ON DIMENSIONAL REDUCTION OVER COSET SPACES*

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(Received November 10, 1989)

Gauge theories defined in higher dimensions can be dimensionally reduced over coset spaces giving definite predictions for the resulting four-dimensional theory. We present the most interesting features of these theories as well as an attempt to construct a model with realistic low energy behavior within this framework.

PACS numbers: 11.15.Tk

1. Introduction

During a few last years there has been an intensive activity in attempts to unify all the fundamental interactions using the notion of higher dimensions which was originated by Kaluza and Klein [1]. The Kaluza-Klein scheme itself had limited success due to its inability to accommodate the observed chiral world and to justify the assumed space-time configurations. A solution to these problems can be attained by the introduction of extra matter fields in higher dimensions [2]. As a bonus it seems now that the introduction of gauge theories in higher than four dimensions provides not only a solution to the Kaluza-Klein problems but also a very useful framework to describe the four-dimensional low energy interactions. It is also well known that the introduction of gauge theories in higher dimensions is very well motivated in superstring theories [3].

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* Presented at the XXIX Cracow School of Theoretical Physics, Zakopane, Poland, June 2–12, 1989.

** Partially supported by the bilateral agreement between Greece and the Federal Republic of Germany.

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In gauge theories defined on a higher dimensional manifold M^D, in order to maintain the four-dimensional Poincare invariance one has first to assume that M^D compactifies to $M^4 \times B$, where B is a compact space. The length scale of B is assumed to be of the order of the Planck length, a fact which makes B unobservable to the real world. The second step that one has to make in order to describe the observable four-dimensional world is to dimensionally reduce the theory or in other words to make the Lagrangian independent of the extra coordinates, and to integrate them out. Thus, the four-dimensional theories should be considered as effective theories obtained from the higher dimensional ones through some kind of dimensional reduction procedure.

A naive and crude way to make the four dimensional Lagrangian independent of the higher than four coordinates is to set to zero the field dependence on the extra coordinates. A much more elegant way is to allow a field dependence and employ a symmetry of the Lagrangian. In that case an obvious choice is to use the gauge symmetry [4]. Then the Lagrangian is independent of the extra coordinates just because it is gauge invariant. In fact, there exists an attractive approach to reduce dimensionally gauge theories which is based on this idea, namely the Coset-Space-Dimensional-Reduction (C.S.D.R.) scheme [4–10, 16]. In this scheme one starts with a pure Yang-Mills–Dirac theory based on a gauge group G and defined on a manifold M^D which compactifies to $M^4 \times S/R$ ($D = 4 + \dim S/R$), M^4 is a four dimensional Minkowski space and S/R a compact coset space. It is further assumed that transformations of the fields under symmetries of S/R are compensated by gauge transformations. This requirement implies definite constraints on the higher dimensional fields and allows a detailed examination of the various features of the effective four-dimensional theory.

One of the most attractive features of the C.S.D.R. scheme is that chiral fermions can be obtained [5] in four dimensions even if the higher dimensional theory is vectorlike. A lot of progress has been made in the construction of realistic models [7–10] resulting from higher dimensional theories via C.S.D.R., despite the very limited freedom of the method. In the process, various techniques have been applied especially the Wilson-flux breaking mechanism [10, 11] using certain discrete symmetries of S/R. However given that gravity is intrinsically involved in the C.S.D.R. one cannot ignore it and eventually has to consider the higher dimensional theory together with gravity [13, 14, 15]. It is interesting that there exists a classical solution of the Einstein-Yang-Mills-Dirac Lagrangian, corresponding to the C.S.D.R. requirements. Nevertheless the existence of the above classical solution introduces new constraints when discrete symmetries are used in order to break the four dimensional gauge group. In addition a large cosmological constant is required in the higher dimensional theory which can be avoided at the cost of introducing torsion in the coset space [15].

In the present paper we shall present an overview of the C.S.D.R. scheme and some applications. Sect. 2 serves as an introduction to some of the features of the scheme. In Sect. 3 we briefly discuss some geometrical and topological properties of the coset spaces which are used in Sect. 4 in an example of the models that can be obtained using the C.S.D.R. scheme. Finally Sect. 5 contains our conclusions.

2. *The C.S.D.R. scheme*

In this Section we present a review of the dimensional reduction procedure known as C.S.D.R. scheme [4–10, 16]. Consider in D-dimensions a Yang-Mills-Dirac theory with gauge group G defined on a manifold M^D which is the direct product of a four-dimensional spacetime M^4 and a compact coset space S/R, $D = 4+d$ where $d = \dim(S/R) = \dim S = \dim R$. The action in D dimensions is:

$$A = \int d^4x d^{D-4}y \sqrt{-g} \left\{ -\tfrac{1}{4} \operatorname{Tr}(F_{MN}F_{K\Lambda})g^{MK}g^{N\Lambda} + \frac{i}{2} \bar{\psi}\Gamma^M D_M \psi \right\}. \tag{1}$$

We use here Greek letters to denote curved indices (while Latin letters later on will denote flat indices); K, Λ, M, N run over the D-dimensional space, μ, ν, λ, over M^4 and α, β, γ, over S/R.

The fields A_M, ψ are assumed to be symmetric in the sense that any transformation under symmetries of S/R (which form the isometry group S of S/R) is compensated by gauge transformations. We note that we can start from any representation F of G for the fermions if no further symmetry as supersymmetry is required. So if $g(s), f(s)$ are gauge transformations in the adjoint and F irreps of G corresponding to the transformations acting on S/R, we require:

$$A_\mu(x, y) = g(s)A_\mu(x, s^{-1}y)g^{-1}(s), \tag{2}$$

$$A_\alpha(x, y) = g(s)\, \Sigma^\beta_\alpha A_\beta(x, s^{-1}y)g^{-1}(s) + g(s)\partial_\alpha g^{-1}(s), \tag{3}$$

$$\psi(x, y) = f(s)\Omega\psi(x, s^{-1}y)f^{-1}(s), \tag{4}$$

where Σ^α_β is the Jacobian matrix for the transformation s and Ω is the Jacobian matrix plus a local Lorentz rotation in the tangent space which is needed for the fermions when they transform in a curved space.

These conditions imply certain constraints that the D-dimensional fields have to obey. The solution of these constraints will provide us with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction. The simplest case occurs when the group S is abelian and $R = I$. In that case one can immediately solve the constraints choosing a gauge where the fields are explicitly independent of y. However in the general case the internal symmetry G and the space symmetry S combine in a non-trivial way as it is dictated by Eqs (2)–(4).

From Eq. (2) it follows that the components $A_\mu(x, y)$ of the initial gauge field $A_M(x, y)$ become after dimensional reduction the four-dimensional gauge fields and furthermore they are independent of y. In addition one can find that they have to commute with the elements of the R subgroup of G. Thus the four-dimensional gauge group H is the centralizer of R in G. We note here that this particular constraint results in a reduction of the number of the degrees of freedom as a consequence of the non-trivial combination of the internal symmetry G and space symmetry S.

Similarly from Eq. (3) the $A_\alpha(x, y)$, denoted by Φ_α from hereon, components of $A_M(x, y)$ become scalar fields in four dimensions. These fields transform under R as a vector $\boldsymbol{v}$ specified by the embedding of $R \subset S$

$$\text{adj}(S) = \text{adj}(R) + \mathbf{v}. \tag{5}$$

Moreover $\Phi_a(x, y)$ act as intertwining operators connecting induced representations of R acting on G and S/R. This implies, exploiting Schur's lemma, that the transformation properties of the fields $\Phi_a(x, y)$ under H can be found if we express the adjoint irrep of G in terms of reps of (R, H)

$$\text{adj}(G) = \sum (r_k, h_k). \tag{6}$$

Then if $\mathbf{v} = \Sigma s_i$, where each s_i is an irrep of R, there is an h_k scalar multiplet for every pair (r_k, s_k) where r_k and s_k are identical irreps.

Turning now to the fermion fields [5] we see from Eq. (4) that they act as intertwining operators between induced representations of R acting on G and the tangent space of S/R, SO(d). The non-trivial solution of the constraints can be obtained if we proceed along the same lines as in the case of scalars. Applying Schur's lemma in this case, one has to decompose F, the representation of G in which the fermions are assigned, under $R \times H$

$$F = \sum (r_k, h_k), \tag{7}$$

then decompose the spinorial representations of SO(d) under R

$$\sigma_d = \sum \sigma_k. \tag{8}$$

It turns out that for every pair (r_k, σ_k) where r_k and σ_k are identical irreps there is an h_k multiplet of spinor fields in four-dimensional theory. In order, however, to obtain chiral fermions in the effective theory we have to impose some further requirements. First, if we start with Dirac fermions there is no possibility to obtain chiral fermions in four dimensions. Next we impose Weyl condition in D dimensions. This requirement forces us to assume that D (and d) is even. In D even we can find a basis of the Γ matrices in which $\Gamma^{D+1} = \gamma^5 \times \gamma^{d+1}$. Imposing the Weyl condition in D dimensions means that ψ has definite chirality under Γ^{D+1}. Let us choose $\Gamma^{D+1}\psi = \psi$. The eigenvalues of γ^5 and γ^{d+1} have now to be simultaneously 1 or -1. In other words in D even ψ can be written as the direct sum of Weyl spinors

$$\psi = \sigma_D + \sigma'_D, \tag{9}$$

with the following SO(4) $\times$ SO(d) branching rules:

$$\sigma_D = (2, 1, \sigma_d) + (1, 2, \sigma'_d); \quad \sigma'_D = (2, 1, \sigma'_d) + (1, 2, \sigma_d). \tag{10}$$

When we apply the Weyl condition we pick up one of the Weyl spinors in the decomposition (9). In order to proceed we have to decompose σ_d and σ'_d under R

$$\sigma_d = \sum \sigma_k; \quad \sigma'_d = \sum \sigma'_k \tag{11}$$

and apply the rules to obtain the unconstrained fields. Since we have already required D to be even there exist two possibilities, either $D = 4n$ or $D = 4n+2$. If $D = 4n$ there are two self-conjugate spinors σ_D and σ'_D, and the only possibility to obtain a chiral four--dimensional theory is to start with F complex. However in $D = 2+4n$ dimensions there is one non-self-conjugate spinor and the other is $\bar{\sigma}_D$. So now instead of Eq. (11) we have the decompositions $\sigma_d = \Sigma\sigma_k$ and $\bar{\sigma}_d = \Sigma\bar{\sigma}_k$. An important aspect to note here is that these decompositions would be the same if R had only real representations, furthermore they are also the same if $\operatorname{rank} R < \operatorname{rank} S$. So we choose S, R in such a way that $\operatorname{rank} S = \operatorname{rank} R$. Suppose now that we start from a vectorlike representation F. Then each term (r_k, h_k) in the decomposition of Eq. (8) will be either self-conjugate or it will have a partner $(\bar{r}_k, \bar{h}_k)$. According to the rule described in Eqs. (7), (8), if we consider σ_d, the four-dimensional left-handed representation $f_L = \Sigma h_k^{(L)}$ is not vectorlike since $\Sigma\sigma_j$ is not self-conjugate. In addition, from $\bar{\sigma}_d$, we have in four dimensions the right-handed representation $f_R = \Sigma\bar{h}_k^{(R)} = \Sigma h_k^{(L)}$. Therefore there will appear two sets of Weyl fermions with the same quantum numbers under H. These two sets of Weyl spinors can be identified if we further impose the Majorana condition. Note that in $D = 2+4n$ the Majorana condition is diagonal with respect to the Weyl one so there is no problem in imposing both at the same time. Furthermore if F is to be real then we have to have $D = 2+8n$ while for F pseudoreal $D = 6+8n$.

Next let us obtain the four-dimensional effective action. Assuming that the metric is block diagonal,

$$g^{MN} = \begin{pmatrix} g^{\mu\nu}(x) & 0 \\ 0 & \frac{1}{L^2} g^{\alpha\beta}(y) \end{pmatrix}, \tag{12}$$

taking into account all the constraints and integrating out the extra coordinates we obtain in four dimensions the following Lagrangian:

$$A = C\int d^4x\{-\tfrac{1}{4}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})+\tfrac{1}{2}\sum_a \operatorname{Tr}(D_\mu\Phi_a D^\mu\Phi_a)-V(\Phi)\}$$

$$+C\int d^4x\,\frac{i}{2}\,\{\bar{\psi}\Gamma^\mu D_\mu\psi+\bar{\psi}\Gamma^a D_a\psi\}, \tag{13}$$

where C is the volume of the coset space and the potential $V(\Phi)$ is given by the formula

$$V(\Phi) = \tfrac{1}{4}\, g^{ac}g^{bd}\operatorname{Tr}\{(f_{ab}^{D}\Phi_D-[\Phi_a, \Phi_b])\,(f_{cd}^{E}\Phi_E-[\Phi_c, \Phi_d])\}, \tag{14}$$

where the indices D and E run over S. Furthermore D_μ, D_a are the appropriate covariant derivatives, $D_\mu = \partial_\mu - iA_\mu$, $D_a = \partial_a - \frac{i}{4}\,\Theta_{abc}\Sigma^{bc} - i\Phi_a$, where Θ_{abc} is the connection of S/R.

The above expression for $V(\Phi)$ is only formal because Φ_a must satisfy the constraints coming from Eq. (3)

$$f_{ai}^{D}\Phi_D-[\Phi_a, \Phi_i] = 0, \tag{15}$$

where the index i runs only in the R-subalgebra of S. These constraints imply that some components Φ_a's are zero, some are constants and the rest can be identified with components of the genuine Higgs fields. When $V(\Phi)$ is expressed in terms of the unconstrained independent Higgs fields, it remains a quartic polynomial which is invariant under gauge transformations of the final gauge group H, and its minimum determines the vacuum expectation value of the Higgs fields.

It turns out that the minimization of the potential is a difficult problem in the analysis of such theories, despite the progress that has been made [7]. However, if S has an isomorphic image S_G within G which contains R_G in a consistent way, then it is possible to allow the Φ_a to become generators of S_G. In this case the potential $V(\Phi)$ is zero, which is clearly the absolute minimum value that it can take. These non-zero vacuum expectation values of the Higgs fields break the symmetry from H to K, where all elements of K commute with the generators of S_G. Thus one deduces that the final unbroken symmetry group K is just the centralizer of S_G in G [6]. Recently it was shown that the fermions of such a theory become all massive after spontaneous symmetry breaking [12]. Therefore this attractive possibility is ruled out in searches to obtain the standard model from C.S.D.R. of a gauge group in higher dimensions. Thus we return to the original problem, namely to derive the scalar potential of the four-dimensional theory in terms of the genuine Higgs fields and minimize it. We should also note here that if the space S/R is symmetric then the potential is such that spontaneous symmetry breaking occurs and so H breaks further [13].

3. *The geometrical and topological properties of coset spaces*

Investigating the geometry and topology of the coset spaces turns out to be quite useful since important new features arise that can be used in physical applications [10]. Let us start by specifying the coset algebra S/R.

We assume that S/R is reductive but non-symmetric in general, so we can divide the generators of S, Q_A in two sets: the generators of R, Q_i ($i = 1, \ldots, \dim R$), and the generators of S/R, Q_a ($a = \dim R+1, \ldots, \dim S$). Then the commutation relations for S are the following:

$$[Q_i, Q_j] = f_{ij}^{k}Q_k, \quad [Q_i, Q_a] = f_{ia}^{b}Q_b, \quad [Q_a, Q_b] = f_{ab}^{i}Q_i+f_{ab}^{c}Q_c. \tag{16}$$

If S/R is symmetric then $f_{ab}^{c} = 0$.

In general the metric of the coset space need not be the usual Cartan-Killing metric of S restricted on S/R. The most general, S-invariant metric on S/R satisfies [17]:

$$f_{ia}^{b}g_{db}+f_{id}^{b}g_{ab} = 0. \tag{17}$$

This condition allows the introduction of different scales (radii) for each R-irrep of the coset. To see this note that $(f_i)_a^b$ serve as a matrix representation of R (one can prove that using the Jacobi identities of the coset) which in general is reducible. One can write $(f_i)_a^b$ in block form, these blocks in a way represent R-irreps and consequently Eq. (17) forces the radii to be the same in each block. It should be noted that the coset indices a, b, c, are raised and lowered employing the metric g_{ab}, so such operations change in general the antisymmetry properties of the structure constants since they involve different radii. We choose the structure constants defined by Eq. (16) to be fully antisymmetric.

The introduction of more than one radii in the coset has drastic consequences in the theory. As we can see from Eq. (14) the metric and therefore the radii, are involved in the calculation of the potential and as we shall see in an example in the next Section they can change the symmetry breaking pattern.

Another important issue that arises from the examination of the topological properties of the coset spaces is connected with the discrete groups that act freely on them. This is related with the use of the Wilson flux breaking mechanism in C.S.D.R. [10]. Dividing $B_0 = S/R$ by a freely acting discrete group $F^{S/R}$ makes $B = B_0/F^{S/R}$ non-simply connected with $\pi_1(S/R) = F^{S/R}$. This means that there will be contours not contractible to a point due to holes in the manifold. It turns out that the surviving fields have to be invariant under the diagonal sum $F^{S/R} \oplus T^G$, where T^G is the subgroup of G which is generated by the Wilson loops [10, 11]. In addition the final unbroken gauge group is the centralizer of T^G in G.

In the following we examine according to Ref. [10] the discrete groups which act freely on coset spaces $B_0 = S/R$, which satisfy rank R = rank S. By freely acting we mean that for every element $g \in F$ except the identity, no points of B_0 remain invariant.

There are two classes of freely acting groups F on B_0. The first of them is just the center $Z(S)$ of S or its subgroups. The action of $Z(S)$ on S/R is clearly free. Furthermore in the decomposition of the adj$S$ under R, every single representation of R and therefore the vector of R transforms to itself under the action of the center. In order to accommodate fermions in the scheme one has also to embed $Z(S)$ in SO(d). An obvious choice of discrete symmetry in SO(d) is the center or a Z_2 subgroup of it. Under this choice every single R-irrep in the decompositions of the vector and the spinor of SO(d) will transform to itself.

The second class of discrete symmetries which act freely on B_0 is $W = W_S/W_R$ where W_S, W_R are the Weyl groups of S, R respectively. In order to see that W acts freely on the coset space S/R, consider the Lie group S and its maximal abelian subgroup A. It is known that the Weyl group W_S acts freely on S/A. As was noted in Section 2, in the physically interesting case, we have in addition that rank S = rank R and one can select A in such a way that $A \subset R \subset S$. If furthermore we extract from W_S the Weyl group of R then $W = W_S/W_R$ acts freely on the coset S/R.

We can illustrate how W acts on the representations of R, when we decompose the adjoint of S under R by an example. Consider the coset G_2/SU(3). The decomposition of G_2 under SU(3) is $14 = 8+3+\bar{3}$. The root diagram of G_2 consists of the root: (*i*) zero with degeneracy two (*ii*) $e_j - e_k$, $j \neq k = 1, 2, 3$ (e_i, $i = 1, 2, 3$ are the three-dimensional unit vectors) which exactly form the root diagram of SU(3) accounting for the 8 in the decomposition and (*iii*) $\pm(3e_j - e_1 - e_2 - e_3)$, $j = 1, 2, 3$ which form 3 and $\bar{3}$. Reflections in planes orthogonal to the long roots give W which in this case is just a Z_2 and transforms 3 to $\bar{3}$ and vice versa. Thus it is an outer automorphism of R. In order to maintain such a transformation in the representations of R when we are looking at the embedding of R in SO(6) we have to allow for outer automorphisms of SU(6) as well. In the above example the spinor and the antispinor of SO(6) decompose under SU(3) as $4 = 1+3$

and $\bar{4} = 1+\bar{3}$ and clearly, if one is considering transformations of SU(3) that interchange 3 and $\bar{3}$, then also 4 has to interchange with $\bar{4}$ in SO(6). Since, however, the outer automorphisms of SO(6) form just a Z_2, one cannot maintain such a property if W is bigger than Z_2.

4. An example

In order to illustrate the C.S.D.R. method and to demonstrate the possibilities as well as the difficulties in constructing realistic models, let us discuss a specific example. Consider a Yang-Mills-Dirac theory with gauge group E_8 defined in ten dimensions, which is *dimensionally reduced over the coset space* $S/R = \mathrm{Sp}(4)/(\mathrm{SU}(2)\times \mathrm{U}(1))_{\text{non-max}}$ [9].

The four-dimensional gauge group H will be, as we discussed, the centralizer of R in G. So we have to embed $\mathrm{SU}(2)\times\mathrm{U}(1)$ into G. This can be done by using the following decompositions:

$$E_8 \supset \mathrm{SU}(5)\times \mathrm{SU}(5); \qquad \mathrm{SU}(5) \supset \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1). \tag{18}$$

We choose $\mathrm{SU}(2)\times\mathrm{U}(1)$ in the latter decomposition to be the image of R in G. It is obvious from the abovet hat $H = C_{E_8}(\mathrm{SU}(2)\times\mathrm{U}(1)) = \mathrm{SU}(5)\times\mathrm{SU}(3)\times\mathrm{U}(1)$ is the four-dimensional gauge group.

In order to determine the surviving scalar fields we have to determine the branching rule of E_8 under $\mathrm{SU}(5)\times\mathrm{SU}(3)\times(\mathrm{SU}(2)\times\mathrm{U}(1))_R$. This is given by:

$$\begin{aligned} 248 = {} & [24, 1;\ 1(0)]+[1, 1;\ 1(0)]+[1, 8;\ 1(0)]+[1, 1;\ 3(0)] \\ & +[1, 3;\ 2(5)]+[1, \bar{3};\ 2(-5)]+[5, 1;\ 1(6)]+[\bar{5}, 1;\ 1(-6)] \\ & +[5, 3;\ 1(-4)]+[5, \bar{3};\ 2(1)]+[\bar{5}, \bar{3};\ 1(4)]+[\bar{5}, 3;\ 2(-1)] \\ & +[10, 1;\ 2(-3)]+[10, 3;\ 1(2)]+[\overline{10}, 1;\ 2(3)]+[\overline{10}, \bar{3};\ 1(-2)]. \end{aligned} \tag{19}$$

The decomposition of the adjoint of Sp(4) under $(\mathrm{SU}(2)\times\mathrm{U}(1))_{\text{non-max}}$ is

$$10 = 3(0)+1(0)+1(2)+1(-2)+2(1)+2(-1). \tag{20}$$

Therefore according to the rules given in Eqs. (5), (6) the surviving fields in the four-dimensional theory transform under $H = \mathrm{SU}(5)\times\mathrm{SU}(3)\times\mathrm{U}(1)$ as a complex $\beta = (\bar{5}, 3)_{-1}$ and a comples $\gamma = (10, 3)_2$.

In order to determine the fermionic content let us also start from the adjoint of E_8, so the decomposition of F under $R\times H$ is again given by Eq. (19). The decomposition of the spinor of SO(6) under $(\mathrm{SU}(2)\times\mathrm{U}(1))_{\text{non-max}}$ is

$$4 = 1(0)+1(2)+2(-1). \tag{21}$$

If we start from Weyl-Majorana spinors at ten dimensions the surviving fermions, according to the rule described in Eqs. (7)–(11), transform as $(\bar{5}, 3)_{-1}$, $(10, 3)_2$, $(24, 1)_0$, $(1, 1)_0$ and

$(1, 8)_0$ (in fact the $(24, 1)_0$, $(1, 1)_0$ and $(1, 8)_0$ become superheavy due to geometrical terms [20]).

Next we need to construct the potential in terms of the unconstrained scalar fields. To proceed we need also to specify the metric we are going to use. We can see from the decomposition of Eq. (20), and from an explicit calculation of Eq. (17) (using the fully antisymmetric structure constants of Sp(4) which are given by: $f_{13}^{6} = f_{14}^{5} = f_{23}^{5} = f_{24}^{6} = 1$, $f_{71}^{6} = -f_{72}^{5} = f_{81}^{5} = f_{82}^{6} = f_{91}^{2} = -f_{95}^{6} = f_{101}^{2} = f_{105}^{6} = 1$, $f_{78}^{9} = f_{103}^{4} = 2$, the indices 7, 8 9 and 10 refer to $SU(2) \times U(1)$ and the rest are in the coset) that the coset space at hand can accept two different radii. Therefore the metric has the form:

$$g_{ab} = \operatorname{diag}(R_1^{-2}, R_1^{-2}, R_2^{-2}, R_2^{-2}, R_1^{-2}, R_1^{-2}). \tag{22}$$

For the construction of the unconstrained scalar fields we need to introduce the E_8 and Sp(4) generators according to the decompositions under $R = SU(2) \times U(1)$. In correspondence with Eq. (20) we introduce the generators:

$$T_{Sp(4)} = \{T^{\varrho}, T_{+}, T_0, T_{-}, T_{-}^{r}, T_{+}^{r}\}. \tag{23}$$

Similarly corresponding to the first of the decompositions of Eq. (18)

$$E_8 \supset SU(5) \times SU(5), \qquad 248 = (5, \overline{10}) + (\bar{5}, 10) + (10, 5) + (\overline{10}, \bar{5}) + (24, 1) + (1, 24)$$

we introduce the following generators of E_8

$$Q_{E_8} = \{Q_{ab}^{i}, Q_i^{ab}, Q^{ija}, Q_{ija}, Q_{I}^{\lambda}, Q_{II}^{\lambda}\}, \tag{24}$$

where $i, j, a, b = 1, \ldots, 5$; $\lambda = 1, \ldots 24$. In addition corresponding to the second decomposition of Eq. (18)

$$\begin{aligned} &E_8 \supset SU(5) \times SU(3) \times SU(2) \times U(1), \\ &248 = \ldots [1, 1;\ 1(0)] + [1, 8;\ 1(0)] + [1, 1;\ 3(0)] \\ &\qquad \ldots + [5, \bar{3};\ 2(1)] + \ldots + [\bar{5}, 3;\ 2(-1)] \\ &\qquad \ldots + [10, 3;\ 1(2)] + \ldots + [\overline{10}, \bar{3};\ 1(-2)] \end{aligned}$$

we introduce the generators:

$$Q_{E_8} = \{\ldots, Q_0, Q^{\lambda}, Q^{v}, \ldots, Q_{ar}^{i}, \ldots, Q_i^{ar}, \ldots, Q^{ija}, \ldots, Q_{ija}, \ldots\}. \tag{25}$$

The commutation relations of T's and Q's can be found in Refs. [9] and [18]. In order to express the formal Higgs potential of Eq. (14) in terms of the genuine Higgs fields β and γ, we need explicit expressions for the scalar fields Φ_s. Again the decomposition in Eq. (20) suggests the following change in the notation of the scalar fields Φ_s:

$$\{\Phi_s\colon = 1, \ldots, 10\} - \{\Phi^{\varrho}, \Phi_{+}, \Phi_0, \Phi_{-}, \Phi_{-}^{r}, \Phi_{r+}\}. \tag{26}$$

Then the solution of the constraints in Eq. (15) in terms of the genuine Higgs fields is

$$\begin{aligned} &\Phi_0 = \sqrt{15}\, Q_0,\ \Phi^{\varrho} = Q^{\varrho},\ \Phi_{+} = \beta^{ija} Q_{ija},\ \Phi_{-} = \beta_{ija} Q^{ija}, \\ &\qquad \Phi_{-}^{r} = \beta_a^i Q_i^{ar},\ \Phi_{r+} = \beta_i^a Q_{ar}^i, \end{aligned} \tag{27}$$

where $\varrho = 1, 2, 3$; $a = 1, 2, 3$; $i, j = 1, \ldots, 5$; $r = 1, 2$. Now we can rewrite the potential given in Eq. (14) taking into account the solution of the constraints given in Eq. (27) as:

$$\begin{aligned} V(\Phi, R_1, R_2) &= (1/2R_1^4)\,\mathrm{Tr}\,(\Phi^\varrho\Phi^\varrho + \Phi_0^2) + (1/R_2^4)\,\mathrm{Tr}\,(\Phi_0^2) \\ &+ (1/R_1^4 - 2/R_2^4)\,\mathrm{Tr}\,(\Phi_+\Phi_-) + 1/R_1^2(1/R_2^2 - 2/R_1^2)\,\mathrm{Tr}\,(\Phi_-^r\Phi_{r+}) \\ &+ (1/2R_1^2)(1/R_1^2 + 2/R_2^2)\,\mathrm{Tr}\,\{\varepsilon_{rs}\Phi_-[\Phi_{s+}, \Phi_{r+}] - \varepsilon^{sr}\Phi_+[\Phi_-^r, \Phi_-^S]\} \\ &+ (1/2R_1^4)\,\mathrm{Tr}\,\{[\Phi_-^r, \Phi_-^s][\Phi_{s+}, \Phi_{r+}] + [\Phi_-^r, \Phi_{s+}][\Phi_-^s, \Phi_{r+}]\} \\ &+ 1/R_1^2R_2^2\,\mathrm{Tr}\,\{[\Phi_-^r, \Phi_-][\Phi_+, \Phi_{r+}] + [\Phi_-^r, \Phi_+][\Phi_-, \Phi_{r+}]\} \\ &+ 1/2R_2^4\,\mathrm{Tr}\,\{[\Phi_-, \Phi_+][\Phi_-, \Phi_+]\}. \end{aligned} \tag{28}$$

Expressing the Φ's in Eq. (17) in terms of the genuine Higgs fields β and γ according to Eq. (16) we obtain the following potential in four dimensions:

$$\begin{aligned} V_{4\text{-dim}} &= (1/g^2)V(\beta, \gamma, R_1, R_2) \\ &= 18/(g^2R_1^4) + 30/(g^2R_2^4) \\ &+ (4/g^2)(1/R_1^4 - 2/R_2^4)\gamma^+\gamma \\ &+ (2/g^2R_1^2)(1/R_2^2 - 2/R_1^2)\beta^+\beta \\ &+ (4/g^2R_1^4)[4(\beta^+\beta)^2 + (1/5)(\beta_\alpha^i\beta_i^\alpha\beta_\delta^j\beta_j^\delta)] \\ &+ (1/5g^2R_1^2R_2^2)[192\beta_\alpha^i\beta_j^\alpha\gamma_{ik\delta}\gamma^{ik\delta}) + 832\beta_\alpha^i\beta_j^\delta\gamma_{ik\delta}\gamma^{kj\alpha} \\ &+ 320\beta_\alpha^i\beta_i^\alpha\gamma_{jk\delta}\gamma^{jk\alpha}] \\ &+ (8/g^2R_2^4)[4\gamma_{ij\alpha}\gamma^{il\alpha}\gamma_{kl\delta}\gamma^{jk\delta} + \gamma_{ij\alpha}\gamma^{ij\delta}\gamma^{kl\delta}\gamma_{kl\alpha} - (\gamma^+\gamma)^2]. \end{aligned} \tag{29}$$

Before we examine further the above potential it is clear that Higgs fields β and γ are not appropriate for breaking the SU(5) part of the theory down to $\mathrm{SU(3)_c} \times \mathrm{SU(2)} \times \mathrm{U(1)}$, since none of them transforms as the adjoint of SU(5). It appears that this is a quite general phenomenon when G.U.T.'s are obtained in four dimensions via C.S.D.R. and for this reason it was suggested [10] the use of Wilson flux mechanism for this breaking. In order to apply the Wilson flux mechanism we consider instead of the manifold $B_0 = \mathrm{Sp(4)}/(\mathrm{SU(2)} \times \mathrm{U(1)})_{\text{non-max}}$ the $B = B_0/Z_2^{S/R}$, where $Z_2^{S/R}$ is the center of Sp(4). In that case the vector and the spinor transform to themselves under the action of the chosen discrete symmetry. Then as we have discussed in Sect. 3 we can embed the $Z_2^{S/R}$ into the gauge group $H = \mathrm{SU(5)} \times \mathrm{SU(3)} \times \mathrm{U(1)}$. We choose to embed it in a discrete subgroup of the U(1) that appears in the decomposition $\mathrm{SU(5)} \supset \mathrm{SU(3)} \times \mathrm{SU(2)} \times \mathrm{U(1)}$ and in such a way that leaves invariant all the matter fields. Therefore all the components of fermionic and scalar fields are $F^{S/R} \oplus T^G$ invariant and therefore the only effect of Wilson flux breaking mechanism is to break H down to $\mathrm{SU(3)} \times \mathrm{SU(2)} \times \mathrm{U(1)}$. We note that since at the quantum level of the four dimensional theory the coupling constants receive separate renormalizations, we can put the couplings of the last $SU(3) \times U(1)$ to zero [19].

Let us next return to the Higgs potential given in Eq. (29). We should emphasize the interesting possibility that the spontaneous symmetry breaking or not of the gauge symmetry depends on the ratio of the two radii of the coset space $Sp(4)/(SU(2) \times U(1))_{\text{non-max}}$. In particular it is important to stress the possibility of vanishing the mass term of the Higgs field which is responsible for the electroweak symmetry breaking. In that case the electroweak symmetry breaking is driven by radiative corrections and therefore can produce the required hierarchy between the compactification and the electroweak scales. It is unfortunate that this interesting possibility cannot be realized in the model under discussion since there exist two Higgs fields β and γ and therefore one has to consider the behavior at the vacuum of both of them. One can easily convince oneself that at most one can arrange (by choosing $\varrho = (R_1/R_2) < (\frac{1}{2})^{1/4}$) that β acquires v.e.v. while γ does not. Then one has the desirable situation that $SU(2) \times U(1)$ breaks down to $U(1)_{em}$, however the order parameter involved is superheavy as compared to $O(100\ \text{GeV})$ of the observed electroweak symmetry breaking.

5. *Conclusions*

The C.S.D.R. method for reducing dimensionally gauge theories, which are defined in higher dimensions, provides us with a very interesting framework for studying the detailed predictions of the resulting four-dimensional theories. Geometrical and topological properties of the coset spaces can be used in mechanisms of spontaneous breaking of the gauge symmetries. Of particular interest is the fact that a choice of radii in certain coset spaces can produce light fermion and scalar fields. In addition the discrete symmetries of coset recently spaces have been classified and can be used in the Wilson flux breaking mechanism. With these qualifications it seems that we are not far from constructing a realistic model, despite the fact that model building in C.S.D.R. is a very constrained game. Then inclusion of gravity in the picture by adding the Einstein Lagrangian to the Yang-Mills-Dirac one in higher dimensions is expected to provide us with compactifing solutions corresponding to the assumed manifold $M^4 \times S/R$ in the C.S.D.R. scheme.

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