MAGNETICALLY QUASI-BOUND SUPERPOSITRONIUM STATES

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The possibility of the existence of magnetically bound positronium states (superpositronium) is investigated starting from the two body Bethe-Salpeter equation. For the total angular momentum $j = 0$ quasi bound energy states are found in form of resonances of the cross-section calculated with the use of the phase of the scattering states. The energies are in agreement with the observed kinetic energy of the disintegration electrons or positrons in the heavy ion collision experiments. From the width of the resonances the life-time of the quasi-bound states can be estimated. Really bound superpositronium states could not be found.

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1. *Introduction*

Following A. O. Barut's idea [1] of the existence of magnetically bound particle-states with respect to elementary particle physics we have investigated the simplest system of this kind, namely the very narrow e^+e^--pair (superpositronium). On the other hand such considerations are specially interesting in view of the recent observations [2] of narrow coincident electron-positron lines from heavy-ion collisions at GSI. The experimental results point to the possibility, that the observed electrons and positrons can be interpreted as the relic of the decay of magnetically quasi-bound e^+e^--states. In this connection we can show until now, that starting from the relativistic two body Bethe-Salpeter theory in case of the quantum number $j = 0$ *quasi*-bound superpositronium states do exist; the energies for the desintegration electrons and positrons lie in the range up to 500 keV in good agreement with the last GSI-experiments [2, 3]. This result is so much the more of interest, as the observed lines cannot be understood by spontaneous pair production in supercritical fields [4]. For the life-time of the magnetically quasi-bound states we find approximately 10^{-20} sec in accordance with the observed width of the electron-positron lines. On the other hand really bound superpositronium states do not exist.

Of course, there are already other attempts for explaining strongly bound e^+e^--pairs [5, 6]; but these papers do not start from the unique Bethe-Salpeter equation but from more or less ad hoc constructed models.

2. *The relativistic 2-body problem*

From the two-time Bethe-Salpeter equation for a system of two spin 1/2-particles one gets the following "one-time Salpeter equation" [7] in the center of mass system ($\hbar = 1,\ c = 1$)

$$[E-(\vec{\alpha}^{(1)}\vec{p}+\beta^{(1)}m^{(1)})-(-\vec{\alpha}^{(2)}\vec{p}+\beta^{(2)}m^{(2)})-\Gamma(\vec{p})V(\vec{r})]\psi(\vec{r}) = 0. \tag{2.1}$$

(E is the energy eigenvalue after separation of time), where for the e^+e^--pair the potential $V(r)$ takes the form

$$V(\vec{r}) = -\frac{e^2}{r}(1-B) \tag{2.1a}$$

with

$$B = \tfrac{1}{2}\left[\vec{\alpha}^{(1)}\vec{\alpha}^{(2)} + \frac{(\vec{r}\vec{\alpha}^{(1)})(\vec{r}\vec{\alpha}^{(2)})}{r^2}\right], \tag{2.1b}$$

which includes the lowest-order retardation. Herein

$$\vec{r} = \vec{r}^{(1)} - \vec{r}^{(2)}, \quad \vec{p} = \frac{1}{i}\frac{\partial}{\partial\vec{r}} \tag{2.2}$$

are the relative coordinates and their momentum operator respectively, whereas the projection operator

$$\Gamma(\vec{p}) = \tfrac{1}{2}\left(\frac{\vec{\alpha}^{(1)}\vec{p}+\beta^{(1)}m^{(1)}}{\sqrt{\vec{p}^2+m^{(1)2}}} + \frac{-\vec{\alpha}^{(2)}\vec{p}+\beta^{(2)}m^{(2)}}{\sqrt{\vec{p}^2+m^{(2)2}}}\right) \tag{2.3}$$

will be approximated in the following by the 16×16 unit matrix. By this procedure equation (2.1) goes over into the Breit equation [8] which is useful as long as the energy between the two Dirac-particles remains small compared with their rest energy. Especially it contains the spin-orbit interaction which is of great importance for the magnetic interaction of e^+e^--pairs. The Dirac-matrices are given by the 16×16 matrices

$$\begin{aligned}
\alpha_1^{(1)} &= \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1, & \alpha_1^{(2)} &= 1 \otimes \sigma_1 \otimes \sigma_3 \otimes 1\\
\alpha_2^{(1)} &= \sigma_1 \oplus 1 \oplus 1 \oplus \sigma_1, & \alpha_2^{(2)} &= \sigma_1 \oplus \sigma_3 \oplus 1 \oplus \sigma_1,\\
\alpha_3^{(1)} &= \sigma_2 \otimes \sigma_1 \otimes 1 \otimes 1, & \alpha_3^{(2)} &= 1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1,\\
\beta^{(1)} &= \sigma_1 \otimes \sigma_1 \otimes 1 \otimes \sigma_3, & \beta^{(2)} &= 1 \otimes \sigma_1 \otimes \sigma_1 \otimes 1,
\end{aligned} \tag{2.4}$$

wherein σ_i and 1 are the Pauli-matrices and the 2×2 unit-matrix respectively.

With regard to the centrally symmetric e^+e^--potential we use in (2.1) furthermore polar coordinates and separate the angular variables. In this way it follows for the 16 components of the radial wave function $\psi(r)$, cf. [9]:

$$\left[\quad +i(\alpha_3^{(1)}-\alpha_3^{(2)})\left(\frac{d}{dr} + \frac{1+\frac{1}{2}(\alpha_1^{(1)}\alpha_1^{(2)}+\alpha_2^{(1)}\alpha_2^{(2)})}{r}\right)\right.$$

$$-i(\alpha_1^{(1)}-\alpha_1^{(2)})\frac{\alpha_1^{(1)}\alpha_2^{(2)}\sqrt{j(j+1)}}{r}-m(\beta^{(1)}+\beta^{(2)})$$

$$+\frac{\alpha}{r}(1-\tfrac{1}{2}[\alpha_1^{(1)}\alpha_1^{(2)}+\alpha_2^{(1)}\alpha_2^{(2)}+2\alpha_3^{(1)}\alpha_3^{(2)}])\Bigg]\psi(r)=0. \tag{2.5}$$

Herein α is Sommerfeld's fine structure constant, $m = m^{(1)} = m^{(2)}$ is the mass of the electron or the positron and $j = 0, 1, 2, \ldots$ are the quantum numbers of the square of the total angular momentum. In consequence of the representation (2.4) the differential equation (2.5) can be splitted into two independent subsystems of 8 equations, from which 4 are differential equations and 4 algebraic ones. The first system reads ($df/dr = f'$)

$$f_2^{(+)\prime}+\tfrac{1}{2}\left(E+\frac{\alpha}{r}\right)f_3^{(-)}=0,$$

$$f_3^{(-)\prime}+\frac{2}{r}f_3^{(-)}-mf_1^{(+)}-\tfrac{1}{2}\left(E+\frac{3\alpha}{r}\right)f_2^{(+)}-\frac{i\sqrt{j(j+1)}}{r}g_4^{(-)}=0,$$

$$g_2^{(+)\prime}+\frac{1}{r}g_2^{(+)}+\tfrac{1}{2}\left(E+\frac{2\alpha}{r}\right)g_3^{(-)}=0,$$

$$g_3^{(-)\prime}+\frac{1}{r}g_3^{(-)}-mg_1^{(+)}-\tfrac{1}{2}\left(E+\frac{2\alpha}{r}\right)g_2^{(+)}-\frac{i\sqrt{j(j+1)}}{r}f_4^{(-)}=0, \tag{2.6a}$$

and

$$\tfrac{1}{2}\left(E+\frac{\alpha}{r}\right)f_4^{(-)}+\frac{i\sqrt{j(j+1)}}{r}g_2^{(+)}=0,$$

$$mf_2^{(+)}+\tfrac{1}{2}\left(E-\frac{\alpha}{r}\right)f_1^{(+)}=0,$$

$$\tfrac{1}{2}Eg_4^{(-)}-\frac{i\sqrt{j(j+1)}}{r}f_2^{(+)}=0,$$

$$mg_2^{(+)}+\tfrac{1}{2}Eg_1^{(+)}=0 \tag{2.6b}$$

with the following combinations of the components of the spinor ψ:

$$f_1^{(+)}=\frac{1}{\sqrt{2}}(\psi_9-\psi_{14}),\quad f_2^{(+)}=\frac{1}{\sqrt{2}}(\psi_{12}-\psi_{15}),$$

$$f_3^{(-)}=\frac{1}{\sqrt{2}}(\psi_9+\psi_{14}),\quad f_4^{(-)}=\frac{1}{\sqrt{2}}(\psi_{12}+\psi_{15}),$$

$$g_1^{(+)} = \frac{1}{\sqrt{2}}(\psi_{11}-\psi_{16}), \quad g_2^{(+)} = \frac{1}{\sqrt{2}}(\psi_{10}-\psi_{13}),$$

$$g_3^{(-)} = \frac{1}{\sqrt{2}}(\psi_{11}+\psi_{16}), \quad g_4^{(-)} = \frac{1}{\sqrt{2}}(\psi_{10}+\psi_{13}). \tag{2.7}$$

The second system of 8 equations has the form

$$f_2^{(-)\prime}-mf_4^{(+)}+\tfrac{1}{2}\left(E+\frac{\alpha}{r}\right)f_3^{(+)} = 0,$$

$$f_3^{(+)\prime}+\frac{2}{r}f_3^{(+)}-\tfrac{1}{2}\left(E+\frac{3\alpha}{r}\right)f_2^{(-)}-\frac{i\sqrt{j(j+1)}}{r}g_4^{(+)} = 0,$$

$$g_2^{(-)\prime}+\frac{1}{r}g_2^{(-)}-mg_4^{(+)}+\tfrac{1}{2}\left(E+\frac{2\alpha}{r}\right)g_3^{(+)} = 0,$$

$$g_3^{(+)\prime}+\frac{1}{r}g_3^{(+)}-\tfrac{1}{2}\left(E+\frac{2\alpha}{r}\right)g_2^{(-)}+\frac{i\sqrt{j(j+1)}}{r}f_4^{(+)} = 0 \tag{2.8a}$$

and

$$mf_3^{(+)}-\tfrac{1}{2}\left(E+\frac{\alpha}{r}\right)f_4^{(+)}-\frac{i\sqrt{j(j+1)}}{r}g_2^{(-)} = 0,$$

$$\left(E-\frac{\alpha}{r}\right)f_1^{(-)} = 0,$$

$$mg_3^{(+)}-\tfrac{1}{2}Eg_4^{(+)}+\frac{i\sqrt{j(j+1)}}{r}f_2^{(-)} = 0,$$

$$Eg_1^{(-)} = 0 \tag{2.8b}$$

with the combination of spinor-components:

$$f_1^{(-)} = \frac{1}{\sqrt{2}}(\psi_1-\psi_6), \quad f_2^{(-)} = \frac{1}{\sqrt{2}}(\psi_4-\psi_7),$$

$$f_3^{(+)} = \frac{1}{\sqrt{2}}(\psi_1+\psi_6), \quad f_4^{(+)} = \frac{1}{\sqrt{2}}(\psi_4+\psi_7),$$

$$g_1^{(-)} = \frac{1}{\sqrt{2}}(\psi_3-\psi_8), \quad g_2^{(-)} = \frac{1}{\sqrt{2}}(\psi_2-\psi_5),$$

$$g_3^{(+)} = \frac{1}{\sqrt{2}}(\psi_3+\psi_8), \quad g_4^{(+)} = \frac{1}{\sqrt{2}}(\psi_2+\psi_5). \tag{2.9}$$

The sign (+), (−) refers to the intrinsic parity π, cf. [9]. Furthermore the components f_1 and f_2 belong to total spin $s = 0$, whereas the other f's and all g's describe states with $s = 1$ [10].

Evidently from (2.8b) it follows:

$$f_1^{(-)} = g_1^{(-)} = 0. \tag{2.10}$$

For solving the remaining 14 equations (2.6a, b) and (2.8a, b) we restrict ourselves in this paper for simplicity to the cases $j = 0$ for which one gets from (2.6b):

$$f_4^{(-)} = g_4^{(-)} = 0. \tag{2.11}$$

Furthermore the remaining 6 equations of each system (2.6) and (2.8) decouple with respect to f's and g's according to which all g's vanish:

$$g_i^{(\pm)} = 0. \tag{2.12}$$

From the remaining 6 non-trivial variables $f_1^{(+)}$, $f_2^{(\pm)}$ for $s = 0$ and $f_3^{(\pm)}$, $f_4^{(+)}$ for $s = 1$ the functions $f_1^{(+)}$ and $f_4^{(+)}$ can be reduced via the algebraic relations (2.6b) and (2.8b) to $f_2^{(+)}$ and $f_3^{(+)}$ respectively:

$$f_1^{(+)} = \frac{2m}{\frac{\alpha}{r} - E} f_2^{(+)},$$

$$f_4^{(+)} = \frac{2m}{\frac{\alpha}{r} + E} f_3^{(+)}, \tag{2.13}$$

whereby the algebraic equations are satisfied simultaneously. Then with the help of (2.13) one obtains from the differential equations (2.6a) and (2.8a) the following determination equations for $f_2^{(\pm)}$ and $f_3^{(\pm)}$:

$$f_2^{(+)\prime} + \tfrac{1}{2}\left(E + \frac{\alpha}{r}\right) f_3^{(-)} = 0,$$

$$f_3^{(-)\prime} + \frac{2}{r} f_3^{(-)} + \frac{2m^2}{E - \frac{\alpha}{r}} f_2^{(+)} - \tfrac{1}{2}\left(E + \frac{3\alpha}{r}\right) f_2^{(+)} = 0 \tag{2.14a}$$

and

$$f_2^{(-)\prime} - \frac{2m^2}{E + \frac{\alpha}{r}} f_3^{(+)} + \tfrac{1}{2}\left(E + \frac{\alpha}{r}\right) f_3^{(+)} = 0,$$

$$f_3^{(+)\prime} + \frac{2}{r} f_3^{(+)} - \tfrac{1}{2}\left(E + \frac{3\alpha}{r}\right) f_2^{(-)} = 0. \tag{2.14b}$$

The two independent systems (2.14a) and (2.14b) describe the e^+e^--pair with $j = 0$ possessing the angular momentum $l = 0$ and $l = 1$ but belonging to different total parity $P = \pi(-1)^l$. The first one belongs to positive total parity, the second one to negative total parity, cf. [9].

In order to demonstrate this more in detail we analyze the system (2.14a) for the interaction free case. Neglecting the terms α/r we obtain immediately:

$$f_2^{(+)\prime} + \tfrac{1}{2} E f_3^{(-)} = 0, \tag{2.15a}$$

$$f_3^{(-)\prime} + \frac{2}{r} f_3^{(-)} - \tfrac{1}{2} E \left(1 - \frac{4m^2}{E^2}\right) f_2^{(+)} = 0. \tag{2.15b}$$

By elimination of $f_3^{(-)}$ or $f_2^{(+)}$ in (2.15b) with the use of (2.15a) we find the radial differential equations of second order (Schrödinger type):

$$f_2^{(+)\prime\prime} + \frac{2}{r} f_2^{(+)\prime} + \frac{E^2}{4}\left(1 - \frac{4m^2}{E^2}\right) f_2^{(+)} = 0,$$

$$f_3^{(-)\prime\prime} + \frac{2}{r} f_3^{(-)\prime} - \frac{2}{r^2} f_3^{(-)} + \frac{E^2}{4}\left(1 - \frac{4m^2}{E^2}\right) f_3^{(-)} = 0. \tag{2.16}$$

Evidently $f_2^{(+)}$ (and because of (2.13) also $f_1^{(+)}$) belongs to the angular momentum $l = 0$ and $f_3^{(-)}$ to $l = 1$. The differential equations (2.16) are valid too for $f_2^{(-)}$ and $f_3^{(+)}$, respectively, following from (2.14b) in the case of negative total parity.

Finally we note that Barut has derived relativistic two-body equations from the action principle in quantum electrodynamics avoiding the Bethe-Salpeter approach [11]. Then one has instead of (2.1) the slightly modified wave equations

$$[E - (\vec{\alpha}^{(1)}\vec{p} + \beta^{(1)} m^{(1)}) - (-\vec{\alpha}^{(2)}\vec{p} + \beta^{(2)} m^{(2)}) - V(\vec{r})]\psi(\vec{r}) = 0 \tag{2.17}$$

with

$$V(\vec{r}) = -\frac{e^2}{r}(1 - \vec{\alpha}^{(1)}\vec{\alpha}^{(2)}). \tag{2.17a}$$

These equations lead to expressions very similar to the differential equations (2.14a) and (2.14b); we have proved that the results obtained from Barut's equation are within the error limit identical with those from (2.14a) and (2.14b).

3. The radial wave equation and effective potential

For discovering the energy states belonging to $j = 0$ ($l = 1$, $s = 1$ and $l = 0$, $s = 0$) with positive total parity, to which we restrict ourselves in this section, we have to solve the differential equations for $f_2^{(+)}$ and $f_3^{(-)}$, that means the system (2.14a), whereas the second system (2.14b) is to be solved by $f_2^{(-)} = f_3^{(+)} \equiv 0$. Then we get by the substitution

$$r = \frac{\alpha}{E} x, \quad E = \varepsilon m \tag{3.1}$$

from (2.14a) the simplified differential equations $\left(' = \dfrac{d}{dx}\right)$:

$$f_2^{(+)\prime} + \frac{\alpha}{2}\,\frac{1+x}{x}\,f_3^{(-)} = 0, \tag{3.2}$$

$$f_3^{(-)\prime} + \frac{2}{x}\,f_3^{(-)} - \frac{\alpha}{2}\left[\frac{x+3}{x} + \frac{4x}{\varepsilon^2(1-x)}\right] f_2^{(+)} = 0. \tag{3.3}$$

Furthermore from (2.13) it follows:

$$f_1^{(+)} = \frac{2}{\varepsilon}\,\frac{x}{1-x}\,f_2^{(+)}. \tag{3.4}$$

For the following (numerical) calculations the pole at $x = 1$ in (3.3) and (3.4) is avoided by the substitution

$$\frac{1}{1-x} \Rightarrow \frac{1-x}{(1-x)^2+\delta^2} = d(x, \delta),$$

$$\lim_{\delta\to 0} d(x, \delta) = \frac{1}{1-x}\,. \tag{3.5}$$

Later we choose δ very small and show, that the result does not depend sensitively on the value of δ. In this way the singularity at $x = 1$ is regularized.

Now we decouple the differential equations (3.2) and (3.3) in such a way, that we eliminate $f_3^{(-)}$ in (3.3) with the use of (3.2). By this procedure we obtain[1]

$$f_2^{(+)\prime\prime} + \frac{3+2x}{x(1+x)}\,f_2^{(+)\prime} + \frac{\alpha^2}{4}\,(x+1)\left[\frac{x+3}{x^2} + \frac{4}{\varepsilon^2}\,d(x, \delta)\right] f_2^{(+)} = 0 \tag{3.6}$$

With the solution $f_2^{(+)}$ of equation (3.6) $f_3^{(-)}$ is given by (3.2) and $f_1^{(+)}$ by (3.4). All other spinorial wave-functions vanish. Then the particle number density takes the form using (2.7):

$$\psi^+\psi = |f_1^{(+)}|^2 + |f_2^{(+)}|^2 + |f_3^{(-)}|^2. \tag{3.7}$$

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[1] Without the substitutions (3.1) and (3.5) the differential equation (3.6) takes the physically more transparent form, cf. (2.16):

$$f_2^{(+)\prime\prime} + \frac{2}{r}\,f_2^{(+)\prime} + \frac{\alpha/r^2}{E+\alpha/r}\,f_2^{(+)\prime} - \frac{2\alpha}{r}\,\frac{m^2}{E-\alpha/r}\,f_2^{(+)} + \frac{\alpha}{r}\left(E + \frac{3}{4}\,\frac{\alpha}{r}\right) f_2^{(+)}$$
$$+ \tfrac{1}{4}E^2\left(1 - \frac{4m^2}{E^2}\right) f_2^{(+)} = 0, \qquad \left(' = \frac{d}{dr}\right).$$

No centrifugal potential appears; the third and the fourth term have magnetic origin, the fifth term contains the Coulomb-potential plus correction. -- For the numerical calculations the form (3.6) is more suitable.

For solving the differential equation (3.6) we bring it at first into the form of a 1-dimensional Schrödinger equation. For this we set

$$f_2^{(+)} = \frac{\sqrt{1+x}}{x^{3/2}} F \tag{3.8}$$

and get from (3.6)

$$F'' - V(x)F + k^2 F = 0 \tag{3.9}$$

with (cf. (3.5))

$$V(x) = -\frac{\alpha^2}{x} + \tfrac{3}{4}(1-\alpha^2)\frac{1}{x^2}$$

$$+\tfrac{3}{4}\frac{1}{(1+x)^2} + \tfrac{3}{2}\frac{1}{x+1} - \tfrac{3}{2}\frac{1}{x} - \frac{2\alpha^2}{\varepsilon^2}\frac{1-x}{(1-x)^2+\delta^2} - \frac{\alpha^2}{\varepsilon^2}\frac{\delta^2}{(1-x)^2+\delta^2} \tag{3.9a}$$

and (cf. (2.16))

$$k^2 = \left(\frac{\alpha}{2}\right)^2 \left(1 - \frac{4}{\varepsilon^2}\right). \tag{3.9b}$$

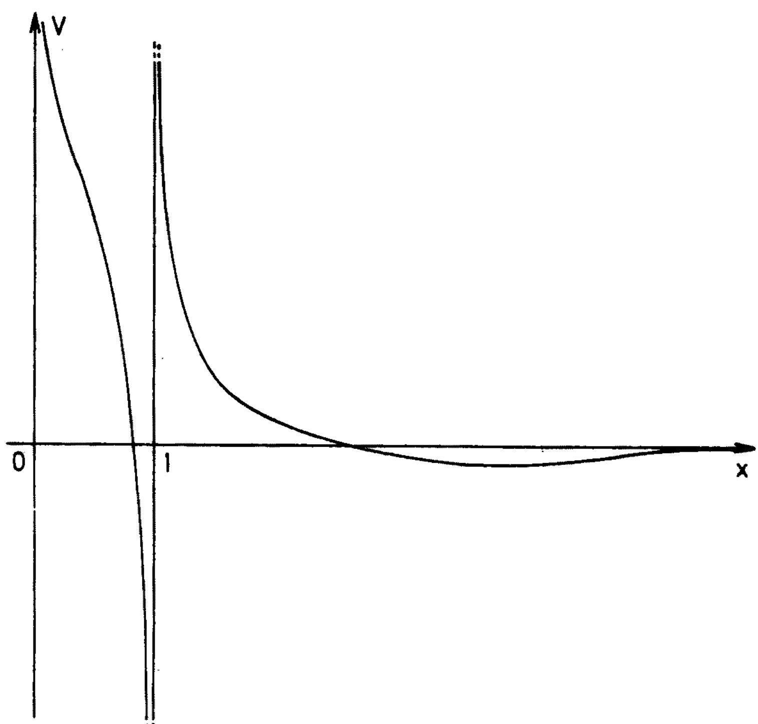

Fig. 1. Qualitative course of the effective potential according to Eq. (3.10)

Equation (3.9a) represents the effective potential: The first term is the Coulomb-potential; all the other terms describe the magnetic interaction. In view of (2.16) no real centrifugal potential is present. However because α^2 and δ^2 are very small quantities the last term in (3.9a) can be suppressed totally and in the second term α^2 can be neglected. In this way for the effective potential it remains:

$$V(x) = -\frac{\alpha^2}{x} + \tfrac{3}{4}\left(\frac{1}{(1+x)^2} + \frac{1}{x^2}\right)$$

$$+\tfrac{3}{2}\left(\frac{1}{1+x} - \frac{1}{x}\right) - \frac{2\alpha^2}{\varepsilon^2}\frac{1-x}{(1-x)^2+\delta^2}. \tag{3.10}$$

The shape of $V(x)$ is shown in Fig. 1. Obviously the last magnetic interaction term in (3.10) is very important in the neighbourhood of $x = 1$, i.e. according to (3.1) in the region of the classical electron radius.

The quantity (3.9b) represents the energy without the rest energy; with $\varepsilon = 2+\hat{\varepsilon}$ we get from (3.9b) for $|\hat{\varepsilon}| \ll 2$ in the first order

$$k^2 = \left(\frac{\alpha}{2}\right)^2 \hat{\varepsilon}. \tag{3.11}$$

According to this bound states are characterized by $k^2 < 0 \leftrightarrow \hat{\varepsilon} < 0$, scattering states by $k^2 > 0 \leftrightarrow \hat{\varepsilon} > 0$.

4. *Scattering states*

4.1. Positive total parity

In view of the observed e^+- and e^--lines in the heavy ions experiments $\hat{\varepsilon} > 0$ is valid in this case. Therefore we investigate at first the scattering states for $j = 0$ and positive total parity and deduce from these the cross-section for e^+e^--scattering. In the case of resonances in the cross-section we can be sure that there exist *quasi*-bound states of the e^+e^--pair with the energy of the resonances.

At first we investigate the asymptotic behaviour of F according to (3.9) and (3.10). For $x \to 0$ we find

$$F'' - \tfrac{3}{4}\frac{1}{x^2}F = 0 \tag{4.1}$$

with the 2 independent solutions:

$$F = \begin{cases} x^{-1/2}, \\ x^{3/2}. \end{cases} \tag{4.2}$$

But only in case of the second solution the spinorial wave function $f_2^{(+)}$ and herewith also $f_1^{(+)}$ and $f_3^{(-)}$ are finite at $x = 0$, see (3.8), (3.4) and (3.2). Therefore the first solution of (4.2) must be excluded.

For $x \to \infty$ we get from (3.9) and (3.10)

$$F''+k^2F = 0 \tag{4.3}$$

with the 2 independent solutions ($k^2 > 0$):

$$F_{(\pm)} = e^{\pm ikx}, \tag{4.4}$$

so that in view of (3.8) the wave function $f_2^{(+)}$ represents asymptotically an ingoing or outgoing spherical wave. Hence we make for the total functions $F_{(\pm)}$ the ansatz

$$F_{(\pm)} = x^{3/2}e^{\pm ikx}f_{(\pm)}(x) \tag{4.5}$$

with the boundary condition:

$$f_{(\pm)}(0) = 1. \tag{4.5a}$$

From (3.9) and (4.5) it follows immediately

$$f_{(-)} = f^*_{(+)}, \tag{4.5b}$$

both relations (4.5a) and (4.5b) are obtained without restriction of generality.

For stationary scattering states the total function F in (3.8) and (3.9) must be real valued. Thus we have to set in view of (4.5) and (4.5b)

$$F = F_{(+)}+F_{(-)} = x^{3/2}(f_{(+)}(x)e^{ikx}+f^*_{(+)}(x)e^{-ikx}), \tag{4.6}$$

that means a superposition of outgoing and ingoing waves. Separating $f_{(+)}$ in its real and imaginary part

$$f_{(+)}(x) = a(x)+ib(x) \tag{4.7}$$

we obtain from (4.6):

$$F = 2x^{3/2}(a \cos kx-b \sin kx). \tag{4.8}$$

In view of (4.5a) and (4.7) we have the boundary condition for a and b:

$$a(0) = 1, \quad b(0) = 0 \tag{4.8a}$$

in agreement with the behaviour of (4.8) at $x = 0$, see (4.2).

It follows from (4.8) that

$$F = -2x^{3/2}\sqrt{a^2+b^2}\sin(kx-\gamma),$$

$$\sin\gamma = \frac{a}{\sqrt{a^2+b^2}}, \quad \cos\gamma = \frac{b}{\sqrt{a^2+b^2}}. \tag{4.9}$$

In view of (4.4), (4.5) and (4.7) $a(x)$ and $b(x)$ will have the asymptotic behaviour

$$\lim_{x\to\infty} \left.\begin{matrix} a(x) \\ b(x) \end{matrix}\right\} \sim x^{-3/2}, \tag{4.9a}$$

so that $\sin\gamma$ and $\cos\gamma$ have asymptotic *constant* values. Herein γ has the meaning of the scattering phase with respect to a free (plane) wave, cf. (2.16).

With the solution (4.9) the non-vanishing wave functions are known. In view of (3.2), (3.4) and (3.8) we have:

$$f_2^{(+)} = \frac{\sqrt{x+1}}{x^{3/2}} F,$$

$$f_1^{(+)} = \frac{2}{\varepsilon} \frac{\sqrt{x+1}}{x^{1/2}} \frac{1-x}{(1-x)^2+\delta^2} F,$$

$$f_3^{(-)} = -\frac{2}{\alpha} \frac{x}{1+x} f_2^{(+)\prime}. \tag{4.10}$$

With the regularization procedure (3.5) all these functions are regular within the range $0 \leqslant x \leqslant \infty$. The asymptotic behaviour for $x \to \infty$ takes with the use of (4.9) and (4.9a) the form (up to the same, but irrelevant factor):

$$f_2^{(+)} \to \frac{1}{x} \sin(kx-\gamma),$$

$$f_1^{(+)} \to -\frac{2}{\varepsilon}\frac{1}{x} \sin(kx-\gamma),$$

$$f_3^{(-)} \to \sqrt{1-\frac{4}{\varepsilon^2}} \frac{1}{x} \sin\left(kx-\gamma-\frac{\pi}{2}\right). \tag{4.10a}$$

Taking into consideration, that according to (2.16) $f_1^{(+)}$ and $f_2^{(+)}$ belong to $l = 0$ and $f_3^{(-)}$ to $l = 1$ the partial cross-section for $j = 0$ with positive total parity is given by the scattering phase γ as follows [12][2]

$$\sigma \sim \frac{1}{\varepsilon^2 k^2} \lim_{x\to\infty} \sin^2\gamma. \tag{4.11}$$

In order to find the asymptotic value of γ, we have to set $F_{(+)}$ according to (4.5) into (3.9) and to solve the differential equation for $f_{(+)}(x)$, i.e. because of (4.7) for $a(x)$ and $b(x)$ with the initial condition (4.8a) for different values of ε. Then σ is obtained by the definitions (4.9) as a function of the energy.

Doing this with the use of the potential (3.10) we neglect the Coulomb-potential α^2/x, because we are interested in the resonances of the cross-section only caused by the magnetic interaction. Furthermore the last term in (3.10) is important only in the neighbourhood of $x = 1$; inspite of this it would produce (as the Coulomb-potential) a logarith-

[2] Because of the substitution (3.1) the wave-vector with respect to r reads $\frac{E}{\alpha}k = m\frac{\varepsilon}{\alpha}k$.

mic asymptotic behaviour of the scattering phase and destroy the property (4.9a). In order to avoid this we damp it for large x-values by the factor $e^{-(x-1)}$. In this way we obtain:

$$f''_{(+)}+\left(\frac{3}{x}+2ik\right)f'_{(+)}-\left[\frac{3}{4}\frac{}{(x+1)^2}+\frac{3}{2}\left(\frac{1}{x+1}-\frac{1}{x}\right)-\frac{3ik}{x}\right.$$
$$\left.-\frac{2\alpha^2}{\varepsilon^2}\frac{(1-x)e^{-(x-1)}}{(1-x)^2+\delta^2}\right]f_{(+)}=0. \tag{4.12}$$

We solve this differential equation in 2 steps. Starting at $x = 0$ we expand $f_{(+)}$ in a power series

$$f_{(+)}=\sum_{n=0}^{\infty} c_n x^n \tag{4.12a}$$

with the initial condition, because of (4.5a) or (4.8a),

$$c_0 = 1. \tag{4.12b}$$

The recurrence formula following from (4.12) and (4.12a) has the form neglecting the damping factor $e^{-(x-1)}$:

$$(1+\delta^2)(n+5)(n+7)c_{n+5} = -[\tfrac{3}{2}+ik(2n+11)$$
$$+\delta^2(\tfrac{3}{2}+2(n+4)(n+6)+ik(2n+11))]c_{n+4}$$
$$+\left[\tfrac{9}{4}+2(n+3)(n+5)-\frac{2\alpha^2}{\varepsilon^2}-2\delta^2(\tfrac{3}{8}+\tfrac{1}{2}(n+3)(n+5)\right.$$
$$\left.+ik(2n+9))\right]c_{n+3}-\left[\frac{2\alpha^2}{\varepsilon^2}-2ik(2n+7-\tfrac{3}{2}\delta^2)\right]c_{n+2}$$
$$-\left[(n+1)(n+3)+\tfrac{3}{4}-\frac{2\alpha^2}{\varepsilon^2}\right]c_{n+1}+\left[\frac{2\alpha^2}{\varepsilon^2}-ik(2n+3)\right]c_n. \tag{4.12c}$$

Starting from (4.12b) all coefficients c_n can be calculated by (4.12c). However, because of the asymptotic behaviour (4.9a) the series (4.12a) is not convergent for large x-values. Therefore we go over at $x = 10^{-2}$ to a numerical integration of (4.12), where the initial values for $f_{(+)}$ and $f'_{(+)}$ are taken from the power series.

The result of the calculation is given in Fig. 2 where the cross-section according to (4.11) is drawn in dependence of the energy $(\varepsilon-2)m$ in keV. In all cases of the numerical calculation the asymptotic value of x is chosen as 2.5×10^4 and the parameter δ is varied between 10^{-5} and 10^{-7} without any noticeable effect. According to the resonances of the cross-section magnetically quasi-bound states should exist with energies given in Table I. Simultaneously the life-time of the quasi-bound states is calculated from the half-width

TABLE I

Energies and life-times for the resonances according to Fig. 2 for positive total parity. The doublet-character of the resonances is resolved. The comparison with the experimental data is performed with the sum energy of the e^+e^--lines

Theory	Energy [keV]	316		388		477		595		754		988	
		314	318	386	390	475	480	591	599	748	760	978	998
Lifetime [sec]		2×10^{-20}		1.4×10^{-20}		1.2×10^{-20}		8×10^{-21}		6×10^{-21}		2×10^{-21}	
Experiment e^+ or e^--lines													
Energy [keV]	sum lines					460		600 [14]		780			
								608±8 [16]	620±8 [16]	748±8 [16]	760±20 [3]		
	single lines					230±5 [2]		300 [2]		390 [2]			
								304	310 [3]	375 [3]	380		
Lifetime [sec]								2×10^{-20}		8×10^{-21}	4×10^{-21}		

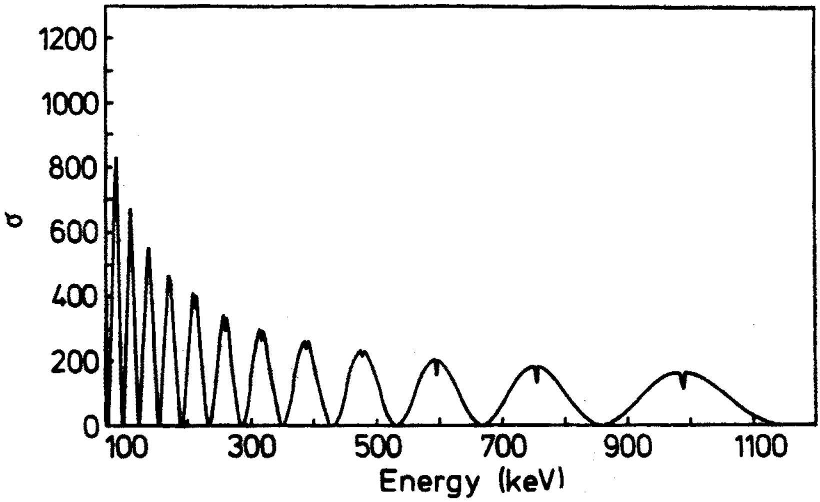

Fig. 2. Cross-section σ (in arbitrary units) in dependence of the energy $(\varepsilon-2)m$ according to (4.11). The maxima characterize the energy of the resonances and the width of the resonances gives the life-time of the quasi-bound states. Obviously, the resonances possess a feinstructure in form of a doublet structure

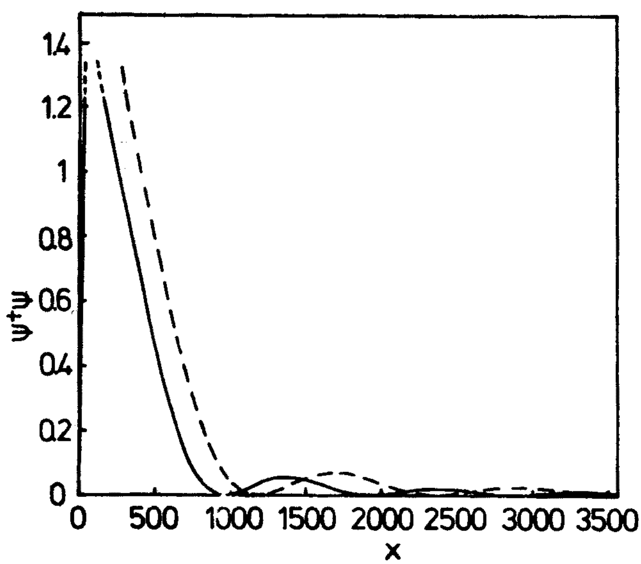

Fig. 3. The course of $\psi^{\dagger}\psi$ in arbitrary units as function of the distance x for the resonance energies of 761 keV (solid line) and 475 keV (dashed line). The maximum values are reached at $x = 1$. For the other resonance-energies one obtaines very similar curves, where with decreasing energies the beginning of the scattering wave is shifted to larger distances

of the resonances. The comparison with the experimental data shows good agreement. In Fig. 3 the density distribution (3.7) is given with the help of (4.10) for some resonance energies.

4.2. Negative total parity

In total analogy to the foregoing section we can analyze the scattering states for $j = 0$ and negative total parity starting from the equations (2.14b). Going over to the second order differential equations we find for $f_3^{(+)}$:

$$\frac{d^2 f_3^{(+)}}{dr^2} + \frac{2}{r}\frac{df_3^{(+)}}{dr} + \frac{3\alpha/r^2}{E+3\alpha/r}\frac{df_3^{(+)}}{dr} - \left[\frac{2}{r_2} - \frac{\alpha E}{r}\right.$$

$$\left. -\frac{3}{4}\frac{\alpha^2}{r^2} + \frac{2\alpha m^2/r}{E+\alpha/r} - \frac{6\alpha/r^3}{E+3\alpha/r}\right] f_3^{(+)} + \frac{E^2}{4}\left(1 - \frac{4m^2}{E^2}\right) f_3^{(+)} = 0. \tag{4.13}$$

Herein the third term as well as the third up to the fifth therm inside the bracket describe the magnetic interaction, whereas the first one inside the bracket is the centrifugal potential and the second one the Coulomb potential. With the substitution (3.1) and

$$f_3^{(+)} = \frac{\sqrt{E+3\alpha/r}}{r} F \tag{4.14}$$

it follows from (4.13) $\left(' = \dfrac{d}{dx}\right)$:

$$F'' - V(x)F + k^2 F = 0 \tag{4.15}$$

with the effective potential

$$V(x) = \frac{2}{x^2} - \frac{\alpha^2}{x} - \frac{3}{4}\frac{\alpha^2}{x^2} + \frac{2\alpha^2/\varepsilon^2}{x+1} - \frac{6}{x^2(x+1)} + \frac{27/4}{x^2(x+1)^2} \tag{4.15a}$$

and the "energy", cf. (3.9b):

$$k^2 = \left(\frac{\alpha}{2}\right)^2 \left(1 - \frac{4}{\varepsilon^2}\right). \tag{4.15b}$$

The first term in (4.15a) is the centrifugal potential and the second one the Coulomb potential; all the other terms describe the magnetic interaction.

The solution of (4.15) regular at $x = 0$ and corresponding to (4.5) has the form

$$F_{(\pm)} = x^{(\sqrt{3}+1/2)} e^{\pm ikx} f_{(\pm)}(x) \tag{4.16}$$

with the boundary condition:

$$f_{(\pm)}(0) = 1. \tag{4.16a}$$

Following the procedure of the equations (4.6) up to (4.9) the stationary scattering states are described by the wave-function

$$F = -2x^{(\sqrt{3}+1/2)}\sqrt{a^2+b^2}\sin(kx-\gamma),$$

$$\cos\gamma = \frac{b}{\sqrt{a^2+b^2}} \tag{4.17}$$

as solution of (4.15). Taking into account that the wave-function (4.14) belongs to $l = 1$, the partial cross section for $j = 0$ with negative total parity is given by the scattering phase γ as follows (cf. (4.11)):

$$\sigma \sim \frac{1}{\varepsilon^2 k^2}\lim_{x\to\infty}\cos^2\gamma. \tag{4.18}$$

The differential equation for $a(x)$ and $b(x)$ follows after insertion of $F_{(+)}$, respectively to (4.16) and (4.15). Doing this we neglect in (4.15a) the Coulomb-potential, because we are interested only in the magnetic interaction; furthermore we damp the magnetic term proportional to $(1+x)^{-1}$ for large x-values by e^{-x}; otherwise the scattering phase γ would have a logarithmic asymptotic behaviour. Thus we find ($f_{(+)} = a+ib$):

$$f''_{(+)} + \left[\frac{2\sqrt{3}+1}{x} + 2ik\right]f'_{(+)} - \left[\frac{2\alpha^2}{\varepsilon^2}\frac{e^{-x}}{x+1} + \frac{27}{4}\frac{1}{(x+1)^2} + \frac{15}{2}\left(\frac{1}{x+1} - \frac{1}{x}\right) - 2ik(\sqrt{3}+\tfrac{1}{2})\frac{1}{x}\right]f_{(+)} = 0. \tag{4.19}$$

This differential equation corresponds exactly to (4.12) and is solved by the same numerical procedure. From the asymptotic behaviour of the solutions for different ε-values the scattering phase γ is obtained according to (4.17) as function of a energy.

The result is given in Fig. 4 where the cross-section according to (4.18) is drawn as function of the energy $(\varepsilon-2)m$. In Table II the energies and the life-times of the resonances are given and compared with the experimental data.

5. *Final remarks*

The comparison with the experimental results in Section 4 shows that there exists an evidence of the e^+e^--resonances in form of e^+ or e^--lines only in the range of 200 keV $\gtrsim E \gtrsim$ 400 keV, whereas from the theory a larger spectrum of resonances follows. The explanation of this fact may be, that at low energies with longer life-times of the resonances an annihilation of the e^+e^--pairs into two photons occurs [17] and that for high energies the absolute value of the cross-sections goes down very rapidly. In consequence of this only in the middle range of energies the resonances are observable. Practically the same results are obtained from Barut's relativistic two-body equations (2.17). Finally we note that in

TABLE II

Energies and life-times for the resonances according to Fig. 4 for negative total parity. The doublet-character of the resonances is resolved. For comparison with the experimental data the sum energy of the e^+e^--lines is used

Theory	Energy [keV]	347		429		528		659		853		1145	
		342	352	424	434	521	536	654	674	838	868	1120	1170
Lifetime [sec]		1.8×10^{-20}		1.6×10^{-20}		1.2×10^{-20}		8×10^{-21}		6×10^{-21}		4×10^{-21}	
Experiment e^+ or e^--lines													
Energy [keV]	sum lines					520	560 ± 12 [14] [15]	660		810 [13] 805 ± 8 [16], 809 ± 8 [16]			
	single lines					260 [2]	280	330 [2]		405			
Lifetime [sec]										$\simeq10^{-20}$			

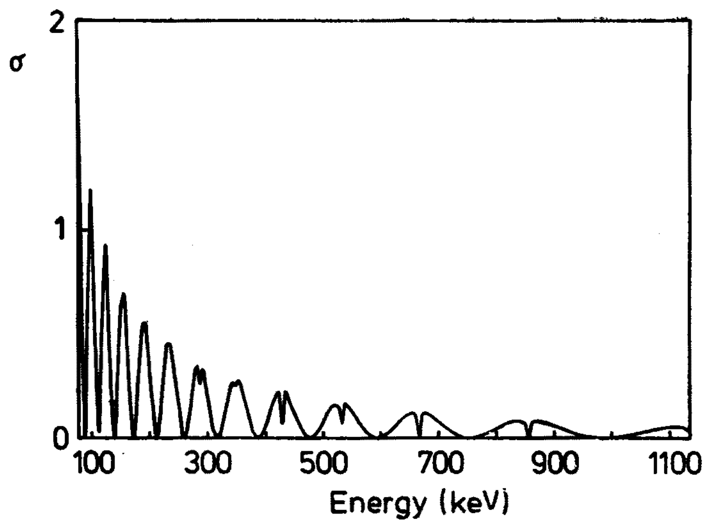

Fig. 4. Cross-section σ (in arbitrary units) in dependence of the energy $(\varepsilon-2)m$ according to (4.18). Obviously, the resonances possess a doublet-structure

consequence of the effective potential (3.9a) and (4.15a) no real magnetically bound states exist with long life-times. However, the application of the integration method used above to the electrically bound system with energies $E < 2m$ leads to the known stable energy-eigenstates of normal positronium.

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