EQUIVALENCE OF DIRAC QUANTIZATION AND SCHWINGER'S ACTION PRINCIPLE IN THE TWO-DIMENSIONAL MASSLESS ELECTRODYNAMICS

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We present a solution for the Schwinger model: quantum electrodynamics in two dimensions for massless fermions, in the canonical Hamiltonian formalism. The equivalence of the Dirac quantization procedure for constrained systems and the Schwinger's action principle is established.

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1. Introduction

In the fifties, two approaches to obtain the canonical (anti)commutation relations in quantum theories involving constraints were developed almost simultaneously by Dirac [1] and Schwinger [2]. It was only last year though that a formal proof establishing the equivalence of these methods appears [3]. Then reason for such a delay probably stems from the quite different techniques employed by these two researchers: while Dirac proposed a modification of the canonical classical theory to get rid of the constraints prior to quantization, the Schwinger's action principle works with quantum states and operators from the beginning.

In this paper we illustrated this equivalence, in the context of quantum field theory, working in a toy model in $(1+1)$ dimensions of space-time which possess an exact solution: the Schwinger model. In Section 2 we present the canonical Hamiltonian formulation of the Schwinger model and make use of the Dirac brackets to map the classical theory in its quantum analogue. We will develop our Hamiltonian formalism starting from the bosonized version of the model. Then we show that the spectrum of the theory consists of a free massive mode with the mass proportional to the coupling constant. These result are shown to follow also from the Schwinger's action principle in Section 3. In Section 4 we include a discussion of the covariance of these operations through the computation of the Poincaré generators and explicit verification of their algebra.

2. *Dirac quantization of the Schwinger model*

Let us start with the Lagrangian density for the Schwinger Model (SM) [4]

$$L = \bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi - \tfrac{1}{4}F_{\mu\nu}^2. \tag{2.1}$$

Using the bosonization dictionary of Kogut and Susskind [5],

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi = \tfrac{1}{2}(\partial_\mu\phi)^2, \tag{2.2a}$$

$$\bar{\psi}\gamma^\mu\psi = \frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu\phi, \tag{2.2b}$$

we obtain the bosonized version for this model as

$$L = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{e}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu\phi A_\mu - \tfrac{1}{4}F_{\mu\nu}^2. \tag{2.3}$$

We apply the Hamiltonian formalism in this bosonized theory. One must be advised that the theory already contains some quantum effects due to the incorporation of the chiral anomaly in the bosonization process.

The canonical momenta obtained from (2.3) are

$$\pi \equiv \frac{\delta L}{\delta\dot{\phi}} = \dot{\phi} + gA_1, \tag{2.4a}$$

$$\pi^\mu \equiv \frac{\delta L}{\delta\dot{A}_\mu} = F^{\mu 0}. \tag{2.4b}$$

The momentum conjugated to A_1 is the electric field on the line

$$E = \pi_1 \tag{2.5}$$

and the canonical Poisson brackets are

$$\{A_0(x), \pi_0(y)\} = \{E(x), A_1(y)\} = \{\phi(x), \pi(y)\} = \delta^2(x-y).$$

From (2.4) we recognize $\pi_0 \approx 0$ as a primary constraint of the system. Here $\approx$ denotes *weak equality* in Dirac's terminology, which just means that all (Poisson) brackets must be calculated before using the constraints, The primary Hamiltonian is

$$H = \int dx\, \tfrac{1}{2}\left[\pi^2 + E^2 + (\partial_1\phi)^2 + g^2A_1^2 - 2g\pi A_1 + 2A_0G + 2\lambda\pi_0\right], \tag{2.6}$$

where λ is a Lagrange multiplier field and also we have defined

$$g = \frac{e}{\sqrt{\pi}}, \qquad G = \partial_1 E + g\partial_1\phi.$$

In order to preserve the constraint π_0 under time evolution a new constraint will be necessary. This requirement leads to the secondary constraint

$$0 \approx \dot{\pi}_0 = G. \tag{2.7}$$

This is the Gauss law constraint. We now have a pair of first-class constraints, which reflects the underlying gauge invariance of the bosonized model. It is simple to verify that the functional

$$Q[\theta] = \int d^2x(\pi_0\dot{\theta}+G\theta) \tag{2.8}$$

is the generator of gauge transformations

$$A_\mu \to A_\mu+\partial_\mu\theta, \quad \phi \to \phi.$$

No further constraints are needed to keep the Gauss law invariant in time. On the other hand the equation of motion for the zero component of the gauge field will fix the value of the Lagrange multiplier λ.

$$\lambda = \dot{A}_0.$$

To apply the Dirac algorithm one must impose two gauge fixing conditions to obtain a set of second-class constraints. Working in the Coulomb gauge

$$\partial_1 A_1 \approx 0$$

we obtain as a secondary constraint

$$\nabla^2 A_0 \approx \partial_1 E. \tag{2.11}$$

The consistency of these gauge constraints is simple to check. Using the gauge function

$$\Lambda = -\frac{1}{\nabla^2}\partial_1 A_1$$

one obtains that

$$\partial_1 A_1' = \partial_1 A_1+\nabla^2\Lambda = 0, \quad \nabla^2 A_0' = \nabla^2 A_0+\nabla^2\dot{\Lambda} = \partial_1 E'$$

as required. It is easy to check that the set of constraints

$$\Omega_1 = \pi_0, \quad \Omega_2 = \partial_1 E+g\partial_1\phi, \quad \Omega_3 = \partial_1 A_1, \quad \Omega_4 = \nabla^2 A_0-\partial_1 E$$

is second-class and has the following inverse matrix

$$C_{\alpha\beta}^{-1} \equiv \langle\Omega_\alpha|\Omega_\beta\rangle^{-1} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\frac{1}{\nabla^2}\delta^2(x-y).$$

The Dirac brackets between any two dynamical variables is defined by

$$\{A, B\}^* = \{A, B\}-\int dzd\omega\{A, \Omega_\alpha\}C_{\alpha b}^{-1}\{\Omega_b, B\}. \tag{2.12}$$

The only non-vanishing Dirac brackets are

$$\{\phi(x), \pi(y)\}^* = \delta^2(x-y),$$

$$\{E(x), \pi(y)\}^* = -g\delta^2(x-y),$$

$$\{A_0(x), \pi(y)\}^* = -g\frac{\partial_1}{\nabla^2}\delta^2(x-y). \tag{2.13}$$

Once we have passed to the Dirac brackets all constraints can be set to zero strongly. After elimination of the redundant variables the Hamiltonian reads

$$H = \int dx\,\tfrac{1}{2}\left[g^2B^2+E^2+\frac{1}{g^2}(\partial_1 E)^2\right], \tag{2.14}$$

where we have introduced a new field $B = A_1 - \frac{1}{g}\pi$ to diagonalize the Hamiltonian. Using (2.13) one observes that E and B form a conjugated pair under DB operation

$$\{E(x), B(y)\}^* = \delta^2(x-y). \tag{2.15}$$

From (2.14) and (2.15) we can see that the electric field E is a free, massive field with mass $g = \frac{e}{\sqrt{\pi}}$

$$\ddot{E} = \{\{E, H\}^*, H\}^* = \nabla^2 E - g^2 E \tag{2.16}$$

or

$$(\partial^2+g^2)E = 0. \tag{2.17}$$

3. *Schwinger's action principle*

Let us show in this Section that the use of Schwinger's action principle produces the same set of commutators as in (2.16). Let G be the generator of unitary transformations as proposed by Schwinger and A be any of the dynamical variables of the problem. The constrained commutators are constructed in the following way

$$[A, G]^* = [A, G] - \int dy\lambda^\alpha(x, y)\delta\Omega_\alpha(y). \tag{3.1}$$

Here Ω_α represents all the constraints of the theory and λ^α are Lagrange multipliers fields to be determined from consistency conditions with the constraints. The first term in the right hand side gives the unconstrained variation for A whereas the second term substracts out the normal component and therefore forces the constrained variation (the left hand side) to stay on the hypersurface defined by the constraints. In this problem the generator G is given by

$$G = \int dy(E\delta A_1 - A_1\delta E + \pi_0\delta A_0 - A_0\delta\pi_0 + \pi\delta\phi - \phi\delta\pi). \tag{3.2}$$

Choosing $A = \phi$, calculating separately the right hand side and the left hand side of (3.1) and determining the Lagrange multipliers by imposing the constraints as consistency conditions one immediately arrives at

$$-i[\phi(x), \pi(y)]^* = \delta^2(x-y). \tag{3.3a}$$

Choosing $A = \pi$ and repeating the procedure above one obtains, after a straightforward algebra that

$$-i[E(x), \pi(y)]^* = -g\delta^2(x-y), \tag{3.3b}$$

$$-i[A_0(x), \pi(y)]^* = -g\frac{\partial_1}{\nabla^2}\delta^2(x-y) \tag{3.3c}$$

which is the quantum version of (2.13). All the conclusions of the previous section follow then from here.

4. Lorentz invariance

In the study presented in the former sections we have worked in the Hamiltonian formalism and a sequence of apparently non-covariant steps were performed on the system. This procedure might have provoked questions related to the explicit Lorentz invariance of the model, which have not yet been proved. It is well known that the model can be solved exactly starting from its Lorentz-covariant field equations and that the spectrum is explicitly relativistic.

In this Section we want to establish the Lorentz invariance for the Schwinger model. Since the Lorentz-invariance of the Hamiltonian formalism is never self-evident, such a symmetry, if present, must be uncovered by the explicit construction the Poincaré generators and verification of their algebra.

In a space-time of dimension two there is only one Lorentz generator, namely $M = M^{01}$ which produces boosts on the line. Therefore no Lorentz algebra is possible. On the other hand we have three generators for the Poincaré group and their algebra can be verified. In this way one is able to confirm that the relativistic invariance is present in the constrained Hamiltonian formalism regardless the sequence of apparent non-covariant steps performed to solve the model.

The three Poincaré generators are

$$H \equiv P^0 = \int dx T^{00}, \tag{4.1a}$$

$$P \equiv P^1 = \int dx T^{01}, \tag{4.1b}$$

$$M \equiv M^{01} = \int dx(tT^{01} - xT^{00} + \pi^\alpha A^\beta \Sigma^{01}_{\alpha\beta}). \tag{4.1c}$$

$T^{\mu\nu}$ is the energy-momentum tensor and $\Sigma^{01}_{\mu\nu}$ is the spin tensor. Using the results of Section 2 and

$$\Sigma^{01}_{\mu\nu} = \varepsilon_{\mu\nu} \tag{4.2}$$

one can write the Poincaré generators as

$$P = \int dx(-B\partial_1 E), \tag{4.3a}$$

$$M = tP - \int dx\left(xH + E\frac{\partial_1}{\nabla^2}E\right). \tag{4.3b}$$

Using the Dirac(or Schwinger) brackets (2.14) it is a straightforward algebra to check that these generators indeed satisfy the Poincaré algebra given by

$$\{P, H\}^* = \{M, M\}^* = 0, \tag{4.4a}$$

$$\{H, M\}^* = P, \tag{4.4b}$$

$$\{P, M\}^* = H. \tag{4.4c}$$

5. *Discussion*

We have studied the Schwinger model under the point of view of constrained systems. We have used both the Dirac bracket approach and the Schwinger action principle to solve the model and perform its quantization. We have in this way showed explicitly the equivalence of both methods as recently proposed by the authors of Ref. [3]. Finally we have verified the covariance for this solution by explicitly constructing the generators of the Poincaré group and checked their algebra.

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