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GLUEBALL MASSES, FINITE SIZE EFFECTS AND BOUNDARY CONDITIONS*

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Finite size effects can be dramatically different for periodic and twisted boundary conditions, as long as we are in small volumes where perturbation theory prevails. In that case glueballs are constituted from gluons; whereas the glueballs feel only periodic boundary conditions, its constituent gluons experience the twist and have to wind at least as many times around the box as there are colours. We review Montecarlo results on glueball masses and string tension with periodic and twisted boundary conditions. For lattice sizes now available they seem to be in reasonable agreement. Various improvements are suggested.

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1. Introduction

Pure glue systems are the simplest to simulate, and careful analysis and set-up of the simulation might give us a clue of the confining mechanism.

In the past seven years the glueball masses have been subject to changes in the mass ratios extracted from the theory by Montecarlo simulations. The bulk of the changes came from going to bigger sized lattices, in other words, from finite size effects. As an example: the tensor to scalar mass ratio grew from 0.9 to 1.6 in the last five years. It is useful to see to what extent we can still expect further change: *not* by going to even larger lattices, but by studying *other* boundary conditions. In this paper I review the evidence that we are now at lattice sizes where these effects do indeed vanish. Also discussed are additional signals that show us whether or not we have entered a "string régime".

2. Twisted versus periodic boundary conditions

Let us take any set of boundary conditions satisfying [1]
(*i*) periodicity for local gauge invariant quantities,
(*ii*) uniqueness of the vector potentials.

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Obviously periodic boundary conditions on the potentials are included; but there are other classes of such boundary conditions, as 't Hooft has emphasized [1]; for the purpose of our discussion we limit ourselves to boxes with a finite length L in three space directions and an infinite extent in the time direction. The classes are then characterized by a 3×3 twist tensor n_{ij} with integer entries mod N (SU(N) is supposed to be the gauge group). In every class there is a representative given by three constant gauge transforms (Γ_x, Γ_y, Γ_z) with the commutation relations:

$$\Gamma_i\Gamma_j\Gamma_i^{-1}\Gamma_j^{-1} = \exp\left(i\frac{2\pi}{N}n_{ij}\right). \tag{2.1}$$

These gauge transformations relate the vector potentials A_μ shifted over a period:

$$A_\mu(\vec{x}+\vec{e}_k L) = \Gamma_k A_\mu(\vec{x})\Gamma_k^{-1} \tag{2.2}$$

and the twist tensor can be related to the magnetic flux vector $\vec{m}$ by:

$$m_k = \tfrac{1}{2}\varepsilon_{kij}n_{ij}. \tag{2.3}$$

This flux originates in the boundary conditions (i) and (ii) and is therefore expected to have no influence on the physical objects like glueballs we create with *local* sources in the large volume limit. But the eigenstates of the Hamiltonian can remember — even in the large volume limit — what boundary conditions were applied! Simple examples are kinks in 1+1 dimensional field theories with spontaneously broken symmetry, and monopoles in 3+1 dimensional field theory. But local sources are supposed to have zero overlap with such states in the infinite volume limit.

So for correlation functions of local sources, the set of intermediate states (i.e. eigenstates of the transfer matrix) that contributes to the correlation should be insensitive to the boundary conditions in the large volume limit. This will be our credo. Let us not forget that spurious states like the ones above are of non-perturbative origin. As we will be concerned with perturbative physics in small volumes in this paper we should not be unduly worried: in small enough boxes (so that perturbation theory applies) one can equally well excite glueballs with suitable combinations of Polyakov loops (see Section 4) as with plaquette combinations. But to extract the large volume physics one better take the local sources.

3. Results with twisted boundary conditions: theory and experiment

Let us first discuss the qualitative picture. Small boxes allow a perturbative expansion. So a glueball state is composed of gluons that propagate in the twisted box with a propagator $G_p(x)$ whose periodicity p is determined by that of the potentials in Eq. (2.2). Let us for simplicity restrict ourselves to SU(2).

Then the simplest possibility one can think of is a twist in say the z-direction: $\vec{m} \doteq (001)$. It is generated by taking $\Gamma_1 = i\sigma_1$, $\Gamma_2 = i\sigma_2$, $\Gamma_3 = 1$ in Eq. (2.1). The periodicity of the vector potentials in directions 1 and 2 is now $2L$ instead of L. This means that the gluon has to propagate twice around the volume in the direction 1 and twice in the direction 2 in order to create a finite size effect. Mathematically we can relate the gluon propagator G_p in a finite periodic box to that in an infinite box through Poisson's formula for a periodic δ-function:

$$\sum_{l\leftarrow Z}^{-} \delta(l-l') = \sum_{n'} \exp(i2\pi n' l).$$

We get:

$$G_p(x) = G_\infty(x) + \sum_{n=\pm 1, \pm 2, \ldots} G_\infty(x+np).$$

In the case at hand $p = 2L$ in 1 and 2 directions, and the gluon has to wind at least twice around the box, as we stated before. This way of presenting is actually a slight oversimplification: a simple counting of degrees of freedom shows that we have 4 times as much spatial degrees of freedom as we started from! The reason is that the boundary conditions (2.2) involve the colour degrees of freedom. The colour degrees of freedom are therefore providing the increase in periodicity, but they would expain only a factor *3* (N^2-1 for SU(N)). The twisted propagator G_p is containing not *all* momenta $\vec{p} = \frac{2\pi}{L}\left(\frac{n_x}{2}, \frac{n_y}{2}, n_z\right)$; in fact the momenta present in the original box, $\vec{p} = \frac{2\pi}{L}\vec{n}$, ($\vec{n}$ *integer*), are all absent; thus the spatial volume becomes only 3 times as big. For details see Ref. [10]. Let us now describe MC experiment and theory.

(a) The first Montecarlo experiment was done by Stephenson and Teper within pure SU(2) gauge theory. They used the twist $\vec{m} = (1, 1, 1)$, which does not break cubic symmetry. The reason for this is the mod 2 property of the magnetic flux. The energy levels for the various cubic-group representations were calculated perturbatively for very small boxes (lowest order perturbation theory). The results are compared in Fig. 1. The values on the vertical axis are obtained from the correlation between suitably chosen local sources (that is, with the corresponding quantum numbers of the cubic representations shown on the horizontal axis). The correlation is followed over two time units, except for the T_2^-, which has only been followed over one time unit, according to the well known formula:

$$ma \leqslant m_{\text{eff}}(\Delta t) = -\log \frac{\text{correlation}(\Delta t)}{\text{correlation}(\Delta t - 1)}.$$

The value of ma still depends on $\beta = \frac{4}{g^2}$, the bare coupling.

(b) The theoretical results were obtained [3] in a continuum box (lattice length $a = 0$). The formula reads to lowest order in the coupling g (see Fig. 2 for the graphs):

$$m_x = \frac{1}{L}\left\{2\pi\sqrt{2} + c_{1,x}\frac{g^2}{4\pi} + c_{2,x}\left(\frac{g^2}{4\pi}\right)^2 + O(g^6)\right\}. \tag{3.1}$$

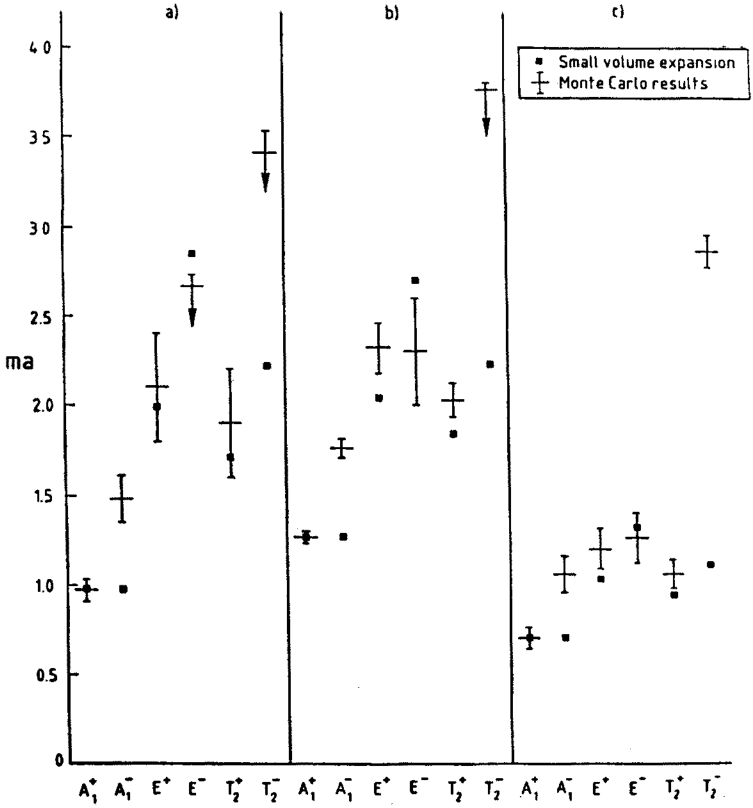


Fig. 1. Montecarlo data (Ref. [2]) for various irreducible representations of the cubic group

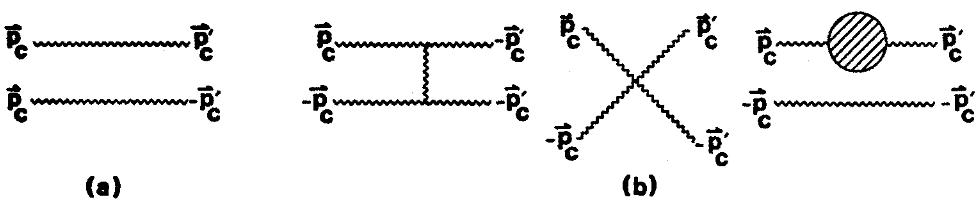


Fig. 2. The graphs contributing to $O(1)$ (Fig. 2a) and to $O(g^2)$ (Fig. 2b) in Eq. (3.2)

The left hand side represents the energy in the center of mass of two gluons with minimal (but *non* zero!) momentum $\pm\vec{p}_c$ $\left(\vec{p}_c = \frac{\pi}{L}(0, 1, \pm 1), (\pm 1, 0, 1), (\pm 1, 1, 0)\right)$. To lowest order (Fig. 2a) the energy equals $2E$, with E being the on-shell energy $E = |\vec{p}_c| = \frac{1}{L}\pi\sqrt{2}$. There are 24 degenerate states with this energy, that do split to order g^2 into levels labelled x, with x running through the representation $A_1^\pm$, $E^\pm$, $T_1^\pm$, $T_2^\pm$ (there are no $A_2^\pm$ levels in this set of 2-gluon levels). The numbers $c_{1,x}$ are given in Table I. The results are plotted in

TABLE I

Values of the coefficient $c_{1,x}$ in Eq. (3.2). The contribution from tree graphs is called $c_{1,xe}$ that from self energy graphs $c_{1,xs}$

$[X]$	$c_{1,xe}$	$c_{1,xs}+4.94$
E^+	32	0.06
E^-	32	0.
A_1^+	24	0.12
T_1^+	16	0.
T_2^+	$-8+8\sqrt{5}$	0.19
T_2^-	0	0.
T_2^+	0	-0.62
E^+	-12	-0.06
T_2^+	$-8-8\sqrt{5}$	-0.10
A_1^-	-64	0.
A_1^+	-64	-0.12

Fig. 1. Let us first note that $c_{1,0^{++}} \leqslant c_{1,E^+} \leqslant c_{1,T_2^+}$ are all negative, so the theory would become unstable if $\frac{g^2}{4\pi} \geqslant \frac{2\pi\sqrt{2}}{c_{1,0^{++}}}$. Second, the formula (3.2) was used in Fig. 1 without taking the lattice spacing into account: the value of the $A_1^{++}(0^{++})$ was used to fix the coupling. The formula (3.2) with lattice spacing taken into account reads:

$$m_x L = \hat{c}_{0,x} + \hat{c}_{1,x}\frac{g^2}{4\pi} + \hat{c}_{2,x}\left(\frac{g^2}{4\pi}\right)^2 + O(g^6). \tag{3.2}$$

The mass m and the size L of the box are now measured in units of the lattice length. The coefficient $\hat{c}_{0,x} = 2\pi\sqrt{2}\frac{E}{|\vec{p}_c|}$ is only lattice dependent through the lattice dependence of the energy E of an on-shell gluon:

$$\left(\sinh\frac{E}{2}\right)^2 = \sum_{i=1}^{3}\left(\sin\frac{p_{ci}}{2}\right)^2$$

This leads to $\hat{c}_0$ deviating a few percent from the value $2\pi\sqrt{2}$ even for $L = 4$. The next coefficient, $\hat{c}$, is given by the two tree graphs and the self-energy correction in Fig. 1b. A moments reflection will convince the reader that the self energy is finite when evaluated on-shell and projected on the physical polarizations. Moreover it affects only the positive parity states A_1^+, T_2^+ and E^+.

The tree graphs alone give deviations from $c_{1,x}$ of up to 35% at $L = 4$ (up to 10% at $L = 8$) from the continuum values at $L = \infty$ shown in Table I.

To order g^4 one finds for large L logarithmic terms that can be absorbed by the renormalization of g^2. The *log's* are the "short distance singularities", the remaining L dependence ("finite size" effect) leads to a finite limit c_2 ("continuum limit"). To all orders in perturbation theory one can perform this procedure and the result is:

$$m_x L = \sum_n c_{n,x} \left(\frac{g^2(L)}{4\pi}\right)^n + \frac{1}{L^2} r_x(L^2, g^2). \tag{3.3}$$

The second term is usually supposed to be negligible; however for $L = 4$ we just saw it contributed up to 35%, 10% at $L = 8$. So trying to interpret the data in Fig. 1 without the second term is dangerous, specially for $L = 4$.

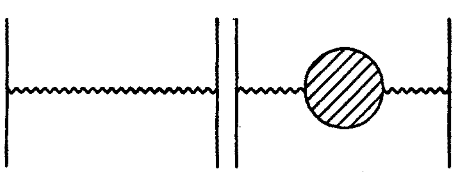
Usually one picks the lowest mass $x = A_1^{++}$ (the 0^{++}) to give us a scale variable $m_{0^{++}} L \equiv z_{0^{++}}$. Ideally one follows then the evolution of a mass ratio $\frac{m_x}{m_{0^+}}$ as a function of diminishing coupling $\frac{g^2}{4\pi}$ and growing L in such a way that z stays constant. In this way the continuum limit is determined for this fixed value of z. Finally the large z limit of the continuum limit is the ratio of the physical masses. Unfortunately the data do not allow such an analysis. Instead, one tries to see an exponential fall off in m with $\beta = \frac{4}{g^2}$, which does not separate out the finite size effects in the second term of (3.3).

4. Polyakov-loops, self-energies and the string régime

The energy $E(L)$ of Polyakov-loops in the twisted box is calculable in perturbation theory, in contrast to the periodic case. In Fig. 3 we have depicted graphs that contribute to low orders. If the Polyakov-loop corresponds to the flux $(e_1, e_2, e_3) \equiv \vec{e}$ then $\vec{p}_c \stackrel{\text{def}}{=} \frac{2\pi}{NL} \vec{e} \times \vec{m}$ is the minimal momentum the loop can have. This is due to the fact that Polyakov loops are *not* periodic [10] except for the case that their electric flux $\vec{e}$ runs parallel to the flux $\vec{m}$. Notice that the Polyakov loops are not local so we are not in contradiction with statement (*i*) in the beginning of Section 2. It is the magnetic flux $\vec{m}$ — due to the twist — that creates through the Poynting vector $\vec{e} \times \vec{m}$ a new momentum scale $\vec{p}_c$. Therefore the graph in Fig. 3a is possible, where a gluon with momentum $\vec{p}_c$ is exchanged; in Fig. 3b the renormalization of that exchanged gluon is shown.

The result for $E_{\vec{e}}(L)$ is:

$$E_{\vec{e}}(L) = |\vec{p}_c| + g^2 \Pi_{\vec{e},\vec{e}}(\vec{p}_c) + O(g^4). \tag{4.1}$$



(a) (b)

Fig. 3. The graphs contributing to $O(1)$ (Fig. 3a) and to $O(g^2)$ (Fig. 3b) in Eq. (4.1)

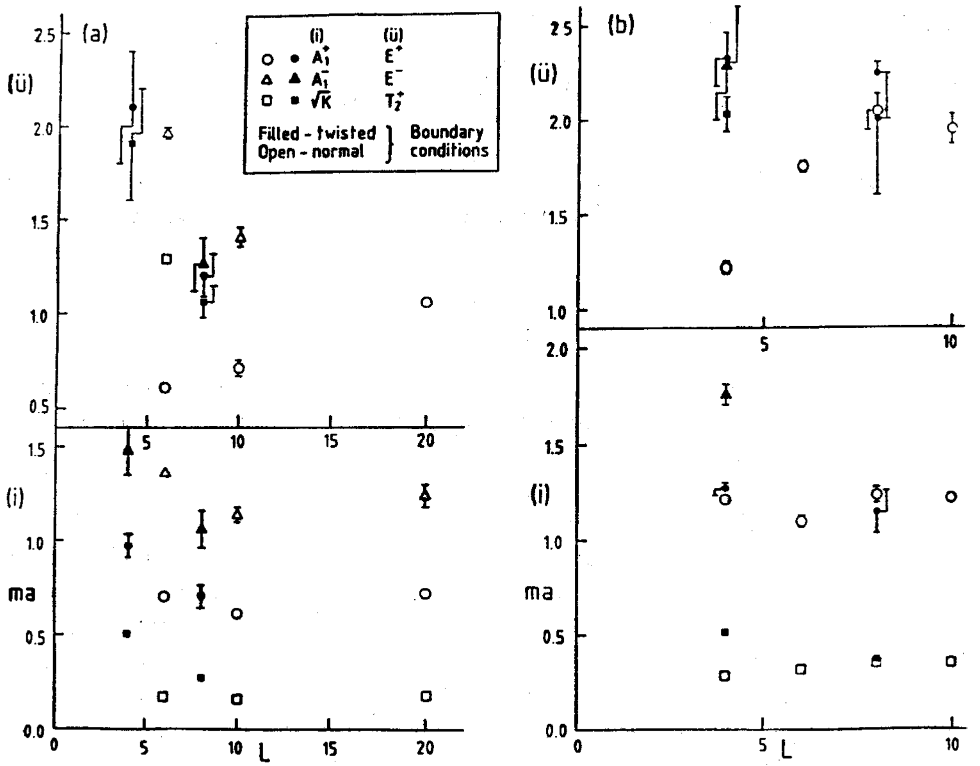


fig 4
Comparison between periodic ("normal") and twisted boundary conditions.

Fig. 4. Comparison between periodic ("normal") and twisted boundary conditions

The self energy Π (Fig. 3(b)) of the gluon is projected onto the normalized polarization vector $\vec{\varepsilon} \parallel \vec{e}$, and evaluated on-shell. To lowest order g^2 this is a finite quantity and appears also in the graphs in Fig. 1, i.e. it also contributes to the masses discussed in Section 3. In fact the graphs in Fig. 1 can be viewed as correlations of appropriate combinations of Polyakov loops.

Teper et al. have measured the string tension $K(L) \equiv \dfrac{E(L)}{L}$ from a loop in the direction of one of the coordinate axes. According to Eq. (4.1), $E(L)L$ depends only on g^2, so $\sqrt{K(L)}$ behaves like $\dfrac{c}{L}$ when (4.1) applies (only for small L!). Surprisingly, this is in good agreement with the $\beta = 2.5$ data (see Fig. 4), also the constant c fits well.

An interesting experiment would be the Montecarlo study of the ratio of three Polyakov loops p_i; one along the main axis, one along the diagonal in say a fixed z plane, and one along a body diagonal [5]. The first two can be perturbatively computed for small box sizes, the third one has $\vec{p}_c = 0$ (since in that case $\vec{e} = (1, 1, 1) = \vec{m}$) and its contributions come from semi-classical configurations in fact instantons with half integer topological charge [6]. Calling the energy and the length of p_i respectively $E_i(L_i)$ and L_i we have in a string régime (large boxes):

$$\frac{E_1}{L_1} = \frac{E_2}{L_2} = \frac{E_3}{L_3} = \text{string tension},$$

whereas for small boxes the three quantities are known perturbatively, as discussed above. Therefore this measurement [5] allows us to assess whether or not we are in the confinement régime.

5. Discussion

The numerical results are crude and our theoretical predictions too. But the qualitative agreement of the prediction and the MC result is encouraging. High statistics measurements of the masses are underway. To know where we stand warrants a measurement of the Polyakov loops as argued in Section 4.

From the comparison between values of masses and loops in twisted and periodic volumes (see Fig. 4) one sees a fair agreement. We feel that this lends additional support to the soundness of the MC approach. Determination of the fourth order coefficient c_2 in equation (3.2) involves the calculation of fourth order scattering amplitudes: we found the background gauge by far the most expedient way of calculating. The relation of these S-matrix amplitudes to the discrete energy levels [9] is straightforward to work out. Computation of its counterpart $\hat{c}_2$, with lattice artifacts present, is a time consuming enterprise.

Last we would like to mention the inclusion of fermions. To eliminate the ambiguity in the fermion field due to the colour twist one introduces in flavour space a twist that counters the ambiguity due to the colour twist. In the case at hand we introduce an $SU(N_f)$ group with an even number of flavours. For $N_f = 2$ and our $SU(2)\vec{m} = (1, 1, 1)$ colour twist we introduce the following boundary condition on the quark field $q(x)$ (colour acts from the left, flavour from the right):

$$q(\vec{x}+e_kL) = \sigma_k q(\vec{x})\sigma_k.$$

This leads to a perturbative behaviour of the quarkfield precisely analogous to that of the gluon; in particular the Fourier components of quark and gluon fields are identical. Thus we find to order g^2 a contribution coming fom the self energy of the gluon due to the massless quark, which is $\frac{1}{4}$ of the contribution by gluons alone (see third column of Table I). Therefore the influence of the quarks is little, at least to this order of perturbation theory. The fourth order calculation is very feasible, and is on its way.

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