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PERTURBATION OF SINE-GORDON SOLITONS ON A FINITE LINE

BY TH. W. RUIJGROK, B. SCHOENMAKER

Instituut voor Theoretische Fysica, Rijksuniversiteit, Utrecht*

AND K. SOKALSKI

Institute of Physics, Jagellonian University, Cracow**

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We study the temporal behaviour of small perturbations of static kink-solutions of the sine-Gordon equation on a finite line. Two classes of solutions are found. The first contains all oscillatory modes of the N kinks present, while the second class describes the scattering of phonons by these kinks. For each solution the oscillation frequency is calculated and plotted as a function of the length of the line.

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1. *Introduction*

The theory of sine-Gordon solitons on a finite line is of importance for a number of phenomena in condensed matter physics. About ten years ago a few papers [1-2] were published in which some exact solutions of the sine-Gordon equation with definite boundary conditions at the end points of the line were exhibited and studied.

To the best of our knowledge, however, no classification of all possible standing waves was ever presented for this system.

It is the purpose of the present paper to give such a classification by showing how small amplitude oscillations of and around static kinks behave. This will be done in the next Section.

In order to prepare the grounds, however, we first give a brief recapitulation of the method used to derive the above mentioned exact solutions.

* Address: Instituut voor Theoretische Fysica, Princetonplein 5, P.O. Box 80.006, 3508 TA Utrecht, The Netherlands.

** Address: Instytut Fizyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland.

In close analogy with the Ansatz of Lamb [3] we try to find a solution of the sine-Gordon equation

$$\Phi_{xx}-\Phi_{tt} = \sin \Phi(x, t) \tag{1.1}$$

in the form

$$\Phi(x, t) = 4 \tan^{-1}[F(x)G(t)]. \tag{1.2}$$

On substitution into (1.1) this gives the following third order polynomial equation in F:

$$[G^3-G^2G''+2G(G')^2]F^3+G^3F''F^2-[2G^3(F')^2+G+G'']F+GF'' = 0, \tag{1.3}$$

where the primes indicate differentiation with respect to x or t. On dividing this equation by G, differentiating with respect to t and dividing by F, we obtain a second order polynomial equation in F:

$$[G^2-GG''+2(G')^2]'F^2+(G^2)'F''F-[2(G^2)'(F')^2+(G''/G)'] = 0. \tag{1.4}$$

In order for these equations to have a common root, the coefficients must satisfy a certain condition, first derived by Sylvester [4]. This could be used to obtain differential equations for $F(x)$ and $G(t)$ separately.

Instead we differentiate Eq. (1.4) with respect to x and divide by $(F^2)'(G^2)'$. This gives an equation in separated form, from which we obtain after integration

$$FF'' = -\mu'F^2+2(F')^2-2\lambda \tag{1.5a}$$

and

$$GG'' = (1-\mu')G^2+2(G')^2+2\kappa, \tag{1.5b}$$

in which μ' is a separation constant and λ and κ are integration constants. Substitution of these expressions for F'' and G'' into Eq. (1.3) again leads to an equation in separated form, from which we find

$$(F')^2 = -\kappa F^4+\mu F^2+\lambda \tag{1.6a}$$

and

$$(G')^2 = -\lambda G^4+(\mu-1)G^2+\kappa. \tag{1.6b}$$

The constants μ and μ' must be equal in order for Eqs. (1.5) and (1.6) to be consistent.

We have given the details of the derivation of Eq. (1.6) because the method is rather general and was used to answer the question whether solutions of the form of Eq. (1.2) exist, but with the factor 4 replaced by some other power of 2. For 2, 8 and 16 the answer is negative, whereas for other factors it is unknown. Also in the theory of surfaces with constant negative curvature [5] only the factor 4 plays a rôle.

Introducing scaling parameters (as in Ref. [1]) through the equations

$$u = \beta x \qquad \tau = \omega t \qquad Af(u) = F(x) \qquad g(\tau) = G(t) \tag{1.7}$$

Eqs. (1.6) take the same form as in Ref. [2]

$$(f')^2 = -(\kappa A^2/\beta^2)f^4+(\mu/\beta^2)f^2+(\lambda/\beta^2 A^2), \tag{1.8a}$$

$$(g')^2 = -(\lambda/\omega^2)g^4+((\mu-1)/\omega^2)g^2+\kappa/\omega^2. \tag{1.8b}$$

They are solved by Jacobi elliptic functions with variables u and τ and moduli k_f and k_g respectively. Since we are interested only in standing waves with N kinks, i.e., with $\Phi(0, t) = 0$ and $\Phi(L, t) = 2\pi N$, we must identify the functions $f(u)$ and $g(\tau)$ with $f(u) = \mathrm{sc}\,(u, k_f)$ and $g(\tau) = \mathrm{dn}\,(\tau, k_g)$. These functions satisfy [6]

$$\left[\frac{d}{du}\,\mathrm{sc}\,(u, k_f)\right]^2 = (1-k_f^2)\mathrm{sc}^4(u)+(2-k_f^2)\mathrm{sc}^2(u)+1 \tag{1.9a}$$

and

$$\left[\frac{d}{d\tau}\,\mathrm{dn}\,(\tau, k_g)\right]^2 = -\mathrm{dn}^4(\tau)+(2-k_g^2)\mathrm{dn}^2(\tau)-(1-k_g^2). \tag{1.9b}$$

Comparison of the coefficients of Eqs. (1.8) and (1.9) gives six relations between the eight parameters κ, μ, λ, A, β, ω, k_f, k_g. Another relation is obtained from the boundary condition at $x = L$. Since $\mathrm{sc}\,(u, k_f)$ varies from 0 to $\pm\infty$ over u-intervals of length $K(k_f)$, the

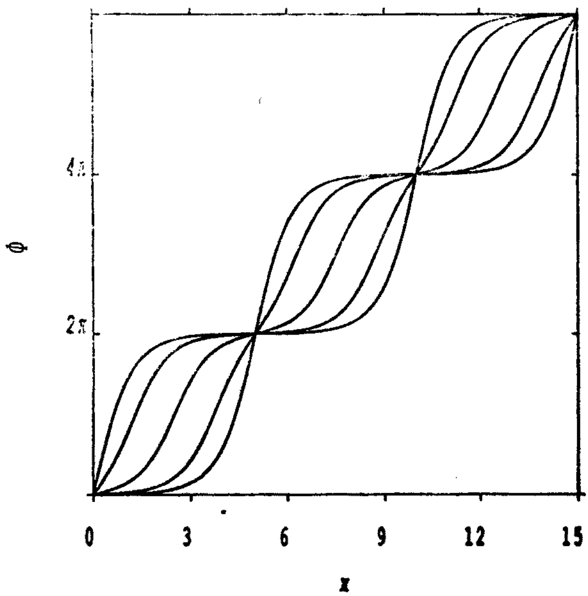

Fig. 1. $\Phi(x, t) = 4\tan^{-1}\,[A\,\mathrm{sc}(\beta x, k_f)\,\mathrm{dn}(\omega t, k_g)]$ for five values of t, covering half a period. $N = 3$, $L = 15$ $A = 0.99$

complete elliptic integral of the first kind, $\Phi(x, t)$ increases by 2π over the corresponding x-interval. For N kinks there is therefore the additional condition $\beta L = NK(k_f)$. So, for given values of L and N there is just one free parameter which controlls the amplitude of the N-kink solution. An example is given in Fig. 1, from which it is clearly seen that consecutive kinks are oscillating with opposite phase. Solutions with other phases, although with small amplitudes, will be constructed in the next Section.

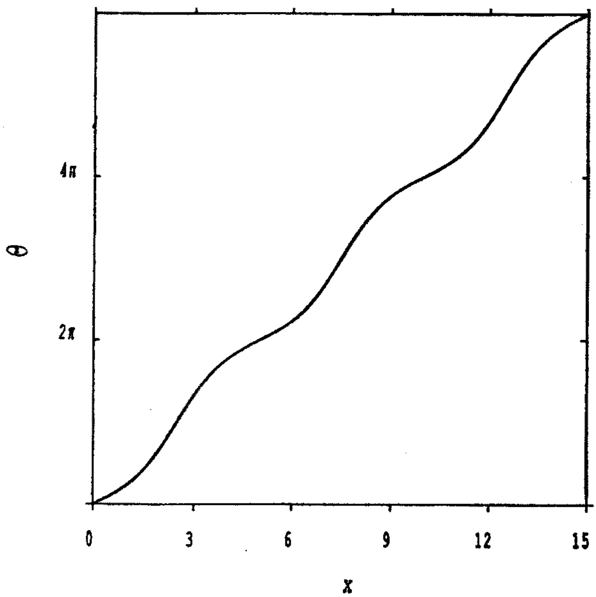

Fig. 2. The static solution Eq. (1.10) for $N = 3$ and $L = 15$

For $k_g = 1$ one has $\mathrm{dn}(\tau, k_g) = 1$, so that we obtain a static N-kink solution in the form

$$\phi(x) = \theta(x) = 4\tan^{-1}[A\,\mathrm{sc}(\beta x, k)], \tag{1.10}$$

where now, for given L and N, the parameters A, β and k are uniquely determined by $A^2 = k'$, $\beta = (1-k')^{-1}$, $k^2+k'^2 = 1$ and $\beta L = NK(k)$.

This static solution, which is unique and has N equidistant kinks, will be used as a starting point in the next Section. For $N = 3$ and $L = 15$ it is shown in Fig. 2.

2. *Perturbation theory*

In this Section we will construct solutions of Eq. (1.1), which are of the form

$$\phi(x, t) = \theta(x)+\psi(x)\cos(\omega t), \tag{2.1}$$

with $\psi(x)$ small compared to the static solution $\theta(x)$. This $\theta(x)$ satisfies $\dfrac{d^2\theta}{dx^2} = \sin\theta(x)$, or after one integration

$$\left(\frac{d\theta}{dx}\right)^2 = 2[c_N(L)-\cos\theta(x)], \tag{2.2}$$

where the integration constant $c_N(L) = 1+\frac{1}{2}\left(\dfrac{d\theta}{dx}\right)_{x=0}$ can be obtained from the explicit soluton (1.10), with the result

$$c_N(L) = 1+8\beta^2 A^2. \tag{2.3}$$

For very large L this $c_N(L)$ approaches one.

Substitution of the form (2.1) for $\Phi(x, t)$ into (1.1) and neglecting higher powers of $\psi(x)$, leads to

$$-\frac{d^2\psi(x)}{dx^2} + \cos\theta(x)\psi(x) = \omega^2\psi(x). \tag{2.4}$$

This is the Schrödinger equation for a particle in the potential $V(x) = \cos\theta(x)$, with boundary conditions $\psi(0) = \psi(L) = 0$. For most x this potential is not very much different from unity, except near each of the N kinks, where $\theta(x)$ makes a full turn of 2π. We therefore expect to find N eigenfunctions which are localized around the positions of the kinks, whereas all other eigenfunctions will have a wavelike character. We will call them bound states and phonon states respectively.

Since the equation (2.4) for $\psi(x)$ is linear we can form linear combinations of the eigenfunctions and describe arbitrary, but small distortions of the static N-kink state.

In order to find the eigenfrequencies ω_n we have to solve Eq. (2.4) numerically. This, however, is greatly facilitated by the fact that the static N-kink solution $\theta(x)$ is monotonically increasing with x. We can therefore use θ as independent variable. In doing so Eq. (2.4) is replaced by

$$2[c_N(L)-\cos\theta(x)]\frac{d^2\psi}{d\theta^2} + \sin\theta\frac{d\psi}{d\theta} - \cos\theta\psi(\theta) = -\omega^2\psi(\theta). \tag{2.5}$$

This is again an eigenvalue problem, now with boundary conditions $\psi(\theta = 0) = \psi(\theta = 2\pi N) = 0$, but without the occurrence of the complicated function $\theta(x)$ of Eq. (1.10). All dependence on L and N is in the number $c_N(L)$, which can be calculated easily, and in the boundary conditions.

If we label the eigenfunctions $\psi_n(\theta)$ with the number of nodes in the interval $0 < \theta < 2\pi N$, the corresponding eigenfrequencies ω_n will increase with n. We expect $\psi_0, \psi_1, \ldots, \psi_{N-1}$ to be localized (bound) states and ψ_n phonon states when $n \geqslant N$.

Among all these states $\psi_{N-1}(\theta)$ plays a special rôle, because it can be given in closed form. It is easily verified that this function and the corresponding eigenvalue are given by

$$\psi_{N-1}(\theta) = \sin\tfrac{1}{2}\theta \quad \text{and} \quad \omega_{N-1}^2 = \tfrac{1}{2}[c_N(L)-1] = 4\beta^2 A^2. \tag{2.6}$$

It has indeed $N-1$ nodes in the interval $(0, 2\pi N)$. For fixed N and large L the first N states become degenerate. In this case $c_N(L) \Rightarrow 1$ and from Eq. (2.6) it then follows that $\omega_n \Rightarrow 0$ for $n < N$. The other (phonon) states, if they really are not localized, should for $L \Rightarrow \infty$ feel a potential $\cos\theta(x)$, which is practically everywhere equal to one. Therefore $\lim_{L\Rightarrow\infty} \omega_n = 1$ should hold for all $n \geqslant N$.

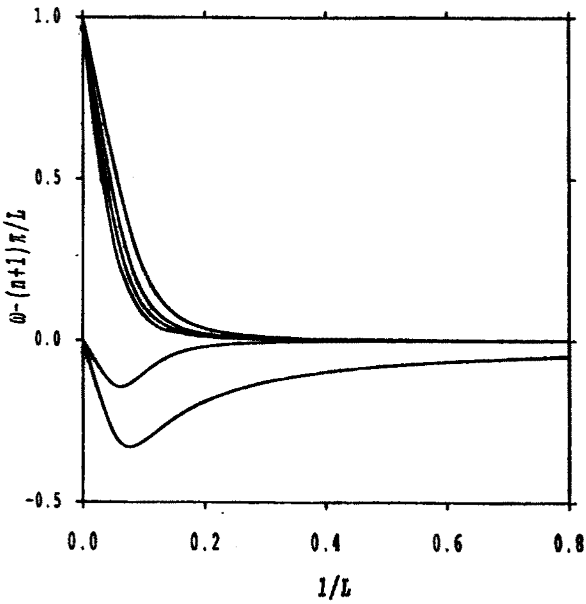

Fig. 3. Frequency shift for $N = 2$ versus $1/L$

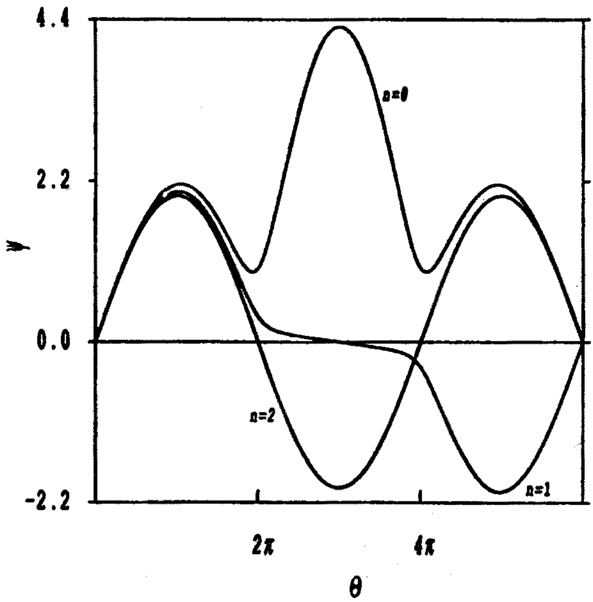

Fig. 4. All bound states $\psi_n(\theta)$ for $N = 3$ and $L = 15$

By numerical calculation this behaviour was verified indeed, as can be seen from Fig. 3, where we have plotted the difference $\omega_n - \omega_n^0$ as a function of $1/L$. The numbers $\omega_n^0 = (n+1)\pi/L$ are the frequencies of the free modes $\psi_n^0(x) = \sin(n+1)\pi x/L$, which are the solutions of the Schrödinger equation (2.4), when the potential $V(x) = \cos\theta(x)$ is replaced by zero. This is a good approximation for small L, since in that case the effect of gravity can be neglected.

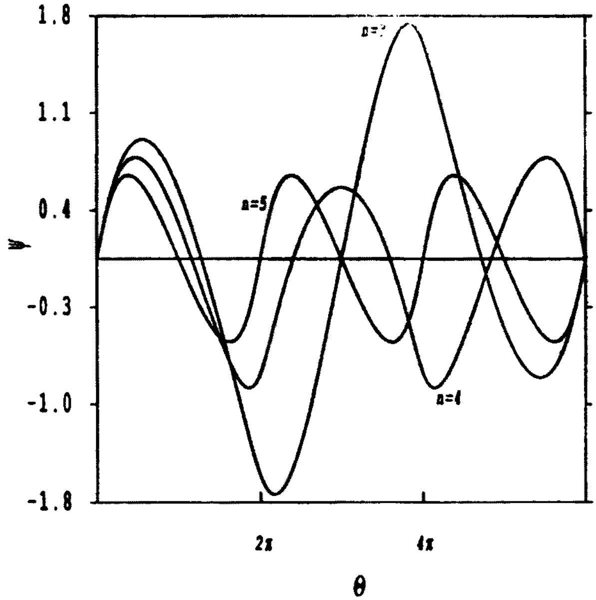

Fig. 5. Some phonon states $\psi_n(\theta)$ for $N = 3$ and $L = 15$

The highest bound state $\psi_{N-1}(\theta)$ is special in another respect too, because it is also obtained when the exact solution

$$\phi(x, t) = 4\tan^{-1}[A\,\mathrm{sc}\,(\beta x, k_f)\,\mathrm{dn}\,(\omega t, k_g)], \tag{2.7}$$

found in the introduction, is expanded in powers of k_g, taking into account that for fixed L and N, the other parameters are functions of this k_g. For $k_g = 0$ the static solution $\theta(x)$ is recovered, while the next nonvanishing term turns out to be proportional to $\sin\frac{1}{2}\theta(x)$. The exact solutions corresponding to the approximation $\Phi(x, t) = \theta(x) + \psi_n(x)\cos\omega t$ for $n \neq N-1$ are not known.

The character of the oscillatory kink modes is illustrated in Fig. 4, where for $L = 15$ and $N = 3$ we have plotted $\psi_n(\theta)$ for $n = 0, 1, 2$. Notice that $\psi_2(\theta)$ is indeed equal to $\sin\frac{1}{2}\theta$.

For $n \geqslant N$ the solutions $\psi_n(x)$ have the appearance of (standing) phonon modes being scattered by N static kinks, which themselves are not influenced by this interaction. Fig. 5 shows $\psi_n(\theta)$ for $n = 3, 4, 5$ and $L = 15$ and $N = 3$.

3. *Conclusions*

In this paper we have shown how a small perturbation of the static N-kink solution of the sine-Gordon equation on a finite line propagates in time.

The prescription for its calculation is very much the same as for the propagation of a Schrödinger wave packet moving in a potential.

The initial disturbance is first written as a linear superposition of the functions $\psi_n(x)$, constructed in the previous Section. Then each of them is multiplied by the corresponding factor $\cos \omega_n t$, after which a resummation produces the state at any later time.

The first N terms in the series describe how energy is transferred to collective motion of the kinks, whereas the remaining part can be interpreted as elastic phonon scattering.

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