DIRAC EQUATION WITH HIDDEN EXTRA SPINS: A GENERALIZATION OF KÄHLER EQUATION. PART TWO*

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A sequence of equations numerated by $N = 1, 2, 3, \ldots$, realizing the Dirac square-root procedure for spin $0 \oplus 1 \oplus \ldots \oplus \frac{1}{2}N$ (N even) or $\frac{1}{2} \oplus \frac{3}{2} \oplus \ldots \oplus \frac{1}{2}N$ (N odd), is further discussed. For $N = 2$ the Dirac-type form of Kähler equation is reproduced. The equation with $N = 3$ is conjectured to be physically distinguished, providing a model for fermion generations.

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As is well known, the equation discovered in 1960 by Kähler [1, 2] linearizes the d'Alemberian differential operator in the space of antisymmetric tensors and so realizes the Dirac square-root procedure for spin $0 \oplus 1$ [1]. This equation can be rewritten also in the equivalent Dirac-type form [5]

$$(\gamma \cdot p - m)\psi = 0, \tag{1}$$

where $\psi = (\psi_{\alpha_1 \alpha_2})$ carries two Dirac bispinor indices, while γ^μ are the usual Dirac 4×4 matrices acting on the first of these indices. The second bispinor index is here free, unless operated on by m. In Ref. [5], this second index was interpreted as being responsible for four fermion generations and so it lost its original connection with Lorentz transformations in the physical spacetime.

Recently, it was observed [6] that the Dirac anticommutation relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu} \tag{2}$$

admit a remarkable sequence of representations given by formulae

$$\Gamma^\mu = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i^\mu, \tag{3}$$

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[1] The Kähler equation was essentially discovered much earlier by Ivanenko and Landau [3], soon after the discovery of Dirac equation [4]. So, it may be called also the Ivanenko-Landau-Kähler equation.

$N = 1, 2, 3 \dots$, where the matrices γ_i^μ, $i = 1, 2, \dots, N$, span the sequence of Clifford algebras defined by the anticommutation relations

$$\{\gamma_i^\mu, \gamma_j^\nu\} = 2\delta_{ij} g^{\mu\nu}. \tag{4}$$

For $N \geqslant 2$ the representations (3) are reducible. In fact, for $N \geqslant 2$ there are such representations of the matrices γ_i^μ, $i = 1, 2, \dots, N$, that

$$\Gamma^\mu = \gamma^\mu \otimes \underbrace{\mathbf{1} \otimes \dots \otimes \mathbf{1}}_{N-1 \text{ times}}, \tag{5}$$

where γ^μ and $\mathbf{1}$ are the usual Dirac 4×4 matrices. Thus, the Dirac equation

$$(\Gamma \cdot p - m)\psi = 0 \tag{6}$$

can be represented as

$$(\gamma \cdot p - m)\psi = 0, \tag{7}$$

where $\psi = (\psi_{\alpha_1 \alpha_2 \dots \alpha_N})$ carries N Dirac bispinor indices of which only the first one is acted on by the Dirac matrices γ^μ. The rest of them are free, unless m operates on this residual set of indices. Of course, the Dirac-type form (1) of the Kähler equation is a particular case of Eq. (7) corresponding to $N = 2$. So, Eq. (6) together with Eqs. (3) and (4) gives us a generalization of the Kähler equation for arbitrary N, realizing the Dirac square-root procedure for spin $0 \oplus 1 \oplus \dots \oplus \frac{1}{2}N$ or $\frac{1}{2} \oplus \frac{3}{2} \oplus \dots \oplus \frac{1}{2}$N, where N is even or odd, respectively.

However, it is important to note that in an external electromagnetic field, where the Dirac equation (6) takes the form

$$[\Gamma \cdot (p - eA) - m]\psi = 0, \tag{8}$$

a particle, if described by ψ, can display only spin 1/2 due to the Dirac anticommutation relations (2). It is true for any $N = 1, 2, 3, \dots$. Thus, the total spin of such a particle is physically divided into an electromagnetically visible part 1/2 and an electromagnetically hidden part $\frac{1}{2} \oplus \frac{3}{2} \oplus \dots \oplus \frac{1}{2}(N-1)$ or $0 \oplus 1 \oplus \dots \oplus \frac{1}{2}(N-1)$, where N is even or odd, respectively.

Hence, we get for Eq. (8) two natural interpretative options, where the physical Lorentz group corresponding to the theory of relativity is generated either by

$$J^{\mu\nu} = L^{\mu\nu} + \tfrac{1}{2} \sum_{i=1}^{N} \frac{i}{2} [\gamma_i^\mu, \gamma_i^\nu] \tag{9}$$

or by

$$J^{\mu\nu}_{\text{visible}} = L^{\mu\nu} + \tfrac{1}{2} \frac{i}{2} [\Gamma^\mu, \Gamma^\nu], \tag{10}$$

with $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and $p_\mu = i\partial/\partial x^\mu$. In the first option m is obliged by the theory of relativity to commute with $J^{\mu\nu}$, while in the second option m must commute with $J^{\mu\nu}_{\text{visible}}$ but not necessarily with $J^{\mu\nu}_{\text{hidden}} = J^{\mu\nu} - J^{\mu\nu}_{\text{visible}}$. Of course, the mass m commutes automatically with $J^{\mu\nu}_{\text{visible}}$ since in the Dirac equation (6) m is assumed to commute with Γ^μ (and is x-independent). Thus, in the second option the electromagnetically hidden spin is not obliged to be connected with the physical spacetime governed by the theory of relativity and so may correspond to some other internal degrees of freedom as it was conjectured in Ref. [5] in the case of $N = 2$. If, however, all matrices γ^μ_i, $i = 1, 2, \ldots, N$, are connected with the physical spacetime (as it is consistent with our construction of the matrices Γ^μ) this second option is logically excluded in favour of the first.

In this case not only Γ^μ but also all other independent linear combinations of γ^μ_i, $i = 1, 2, \ldots, N$, are connected with the physical spacetime. Defining the new matrices

$$\Gamma^\mu_1 = \Gamma^\mu, \Gamma^\mu_2, \ldots, \Gamma^\mu_N \tag{11}$$

by means of the Euler linear combinations of $\gamma^\mu_1, \gamma^\mu_2, \ldots, \gamma^\mu_N$ we get for them the anticommutation relations of the type (4):

$$\{\Gamma^\mu_i, \Gamma^\nu_j\} = 2\delta_{ij} g^{\mu\nu}. \tag{12}$$

For instance, for $N = 3$

$$\Gamma^\mu_1 = \frac{1}{\sqrt{3}}(\gamma^\mu_1 + \gamma^\mu_2 + \gamma^\mu_3).$$

$$\Gamma^\mu_2 = \frac{1}{\sqrt{2}}(\gamma^\mu_1 - \gamma^\mu_2),$$

$$\Gamma^\mu_3 = \frac{1}{\sqrt{6}}(\gamma^\mu_1 + \gamma^\mu_2 - 2\gamma^\mu_3). \tag{13}$$

Note that in Eq. (9) we have now

$$\tfrac{1}{2}\sum_{i=1}^{N} \frac{i}{2}[\gamma^\mu_i, \gamma^\nu_i] = \tfrac{1}{2}\sum_{i=1}^{N} \frac{i}{2}[\Gamma^\mu_i, \Gamma^\nu_i], \tag{14}$$

so that

$$J^{\mu\nu}_{\text{hidden}} = J^{\mu\nu} - J^{\mu\nu}_{\text{visible}} = \tfrac{1}{2}\sum_{i=2}^{N} \frac{i}{2}[\Gamma^\mu_i, \Gamma^\nu_i]. \tag{15}$$

In this case m must commute with $J^{\mu\nu}_{\text{hidden}}$ since it is obliged to commute with $J^{\mu\nu}$ (and is assumed to commute with $\Gamma^\mu_1 = \Gamma^\mu$). Then, the mass m may depend on $\Gamma^\mu_2, \ldots, \Gamma^\mu_N$ but only via their invariants under the hidden Lorentz group (and is independent of Γ^μ_1).

From Eq. (8) and its Hermitian conjugate we can deduce that

$$\frac{\partial}{\partial x^\mu}(\psi^\dagger \Gamma^0_1 \Gamma^\mu_1 \psi) = 0 \tag{16}$$

as well as

$$\eta_{N-1} \frac{\partial}{\partial x^\mu} (\psi^\dagger \Gamma_1^0 \Gamma_2^0 \dots \Gamma_N^0 \Gamma_1^\mu \psi) = 0, \tag{17}$$

the second conclusion being valid only when m commutes with the matrix $\Gamma_1^0 \Gamma_2^0 \dots \Gamma_N^0$ and N is odd, $N = 1, 3, 5, \dots$ (the first one is true, for m commutes with Γ_1^0). Here, the phase factor η_{N-1} makes the operator $\eta_{N-1}\Gamma_2^0 \dots \Gamma_N^0$ Hermitian. The conserved current appearing in Eq. (16) is not covariant under the full Lorentz group generated by $J^{\mu\nu}$, though it is covariant under the visible Lorentz group generated by $J^{\mu\nu}_{\text{visible}}$. In contrast, the current in Eq. (17) is covariant under the full Lorentz group. We can see that under the assumption that the physical Lorentz group corresponding to the theory of relativity is generated by $J^{\mu\nu}$, the chance for probability interpretation of ψ is restricted only to the case of N odd, $N = 1, 3, 5, \dots$ (thus, the case of $N = 2$ corresponding to the original Kähler equation is then excluded). Note that in the case of N odd and m commuting with $\Gamma_1^0 \Gamma_2^0 \dots \Gamma_N^0$ the Hermitian operator $\eta_{N-1}\Gamma_2^0 \dots \Gamma_N^0$ (whose square is the unit operator) is a constant of motion, so one can impose on ψ the condition

$$\eta_{N-1}\Gamma_2^0 \dots \Gamma_N^0 \psi = \psi \tag{18}$$

which guarantees that

$$\eta_{N-1}\psi^\dagger \Gamma_2^0 \dots \Gamma_N^0 \psi > 0, \tag{19}$$

where the spatial integral of the left-hand side is constant in time due to Eq. (17).

Above, we restricted ourselves to the formal structure of the Dirac-type form of the generalized Kähler equation given by formulae (8), (3) and (4). The question of the possible physical interpretation of this handsome equation realizing the Dirac square-root procedure in a general way is entirely open.

In this paper we would like to make the conjecture that the generalized Kähler equation with $N = 3$ is in a way physically distinguished, giving us a model for fermion generations i.e., for the first-generation leptons (ν_e or e^-) or quarks (u or d) as well as their higher--generation replicas.

For $N = 3$ we can use the following representation for Γ_i^μ, $\Gamma_i^5 = i\Gamma_i^0\Gamma_i^1\Gamma_i^2\Gamma_i^3$ and $\Sigma_i^k = \frac{1}{2}\varepsilon^{klm}\frac{i}{2}[\Gamma_i^l, \Gamma_i^m] = \Gamma_i^5\Gamma_i^0\Gamma_i^k$:

$$\begin{aligned} \Gamma_1^\mu &= \gamma^\mu \otimes \mathbf{1} \otimes \mathbf{1}, \quad & \Gamma_1^5 &= \gamma^5 \otimes \mathbf{1} \otimes \mathbf{1}, \\ \Gamma_2^\mu &= \gamma^5 \otimes i\gamma^5\gamma^\mu \otimes \mathbf{1}, \quad & \Gamma_2^5 &= \mathbf{1} \otimes \gamma^5 \otimes \mathbf{1}, \\ \Gamma_3^\mu &= \gamma^5 \otimes \gamma^5 \otimes \gamma^\mu, \quad & \Gamma_3^5 &= \mathbf{1} \otimes \mathbf{1} \otimes \gamma^5 \end{aligned} \tag{20}$$

and

$$\begin{aligned} \Sigma_1^k &= \sigma^k \otimes \mathbf{1} \otimes \mathbf{1}, \\ \Sigma_2^k &= \mathbf{1} \otimes \sigma^k \otimes \mathbf{1}, \\ \Sigma_3^k &= \mathbf{1} \otimes \mathbf{1} \otimes \sigma^k, \end{aligned} \tag{21}$$

where $\gamma^\mu, \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\mathbf{1}$ and $\sigma^k = \frac{1}{2}\varepsilon^{klm}\frac{i}{2}[\gamma^l, \gamma^m] = \gamma^5\gamma^0\gamma^k$ are the usual Dirac 4×4 matrices. Hence, in the case of $N = 3$ the wave function $\psi = (\psi_{\alpha_1\alpha_2\alpha_3})$ describes two Dirac particles with hidden spin 0 and two Dirac particles with hidden spin 1 since there are two eigenvalues ± 1 of the operator $\Gamma_2^5\Gamma_3^5$ commuting with the operators of hidden spin $\frac{1}{2}(\vec{\Sigma}_2+\vec{\Sigma}_3)$ and hidden parity $\eta_2\Gamma_2^0\Gamma_3^0$ ($\eta_2 = i$), where the only acceptable eigenvalue of the latter operator is $+1$ due to the condition (18) (true in our first interpretative option). Thus, for instance, in the case of the electron family one may try to make the following tentative identification:

	N	Visible spin	Hidden spin	$\Gamma_2^5\Gamma_3^5$	$\vec{\Sigma}_2\cdot\vec{\Sigma}_3$
e^-	3	1/2	0	$+1$	-3
μ^-	3	1/2	0	-1	-3
τ^-	3	1/2	1	$+1$	1
$?^-$	3	1/2	1	-1	1

So, the fourth fermion generation is here predicted unavoidably.

If two hidden-spin triplets τ^- and $?^-$ existed in the electron family, there would also exist the corresponding hidden-spin triplets ν_τ and $?^0$ in the neutrino family. Then, because of the separate conservation of hidden spin (holding in our first interpretative option) the decay rates for $W^- \to \tau^-(?^-)+\bar{\nu}_\tau(\overline{?^0})$ and $\tau^-(?^-) \to \mu^-+\bar{\nu}_\mu+\nu_\tau(?^0)$ get no multiplicity factors coming from the hidden spin. This is due to the hidden-spin singlet character of W^- following from the universality of electroweak gauge bosons in the standard model. In the case of τ^- the absence of such factors is consistent with the actual experimental data [7].

On the second-quantization level of the theory, the generalized Kähler fermions with $N = 3$ can be easily included into the scheme of standard model instead of four generations of sequential fermions. Then $m = 0$ and fermion masses arise from a more or less involved Higgs mechanism and from radiative corrections.

For instance, in the case of leptons there are a weak isospin doublet with $N = 3$,

$$\psi_L = \begin{pmatrix} \psi_L^{(e)} \\ \psi_L^{(\nu)} \end{pmatrix}, \tag{22}$$

and a weak isospin singlet with $N = 3$, $\psi_R^{(e)}$, where $\psi_{L,R} = \frac{1}{2}(1\mp\Gamma_1^5)\psi$ (and, of course, $\psi = (\psi_{\alpha_1\alpha_2\alpha_3})$). Then, one may consider the following phenomenological lepton-higgs coupling:

$$\psi^\dagger\Gamma_1^0(h_S\phi_S+h_P\phi_P\Gamma_2^5\Gamma_3^5+h_V\phi_V i\Gamma_2^\mu\Gamma_{3\mu}$$
$$+h_A\phi_A i\Gamma_2^5\Gamma_3^5\Gamma_2^\mu\Gamma_{3\mu}+h_T\phi_T\tfrac{1}{2}\Sigma_2^{\mu\nu}\Sigma_{3\mu\nu})\psi+\text{h.c}, \tag{23}$$

where

$$\phi_S = \begin{pmatrix} \phi_S^+ \\ \phi_S^0 \end{pmatrix}, \quad \text{etc.,} \tag{24}$$

are five higgs weak-isospin doublets and h_s, etc. stand for Yukawa coupling constants. Here, $\Sigma_i^{\mu\nu} = \frac{i}{2}[\gamma_i^\mu, \gamma_i^\nu]$. In Eq. (23) $\eta_2\Gamma_2^0\Gamma_3^0$ ($\eta_2 = i$) is put equal to 1 due to the condition (18). Making use of the formulae

$$i\Gamma_2^\mu\Gamma_{3\mu} = i\Gamma_2^0\Gamma_3^0(1-\Gamma_2^5\Gamma_3^5\vec{\Sigma}_2\cdot\vec{\Sigma}_3) \tag{25}$$

and

$$\tfrac{1}{2}\Sigma_2^{\mu\nu}\Sigma_{3\mu\nu} = (1+\Gamma_2^5\Gamma_3^5)\vec{\Sigma}_2\cdot\vec{\Sigma}_3, \tag{26}$$

we obtain from Eq. (23) (in the tree approximation) the following parametrization of charged-lepton masses:

$$\begin{aligned} m_e &= h_S v_S + h_P v_P + 4h_V v_V + 4h_A v_A - 6h_T v_T, \\ m_\mu &= h_S v_S - h_P v_P - 2h_V v_V + 2h_A v_A, \\ m_\tau &= h_S v_S + h_P v_P + 2h_T v_T, \\ m_{?^-} &= h_S v_S - h_P v_P + 2h_V v_V - 2h_A v_A, \end{aligned} \tag{27}$$

where $v_s = \langle\phi_s^0\rangle_{\text{vacuum}}$, etc., while the neutrinos ν_e, ν_μ, ν_τ, $?^0$ get zero masses. Since $m_e \simeq 0$, Eq. (27) gives

$$m_\tau \simeq \tfrac{4}{3}(h_S v_S + h_P v_P + h_V v_V + h_A v_A). \tag{28}$$

For instance, if $h_V v_V \simeq -k_A v_A$, Eqs. (27) and (28) reduce to

$$\begin{aligned} m_\mu &\simeq h_S v_S - h_P v_P - 4h_V v_V, \\ m_\tau &\simeq \tfrac{4}{3}(h_S v_S + h_P v_P), \\ m_{?^-} &\simeq h_S v_S - h_P v_P + 4h_V v_V \end{aligned} \tag{29}$$

and hence

$$\begin{aligned} h_S v_S &\simeq \tfrac{1}{4}(m_{?^-} + m_\mu + \tfrac{3}{2} m_\tau) > 0, \\ -h_P v_P &\simeq \tfrac{1}{4}(m_{?^-} + m_\mu - \tfrac{3}{2} m_\tau) > 0, \\ h_V v_V &\simeq \tfrac{1}{8}(m_{?^-} - m_\mu) > 0, \end{aligned} \tag{30}$$

where it is expected that $m_{?^-} \gg m_\tau$ since $?^-$ is not seen experimentally yet.

Of course, the most characteristic feature of generalized Kähler fermions with $N = 3$ as discussed in this paper is the existence of hidden-spin triplets. However, their multiplicity coming from the hidden spin is usually invisible due to the hidden-spin conservation. In fact, a question arises, in what a situation it can be visible. An answer is that, in principle, it can be visible in the annihilation process of tauonium $\tau^-\tau^+$ into photons since the bound system $\tau^-\tau^+$ may get three different hidden spins 0, 1, 2 (if τ^- is really a hidden-spin triplet), of which only the state 0 can annihilate into photons that are hidden-spin singlets. This gives an additional factor 1/3 in the radiative life-time fo an average tauonium $\tau^-\tau^+$ in comparison with the positronium e^-e^+ or muonium $\mu^-\mu^+$ which get only one hidden spin 0.

Finally, let us note that in general the condition (18) imposed on ψ spoils the covariance of the wave function under the hidden boosts, though the covariance under hidden spatial rotations as well as under all visible Lorentz transformations is not spoilt. The violation of covariance under hidden spatial rotations would generally introduce mixing of lepton generations. The last observation may be relevant for quarks, where generation mixing really appears.

APPENDIX

At the very end we would like to add a remark on the previous interpretation of the generalized Kähler equation with $N = 1, 3, 5, \ldots$ in terms of fermion generations, as it was presented in the part one of this paper [6]. The principle mentioned there, needed to terminate the sequence $N = 1, 3, 5, \ldots$, could be actually provided by a new "intrinsic Pauli principle" requiring that the wave functions $\psi = (\psi_{\alpha_1\alpha_2\ldots\alpha_N})$, $N = 1, 3, 5, \ldots$, must be antisymmetrical in the Direc bispinor indices $\alpha_2, \ldots, \alpha_N$. In fact, in contrast to the Dirac bispinor index α_1 describing the visible spin 1/2 (and visible chirality ± 1) of our generalized Kähler fermions $N = 1, 3, 5, \ldots$, the indices $\alpha_2, \ldots, \alpha_N$ refer to N-1 hidden spins 1/2 (and N-1 hidden chiralities ± 1) which can be considered as physically identical (of course, $[\vec{\Sigma}_i, \vec{\Sigma}_j] = 0$ for $i \neq j$ and $[\Gamma_i^5, \Gamma_j^5] = 0$ define independent indices $\alpha_1, \alpha_2, \ldots, \alpha_N$). Then, the sequence of N must terminate at $5: N = 1, 3, 5$. Moreover, for $N = 1$ and for $N = 5$ there exists only one Dirac particle with total hidden spin 0 and total hidden chirality $+1$, while for $N = 3$ there are only two Dirac particles with total hidden spin 0 and total hidden chiralities $+1$ and -1 (here, the condition (18) is, of course, applied). Thus, in this case there appear together four versions of any lepton and quark, all with total hidden spin 0. Hence, four fermion generations should exist. Assuming in addition that only fermions with total hidden chirality $+1$ can interact, the effective number of fermion generations may be reduced to three, what is the presently favored figure.

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