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A MECHANISM OF COMPACTIFICATION*

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A mechanism of compactification is suggested. As it was recently shown by Fukaya some limiting procedures on a subset of Riemannian manifolds are possible. They can lead to physical space-time with almost vanishing cosmological constant in a multidimensional theory. Conditions preventing decompactification are discussed. An example showing that twisted boundary conditions can stabilize the situation is given.

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To explain why our Universe is four-dimensional is one of the main goals of contemporary theoretical physics. Of course, one may answer that the Universe is four-dimensional because only four-dimensional universes allow human beings to exist. But then the question may be reformulated in an obvious way. In the Kaluza-Klein approach [1] the space-time is higher-dimensional and symmetries in the additional dimensions are the sources of gauge interactions. In string theories, [2], the situation is similar. In fact compactification of superstring theories involves Kaluza–Klein program at least in their field theory limit. The peculiar feature of string theory is that they can be constructed in a consistent way at least, in no less than in 26 dimensions (bosonic string) or 10 dimensions (fermionic ones). So some light is apparently shed on the dimensionality of space-time. Why only four dimensions are chosen to form the world we live in we still do not know.

Recently, Fukaya has proven some theorems [3–5], which we think may be of great importance in trying to understand "collapsing" of Riemannian manifold to a lower dimensional one. We would like to use them to understand at least some aspects of the vanishing cosmological constant. This assumption is justified by our experimental knowledge, although theoretical explanation of this fact is very difficult and still lacking. The vanishing of Λ is usually connected with the fact that the vacuum energy vanishes [6]. Field theoretical calculations [6–10] say that the value of vacuum energy can be expressed in terms of the ζ-functions of the appropriate (Dirac, Laplace-Beltrami...) field operator:

$$\zeta(s) \equiv \sum_k \lambda_k^{-s}. \tag{1}$$

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where λ_k denotes the k-th (non-zero) eigenvalue of the operator in questions. In the case of scalar electrodynamics the vacuum energy is given by (one loop approximation) [7–9]:

$$\varepsilon_0 = \frac{h}{64\pi^2} \sum_r \left[\lambda_r^2 \ln \frac{\lambda_r}{\Lambda^2} - \tfrac{25}{6} \right] = (c\zeta_\Delta(2) + \zeta'_\Delta(2)) \frac{h}{64\pi^2}, \tag{2}$$

where ζ_Δ denotes the ζ-function of the scalar Laplace-Beltrami operator on the internal space and Λ is the renormalization constant. Dimensional regularization has been applied to remove the ultraviolet divergencies (cf. [8]). This regularization introduces μ^2 as the mass square dimension value. Physical parameters should be independent of μ^2. Counterterms could be introduced in such a way that ε_0 is independent of μ^2. The expression (2) contains some information about topology of the internal space [9]. Apparently, when the spectrum of the Laplace–Beltrami operator can be "removed" in some way the vacuum energy (cosmological constant) vanishes.

As it was shown by Fukaya [3–5], a subspace of a Riemannian manifold can be given a topology so that the eigenvalues $\lambda_k(\mathrm{M})$ of the Laplace–Beltrami operator Δ_M on a manifold M can be extended to continuous functions. This topology is not a natural one and we refer the reader to [3–5] for details. The continuity of the eigenvalues $\lambda_k(\mathrm{M})$ can lead in some cases to interesting conclusions. Suppose that the space-time is of the form $\mathrm{M} \times \mathrm{N}$, where M and N are Riemannian manifolds with metric tensors g_M and g_N. Let us discuss some examples of compactification.

Example 1. Toroidal compactification. Let N be the n-dimensional torus $T^n = \boldsymbol{R}^n/Z^n$. Put $g^\varepsilon_{\mathrm{M}\times\mathrm{T}^n} = g_\mathrm{M} \oplus \varepsilon^2(\mathrm{d}t_1^2 \oplus \ldots \oplus dt_n^2)$. Then ($\lambda_k$ denotes the k-th eigenvalue of the Laplace–Beltrami operator)

$$\{\lambda_k(M \times \mathrm{T}^n, g^\varepsilon_\mathrm{M})\} = \left\{\lambda_k(\mathrm{M}) + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} l_i^2 | l_i \in \mathrm{N}\right\} \tag{3}$$

and we have $\lim_{\varepsilon\to 0} \lambda_k(\mathrm{M}\times\mathrm{T}^n, g^\varepsilon_{\mathrm{M}\times\mathrm{T}^n}) = \lambda_k(\mathrm{N})$.

Example 2. Compactification on n-sphere $\mathrm{N} = \mathrm{S}^n$. Let $g^\varepsilon_{\mathrm{M}\times\mathrm{N}} = g_\mathrm{M} \oplus \varepsilon^2(g_{\mathrm{S}^n})$. Then we have

$$\{\lambda_k(\mathrm{M}\times\mathrm{s}^n, g^\varepsilon_{\mathrm{M}\times\mathrm{S}^n})\} = \left\{\lambda_k(\mathrm{M}) + \frac{1}{\varepsilon^2} m(m+n-1)|_{m\in\mathrm{N}}\right\}$$

and also we have $\lim_{\varepsilon\to 0} \lambda_k(\mathrm{M}\times\mathrm{S}^n) = \lambda_k(\mathrm{M})$.

In both examples the additional (internal) part of the spectrum is removed to infinity! This is a mathematical formula which describes the fact that the massive modes are so heavy that they are invisible in the low energy world. But the above examples suggest that they may be in fact removed. It is obvious that in these cases cosmological constant vanishes because only the zero modes survive the compactification.

Example 3 (see [11]). Let (M, g) be a Riemannian manifold on which U(1) acts freely and isometrically. Let g^ε denote the Riemannian metric such that

$$g^\varepsilon(v, v) = \begin{cases} \varepsilon g(v, v) \text{ if } v \text{ is tangent to a U(1)-orbit} \\ g(v, v) \text{ if } v \text{ is perpendicular to a U(1)-orbit.} \end{cases} \tag{5}$$

Then $\lim_{\varepsilon \to 0} (M, g_\varepsilon) = (M/U(1), g')$ for some metric g' and we obtain a fiber bundle $S^1 \to M \to M/S^1$. This example can be generalized to a very interesting theorem [3–5].

From the physical point of view the parameter ε in examples 1 and 2 can be treated as a variational parameter and the physical space-time is chosen by demanding that cosmological constant has a minimum as a function of ε and, in fact, is almost vanishing. This can be shown as follows (cf. [6–9]). Introduce a parameter ν^2 by the relation

$$\lambda = \nu^2 \tilde{\lambda}. \tag{6}$$

In the cases discussed in examples 1–3

$$\nu^2 \sim \frac{1}{\varepsilon^2} \tag{7}$$

and (2) takes the form

$$\varepsilon_0(\nu^2) = \frac{h}{64\pi^2}\left[\nu^4\zeta(-2) + \nu^4\zeta(-2)\ln\left(\frac{\nu^2}{\Lambda^2}\right)\right]. \tag{8}$$

From the physical point of view collapsing of the internal space to a single point is unacceptable. This is reflected in (8) by the proportionality of ε_0 to ν^4 (in a Kaluza–Klein-like interpretation $\nu^2 \sim R^{-2}$, R being the radius of the internal space). We should look for a minimum of $\varepsilon_0(\nu^2)$ at $\nu^2 \neq 0$. If $\zeta(-2) = 0$ then $\zeta'(-2) = 0$ or $\nu^2 = 0$. The latter case, obviously, corresponds to decompactification (cf. [7]). If $\zeta(-2) \neq 0$ then $\varepsilon_0(\nu^2)$ has extremum at

$$\nu^2_{\min} = \Lambda^2 \exp\left(1/2 - \frac{\zeta(-2)}{\zeta(-2)}\right), \tag{9}$$

$$\varepsilon_0(\nu^2_{\min}) = -\tfrac{1}{2}\Lambda^4\zeta(-2)\exp\left(-1 - \frac{2\zeta(-2)}{\zeta'(-2)}\right). \tag{10}$$

The above formula expresses $\varepsilon_0(\nu^2_{\min})$ in terms of ζ and ζ' and consequently in terms of the spectral invariants a_i [9, 12]. We can say that the vacuum energy (cosmological constant) prevents the internal space from collapsing to a single point because in that case it tends to infinity ($\nu^2 \to \infty$).

Let us discuss compactification on the torus $T^n = (S^1)^n$. The eigenvalues on $(S^1)^n$ have the general form $\lambda_n = \sum_i n_i^2$. If we impose twisted boundary conditions on the fields

$$\psi(\phi_i + 2\pi n_i) = e^{2\pi i n_i \alpha_i}\psi(\phi_i), \tag{11}$$

then $\lambda_n \to \lambda_n^{\alpha_i} = (n_i - \alpha_i)^2$. The zeta function takes the form [12]:

$$\zeta^{\alpha_i}(s) = \sum_n (n_i - \alpha_i)^{-2s} = \zeta_{HLH}(2s, -\alpha_i), \tag{12}$$

where $\zeta_{HLH}(s, \omega) = \sum_n (n+\omega)^{-s}$ is the generalized zeta function. We have [12]

$$\zeta^{\alpha_i}(-2) = \zeta_{HLH}(-4, \alpha_i). \tag{13}$$

$\zeta_{HLH}(-4, 0)$ can be expressed in terms of the Bernouli polynomials [12]. It follows that

$$\zeta_{HLH}(-4, 0) = 0. \tag{14}$$

This means that compactifications on a torus are unstable unless we choose twisted boundary conditions. Twisted boundary conditions emerge in a natural way on orbifolds. The analysis given suggest that compactification on orbifolds may be stable. In principle, we have the possibility of comparing different orbifolds. The orbifold with lower vacuum energy should be preferred in compactification process. This can be interpreted as a phase transition (an analogue to crystalization in solid state physics). The twisted sectors correspond to nonequivalent topological sectors (anyons [13]). This is under investigation.

Apparently some questions arise. Firstly, Fukaya's analysis deals with a Riemannian manifold while the space-time is a pseudo-Riemannian one. The above analysis can be repeated if we suppose that the space-time has the structure $\mathbf{R} \times M^{(3)} \times N^{(int)}$, where $\mathbf{R}$ represents the time axis. A genuine pseudo-Riemannian analysis should also be given but as far as the internal space has no time-like part, the conclusions are the same. Secondly, the "route of collapsing" should be found, that is the appropriate internal spaces. The approach of [9] may turn out to be useful as it tries to relate the vacuum energy to topological invariants. Nevertheless, we think that the examples given above shed some light on the models of compactification discussed in the literature. They may also explain how a possible bigger internal space may collapse to the one demanded by the Kaluza–Klein or string approach.

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