended to include higher genus surfaces, the same can be done in (3) by simply enlarging the class of triangulations $\mathscr{T}$. Finally, the weight $\varrho(T)$ should be chosen appropriately (see [1] for a discussion). Here we will simply take $\varrho(T) = 1$ as this simple and natural choice has shown to be sufficient to produce results identical to the ones derived by Knizhnik et al.

The idea behind the regularized version of the loop Green functions given by (3) has been discussed in detail in the original articles [1, 2, 3]. Let us only mention here that it is

supposed to represent an *explicit* invariant sum over all relevant surfaces. No reparametrization invariance is to be taken care of by gauge fixing. In this way it is a kind of Regge calculus. Relevant physical excitations like the Liouville mode is automatically included, while non-physical excitations should disappear when we approach the critical point β_0 in the right way. This critical point is the value of the "bare" coupling constant β where (3) diverges.

Let us define the observables that will be of interest for us in the scaling limit $\beta \to \beta_0$. The mass gap $m(\beta)$ is defined by the exponential fall off of the two-point function (the two-loop Green function where the loops are contracted to points):

$$G_\beta(x, y) \sim e^{-m(\beta)|x-y|} \quad \text{for} \quad |x-y| \to \infty \tag{4}$$

and the critical exponent ν for the mass gap is defined by

$$m(\beta) \sim (\beta-\beta_c)^\nu \quad \text{for} \quad \beta \to \beta_c. \tag{5}$$

General scaling arguments show that ν is related to the Haussdorf dimension of the surface d_H by

$$d_H = 1/\nu. \tag{6}$$

Finally the susceptibility χ is defined by integrating over the two-point function

$$\chi(\beta) = \int dx G_\beta(x, y) \tag{7}$$

and its critical exponent γ is determined by

$$\chi(\beta) \sim (\beta-\beta_c)^{-\gamma} \quad \text{for} \quad \beta \to \beta_c. \tag{8}$$

Since we get the integrated two-point function by differentiating the one-point function with respect to β we have the following behaviour of the one-point function $G_\beta(x_0)$ near β_c:

$$G_\beta(x_0) \sim \sum (\beta_c/\beta)^{dN/2} N^{\gamma-2} \sim (\beta-\beta_c)^{-(\gamma-1)}. \tag{9}$$

It can be proven that $\gamma \leqslant 1/2$ for any reasonable choice of $\varrho(T)$ [1, 7].

The only additional thing we need to recall is the mapping of the model on an equivalent model where the summation is over ϕ^3-graphs rather than over triangulations. This mapping is established by noting that the dual graph to a triangulation is a ϕ^3-graph and further that topology is preserved in this mapping. This means especially that the planar triangulations are mapped into the planar ϕ^3-graphs which is all we need.

The one-point function $G_\beta(x_{i_0})$ has by translational invariance no dependence on the distinguished point x_{i_0}. It is explicitly given by

$$G_\beta(x_{i_0}) = \sum_{T \in \mathcal{T}} \int \prod_{i \in T/\{i_0\}} dx_i \exp\left(-\beta \sum_{\langle ij \rangle} (x_i - x_j)^2\right). \tag{10}$$

The transformation to the dual representation leads to an equivalent representation

$$G_{\tilde{\beta}}(k_{t_0}) = \sum_{\phi^3 \in \mathcal{F}} \int \prod_{t \in \phi^3/\{t_0\}} dk_t \exp\left(-\tilde{\beta} \sum_{\langle tt' \rangle} (k_t - k_{t'})^2\right), \tag{11}$$

where ϕ^3 denotes a planar ϕ^3-graph, $\mathscr{F}$ the set of all planar ϕ^3-graphs and t, t' points in the given ϕ^3-graph (triangles in the original triangulation). Finally the summation in the exponent is between neighbour points in the ϕ^3-graph. By performing the k integrals explicitly we get an asymptotic expansion for a large number of vertices N in the ϕ^3-graphs:

$$G_{\tilde{\beta}} \sim \sum_N N^{\gamma-2} \exp(\tilde{\beta}_c - \tilde{\beta})N. \tag{12}$$

The "partition function" (11) is the one we want to simulate in order to determine the critical exponents γ and ν. The reason for choosing the dual ϕ^3 graphs rather than the triangulations is just convenience in terms of computer programs plus the fact that the simulations untill now have used triangulations. In this way one would get a more independent determination of the critical exponents.

3. The algorithm

The main problem is to determine the "partition function"

$$G_{\tilde{\beta}} \sim \sum_N Z_N \exp(-\tilde{\beta}N), \tag{13}$$

where the entropy factor Z_N according to (12) is given by

$$Z_N \sim N^{\gamma-2} \exp(\tilde{\beta}_c N). \tag{14}$$

Usually the determination of the partition function is not the best question to address by Monte Carlo, although it is made somewhat easier in this case by the discreteness of N. In order to extract γ we have to find an algorithm which preserves detail balance and which induce transitions between states of N and N-2 vertices. This has been done [7, 8, 12]. This kind of "grand canonical" updatings has certain disadvantages, like fine-tuning problems of the chemical potential $\tilde{\beta}_c$. In order to circumvent this and in order to get a completely independent simulation a new algorithm has been suggested [15]. This algorithm is in spirit the same as has already been used by Bhanot et al. in lattice gauge theories to determine the partition function [16], although here it has of course been redesigned to the particular problem which we address.

Let us denote the set of states consisting of N vertices by S_N. The individual states are denoted by $x, y, \ldots$ They depend on the positions of the N vertices and on the ϕ^3-graph. Their statistical weight is denoted $W_x^{(N)}$ and can be read off from (11). Suppose we have some operations which allow transitions from $S_N \to S_{N-2}$ and vice versa with transition probabilities $V_-(y \to x)$ and $V_+(x \to y)$ chosen in such a way that detailed balance is fulfilled:

$$W_x^{(N-2)} V_+(x \to y) = W_y^{(N)} V_-(y \to x). \tag{15}$$

By definition

$$\sum_{y \in S_N} W_y^{(N)} = Z_N \tag{16}$$

and it follows from (15) and (16) (see [15] for details) that

$$\frac{Z_{N-2}}{Z_N} = \left\langle \sum_{x \in S_{N-2}} \frac{V_-(y \to x)}{\sum_{z \in S_N} V_+(x \to z)} \right\rangle_N, \tag{17}$$

where $\langle \mathcal{O} \rangle$ denotes the expectation value of the observable $\mathcal{O}$ if the number of vertices are kept fixed:

$$\langle \mathcal{O} \rangle_N = \frac{1}{Z_N} \sum_{x \in S_N} \mathcal{O}_x W_x^{(N)}. \tag{18}$$

In order to apply the general formula in our case we have chosen to let transitions between ϕ^3-graphs $G^{(N)} \in S_N$ and $G^{(N-2)} \in S_{N-2}$ be the contraction of a 3-point loop to a vertex and the replacement of a vertex with a 3-point loop. The detailed formula for Z_{N-2}/Z_N can now be worked out according to (17) (see [15] for details). The main point is that we have an explicit operator $\mathcal{O}_\Delta$ given by the left hand side of (17) of which we can calculate expectation values (18) on the set S_N of planar ϕ^3-graphs consisting of N vertices. The weight used for the individual ϕ^3-graphs is determined by (11). The method used for calculating these averages is the standard Monte Carlo flip method [3] which has been used extensively to determine the mean square extension of the surfaces. Now we can in addition measure Z_{N-2}/Z_N by (17) and (18):

$$\frac{Z_{N-2}}{Z_N} = \langle \mathcal{O}_\Delta \rangle_N \tag{19}$$

and this allow us to determine γ since (14) gives

$$\log\left(\frac{Z_{N-2}}{Z_N}\right) \approx -\tilde{\beta}_c + (2-\gamma)/N + \mathcal{O}(1/N^2). \tag{20}$$

The only practical problem left is the problem of an effective computer code. The flip operation is notorically difficult to vectorize. We have partially circumvented this problem by vectorizing in the number of systems which we simulate. The details of this algorithm will be described elsewhere [17], but the philosophy is quite simple. Although the flip operation itself cannot be vectorized since one has to make a non-local check whether it is allowed or not, everything else *can* be vectorized. All these additional operations are now vectorized simply by having many systems.

4. Results

The simulation of the system was done in 1 to 6 dimension. Typically 500 systems were simulated in parallel and the number of vertices in the ϕ^3-graphs were in the range 4–200. The number of MC-updatings for each system at a fixed N was typical of the order of 100.000 sweeps. Half of these were heatbath updates of the coordinates, the other half

were change in the ϕ^3-graphs according to the flip algoritm. The total computer time used was of the order of 100 Cray-XMP hours.

The measurement of the $Z_{(N-2)}/Z_N$ was done by measuring the expectation value of the operator $\mathcal{O}_\Delta$ for a fixed N. The statistics was quite good even for large N, presumably because $\mathcal{O}_\Delta$ is a local operator, and even if the fluctuations of the surfaces become larger with increasing N, the fact that we take an expectation value of a local operator counteract this to some extent. In fact the values are such that we can consistently include $1/N^2$ corrections in the fits (20). This is absolutely necessary if one wants to reduce the systematic errors in the fit determination of γ. In this respect the simulation is much superior to, for instance, the one in [11].

The values found for γ by standard χ^2 fitting is as follows:

d	1	2	3	4	5	6
γ	−0.35	−0.22	−0.10	0.01	0.08	0.13

The error bars are of the order ± 0.05. We see that the values extrapolate nicely down to $\gamma = -1/2$ for $d = 0$ as it should. However there is the immediate problem that we do not get the correct value ($\gamma = 0$) for $d = 1$. This can be partly understood by the fact that there are logarithmic corrections to the asymptotic expansion (15) in the case $d = 1$ which can be solved explicitly.

$$Z_N \approx (\exp \tilde{\beta}_c N)/N^2 \log N \quad \text{for} \quad N \to \infty. \tag{21}$$

Including this $\log N$ factor in the fit brings the value of γ closer to zero but it is still below zero and the asymptotic form (21) is actually inconsistent with the data ($\chi^2 \approx 20$). What is worse is that we have no chance of determining whether this kind of logarithmic correction is present for $d > 1$. The fit cannot determine both γ and κ in a possible term $(\log N)^\kappa$.

If we assume there is no logarithmic correction we get a perfectly consistent fit with a χ^1 around 1 and the best that can be said is that the data are *not* consistent with the assumption the γ stays zero and only the exponent κ in $(\log N)^\kappa$ changes for d in the range 1 to 6.

The same somewhat disappointing results are obtained for the fit to the Haussdorf dimension d_H. At $d = 1$ it is known that the size of the surface should grow as $\log N$. The fit favours $(\log N)^\alpha$ behaviour where $\alpha \approx 0.5$ while for higher dimensions d a power behaviour N^{1/d_H} is favoured, with d_H decreasing with d. This is essentially in agreement with old results, but a warning should be given: It is notorically difficult to distinguish between large d_H and a logarithmic behaviour. For $d > 3$ it is clear that a power behaviour is preferred, however.

To summarize: If we take the Monte Carlo generated data on face value the most naive analysis shows no sign of a phase transition at $d = 1$. However, we are still analysing the data, and a more sofisticated treatment might reveal some sign of interesting behaviour near $d = 1$.

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