ON THE "SQUARE ROOT" OF THE DIRAC EQUATION WITHIN EXTENDED SUPERSYMMETRY

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We discuss the possibilities of extraction of the "square root" of the Dirac equation within N-extended supersymmetry for construction of the more fundamental dynamical theory. The "square root" of the Dirac operator can be defined in N-extended superspace for $N \leqslant 2$, but it is imposible, in the framework of the standard demands to the field theory, to build a new dynamical model with it.

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It is notorious that the original Dirac's method, by which he derived his famous equation for the electron wave function, is based on the idea of factorization of the Klein--Gordon equation [1]. As a result, a new dynamical first-order equation is obtained from the dynamical second-order equation. The obtained equation is more fundamental in the sense that any wave function, which satisfies the (free) Dirac equation, automatically satisfies the (free) Klein-Gordon equation.

The remarkable simplicity and refinement of the Dirac's method are stimulating further attempts to extract a "square root", this time just of the Dirac equation. In particular, in a recent work [2] it was proposed to build a new theory by extraction of the "square root" of the Dirac equation in superspace of simple supersymmetry.

In this work we generalize a consideration of [2] to the case of $N = 2$ extended supersymmetry. It is amusing that analogous generalization to the case with $N > 2$ apparently does not exist. In $N = 2$ extended superspace, just as in $N = 1$ superspace, there exists an operator A built of covariant derivatives with dimension 1/2 (in units of mass), which can be considered as a "square root" of the Dirac operator $\hat{D}$. However, the use of this operator A for obtaining free field equations leads only to constraints which do not carry dynamical information. The interpretation of the operator A as of an auxiliary one, when used for writing down the dynamical system of equations for (free) superfields in the first-order formalism, in fact returns us back to the Dirac equation for superfields. Such "horizontal" supersymmetrization of the Dirac equation $\hat{D}\psi(x) = 0$ by naive replacement of the space-time argument x^μ by a total set of coordinates of (extended) superspace $z^M = (x^\mu, \theta^\alpha_i, \bar\theta^i_{\dot\alpha})$ is noncanonical and incompatible with usual principles of field theory construction, which displays itself, for instance, in appearance of unphysical states and in violation of ordinary spin-statistics connection. The proper supersymmetrization is "vertical", i.e. the spinor field $\psi(x)$ is included in a supermultiplet with other fields, for example scalar fields, equations of motion of which are the second-order Klein-Gordon ones, whereas equations of motion for $\psi(x)$ (Dirac equations) are the first-order ones [3]. The rest of the paper is devoted to expounding of details.

Let us consider an N-extended Poincaré superalgebra in four-dimensional space-time, which is defined by the following (anti) commutation relations [4] (the commutators with Lorentz generators are omitted):

$$
\begin{gathered}
\{Q^i_\alpha, Q^j_\beta\} = \{\bar Q_{\dot\alpha i}, \bar Q_{\dot\beta j}\} = 0, \\
\{Q^i_\alpha, \bar Q_{\dot\beta j}\} = 2\delta^i_j \sigma^\mu_{\alpha\dot\beta} P_\mu, \\
[Q^i_\alpha, B_r] = (b_r)^i_j Q^j_\alpha, \ [\bar Q_{\dot\alpha i}, B_r] = -\bar Q_{\dot\alpha j}(b_r)^j_i, \\
[P_\mu, B_r] = [P_\mu, Q^i_\alpha] = [P_\mu, \bar Q_{i\dot\alpha}] = 0, \\
[P_\mu, P_\nu] = 0, \\
[B_r, B_s] = i c^t_{rs} B_t.
\end{gathered} \tag{1}
$$

The initial letters of Greek alphabet are everywhere used for designation of spinor indices ($\alpha = 1, 2$; $\dot\alpha = \dot 1, \dot 2$), the letters from the middle of Greek alphabet — for vector indices ($\mu = 0, 1, 2, 3$), the letters from the middle of Latin alphabet — for indices of internal symmetry $U(N)$ ($i = 1, 2, \ldots, N$).

The B_r are Hermitian generators of internal symmetry $U(N)$ with structure constants c^t_{rs}. The rule of Hermitian conjugation for spinor charges Q reads

$$(Q^i_\alpha)^+ = \bar Q_{\dot\alpha i}. \tag{2}$$

Thus, if Q^i transform in some (usually fundamental) representation of $U(N)$, then $\bar Q_i$ transform in complex conjugated representation. Note, that the fundamental representation of $U(N)$ is real only for $N \leqslant 2$.

Supersymmetry is a "square root" of space-time Poincaré symmetry in the sense of (1). So it is natural to construct such a linear operator A of dimension 1/2, which would be a "square root" of the Dirac operator (of dimension 1) in extended superspace with coordinates $z^M = (x^\mu, \theta^\alpha_i, \bar\theta^j_{\dot\alpha})$.

In the two-component formalism used here for spinors the (free) Dirac equation is given by

$$\begin{pmatrix} i\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu & m \\ m & i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \end{pmatrix} \begin{pmatrix} \varphi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \equiv \hat D \begin{pmatrix} \varphi \\ \bar\chi \end{pmatrix} = 0. \tag{3}$$

Now let us demand a total set z^M of coordinates of N-extended superspace be an argument of wave functions φ and $\bar{\chi}$ ("horizontal" supersymmetrization). However, the operator $\hat{D}$ in no way affects the anticommuting coordinates of superspace. At the same time in the superspace there exist derivatives with lower dimension. That are just the well--know spinor covariant derivatives [4] which satisfy an algebra

$$\{D^i_\alpha, D^j_\beta\} = \{\bar{D}_{\dot{\alpha}i}, \bar{D}_{\dot{\beta}j}\} = 0,$$
$$\{D^i_\alpha, \bar{D}_{\dot{\beta}j}\} = 2\delta^i_j \sigma^\mu_{\alpha\dot{\beta}} i\partial_\mu \tag{4}$$

and admit the following realization in superspace:

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} - \frac{i}{2}\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}i}\partial_\mu, \tag{5}$$

$$\bar{D}^{\dot{\alpha}}_i = \frac{\partial}{\partial\bar{\theta}^i_{\dot{\alpha}}} - \frac{i}{2}\tilde{\sigma}^{\mu\dot{\alpha}\beta}\theta_{\beta i}\partial_\mu.$$

In $N = 1$ superspace a suitable operator A is known to exist [2]:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{D}^{\dot{\alpha}} & D^\alpha \\ -D_\alpha & \bar{D}_{\dot{\alpha}} \end{pmatrix}, \quad A^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} D^\alpha & -\bar{D}_{\dot{\alpha}} \\ \bar{D}^{\dot{\alpha}} & D_\alpha \end{pmatrix}. \tag{6}$$

Hence

$$AA^+ = \begin{pmatrix} i\tilde{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu & M_{(1)} \\ M_{(1)} & i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \end{pmatrix}, \tag{7}$$

where a Hermitian operator $M_{(1)}$,

$$M_{(1)} = \tfrac{1}{2}(\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} + D^\alpha D_\alpha) \tag{8}$$

plays the role of mass when acting on chiral superfield, $M^2_{(1)} = p^\mu p_\mu$.

The attempts to construct an analogous operator in N-extended superspace find difficulties, because D^i and D_j transform in unequivalent representations for $N \geqslant 3$. For $N = 2$ it is possible to raise and lower indices of internal symmetry by means of antisymmetrical invariant metric ε_{ij}, ε^{ij}, so that [5]

$$(D^i_\alpha)^+ = \bar{D}_{\dot{\alpha}i}, \quad (D^\alpha_i)^+ = -\bar{D}^{\dot{\alpha}i}. \tag{9}$$

The sought for operator in $N = 2$ superspace has a form

$$A = \tfrac{1}{2}\begin{pmatrix} \bar{D}^{\dot{\alpha}i} & D^{\alpha j} \\ D_{\alpha j} & \bar{D}_{\dot{\alpha}i} \end{pmatrix}, \quad A^+ = \tfrac{1}{2}\begin{pmatrix} -D^\alpha_i & -\bar{D}^j_{\dot{\alpha}} \\ \bar{D}^{\dot{\alpha}}_j & D^i_\alpha \end{pmatrix}. \tag{10}$$

Consequently,

$$AA^+ = \begin{pmatrix} i\tilde{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu & M^{ij}_{(2)} \\ M_{(2)ij} & i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \end{pmatrix}, \tag{11}$$

where the role of mass is played by an operator

$$M^{ij}_{(2)} = \tfrac{1}{4}(\bar{D}^{i}_{\dot\alpha}\bar{D}^{\dot\alpha j}+D^{\alpha i}D^{j}_{\alpha})$$
$$\equiv \tfrac{1}{4}(\bar{D}^{ij}+D^{ij}). \tag{12}$$

Having obtained a linear differential operator A, it is natural to consider a "new" set of linear equations:

$$AB = 0 \quad \text{or} \quad A^{+}F = 0 \tag{13}$$

In $N = 1$ superspace the superfields B and F have the form [2]

$$B = \begin{pmatrix} \Phi \\ V^{\dot\alpha}_{\alpha} \end{pmatrix}, \quad F = \begin{pmatrix} \psi_{\alpha} \\ \bar{\kappa}^{\dot\alpha} \end{pmatrix}, \tag{14}$$

therefore (3) can be displayed as follows [2]:

$$\bar{D}^{\dot\alpha}\Phi + D^{\alpha}V^{\dot\alpha}_{\alpha} = 0, \qquad -D_{\alpha}\Phi + \bar{D}_{\dot\alpha}V^{\dot\alpha}_{\alpha} = 0, \tag{15a}$$

$$D^{\alpha}\psi_{\alpha} - \bar{D}_{\dot\alpha}\bar{\kappa}^{\dot\alpha} = 0, \qquad \bar{D}^{\dot\alpha}\psi_{\alpha} + D_{\alpha}\bar{\kappa}^{\dot\alpha} = 0. \tag{15b}$$

There are several possibilities to choose B in $N = 2$ superspace

$$B_1 = \begin{pmatrix} \Phi_1 \\ V^{\dot\alpha i}_{1j\alpha} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \Phi^{j}_{2i} \\ V^{\dot\alpha}_{2\alpha} \end{pmatrix}, \quad B_3 = \begin{pmatrix} \Phi_{3i} \\ V^{\dot\alpha}_{3\alpha j} \end{pmatrix} \tag{16}$$

and, accordingly, F

$$F_1 = \begin{pmatrix} \psi_{1\alpha} \\ \bar{\kappa}^{\dot\alpha}_{1ij} \end{pmatrix}, \quad F_2 = \begin{pmatrix} \psi^{ij}_{2\alpha} \\ \bar{\kappa}^{\dot\alpha}_{2} \end{pmatrix}, \quad F_3 = \begin{pmatrix} \psi^{j}_{3\alpha} \\ \bar{\kappa}^{\dot\alpha}_{3i} \end{pmatrix}. \tag{17}$$

A replacement $A \leftrightarrow A^{+}$ does not bring anything essentially new. When constructing the equations, a minimal number of spinor indices in the formulae (14)–(17) was used.

At first sight the equations (13) look more fundamental, than Dirac equation, because if, for example, F satisfies (13), then

$$\hat{D}F = AA^{+}F = 0, \tag{18}$$

thinking that M assumes one if its eigenvalues. It is not difficult to write down a suitable action in $N = 1$ superspace

$$S_1 = -\int d^4x d^4\theta [F^{+}AB + \text{h.c.}] \tag{19a}$$

or in $N = 2$ superspace

$$S_2 = -\int d^4x d^8\theta [F^{+}AB + \text{h.c.}] \tag{19b}$$

and even introduce a gauge interaction in supersymmetrical way by means of further covariantization of covariant derivatives with respect to $N = 1$ or $N = 2$ Yang-Mills coupling

$$D_\alpha^{(i)} \to D_\alpha^{(i)} = D_\alpha^{(i)} + iA_\alpha^{(i)}, \qquad \bar{D}^{\dot\alpha}_{(i)} \to \bar{D}^{\dot\alpha} = \bar{D}^{\dot\alpha}_{(i)} - i\bar{A}^{\dot\alpha}_{(i)}, \tag{20}$$

taking account of a standard set of constraints on anticommutators of gauge-covariant derivatives in superspace [6, 7].

However, the equations (13) are, in fact, only constrainst, which express some components of the field through others and they do not lead to dynamical equations of motion. We have verified this statement by writting out explicitly the superfield equations (15) in components and solving the arising linear dependencies. We do not give the proof here, because it is extremely cumbersome and elementary. Analogous statement takes place for $N = 2$.

One can obtain a dynamical equation connecting, for instance, the fields B and F. On the grounds of dimensional considerations, the only version of such a connection is given by

$$B = A^+F, \tag{21}$$

which corresponds to Lagrangian density in superspace

$$L = -(F^+AB + \text{h.c.}) + B^+B. \tag{22}$$

The theory (22), as a matter of fact, is writing down of a theory with

$$L = -F^+(AA^+)F \tag{23}$$

in the first-order formalism, which returns us to "horizontal" supersymmetrization of the Dirac equation.

The summary of our discussion consists in conclusion that there are possible only two supersymmetrizations of the Dirac equation: "horizontal" (supersymmetrization of coordinates) and "vertical" (supersymmetrization of fields), or both of them together. One can realize the latter on graded supermanifolds considered recently in [8]. The use of "horizontal" supersymmetrization is always accompanied by an appearance of numerous "superfluous" fields with noncanonical dimensions and ghosts, which considerably hamper the use of such theories for physical applications.

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