APPLICABILITY OF REFINED BORN APPROXIMATION TO NON-LINEAR EQUATIONS

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A computational method called "Refined Born Approximation", formerly applied exclusively to linear problems, is shown to be successfully applicable also to non-linear problems enabling us to compute bifurcations and other irregular solutions which cannot be obtained by the standard perturbation procedures.

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A generalization of the usual perturbation calculus called "Refined Born Approximation" (RBA) [1] was applied in the last few years successfully to linear equations, especially to solve the Lippman-Schwinger equation, and was shown to be particularly useful in molecular physics [2]. The aim of this paper is to demonstrate the applicability and usefulness of the RBA method for non-linear problems, especially for finding solutions that describe effects not attainable by the usual perturbation calculus, like bifurcations, instantons, etc.

The RBA method consists in performing iterations starting, instead of with free waves as zero-order terms with suitable linear functions of parameters to be fixed later by some self-consistency requirements or by some variational principles.

Let us first explain the RBA method in its simplest version with one parameter and one iteration only. Consider the following linear integral equation

$$\varphi(x) = \exp(ipx)+g\int dx'K(x-x')\varphi(x') \tag{1}$$

in an arbitrary number of dimensions, with e^{ipx} denoting a free wave and $K(x)$ denoting a kernel, while g is a coupling constant. The usual Born approximations start with putting as a zero-order term to the right hand side a free wave. In the RBA method we take as the zero-order approximation

$$\varphi^{(0)}(x) = \alpha\exp(ipx), \tag{2}$$

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where α is a parameter. The first order approximation is then

$$\varphi^{(1)}(x) = \exp(ipx) + g\alpha J(x), \tag{3}$$

where

$$J(x) = \int dx' K(x-x') \exp(ipx). \tag{3'}$$

If we were clever enough to guess the exact solution and put it as the zero-order term into the right hand side of (1) then the first order term $\varphi^{(1)}(x)$ would be identical with $\varphi^{(0)}(x)$ at every point x. Since (2) is certainly not an exact solution, we may approximate it by disposing with the parameter α so as to get agreement between $\varphi^{(0)}$ and $\varphi^{(1)}$ at least at a single point. By continuity argument they would almost coincide also in a finite neighbourhood of this point and, consequently, would approach the solution satisfactorily.

One of the simplest possibilities is to equate the two approximations at the origin $x = 0$, i.e. to put

$$\varphi^{(1)}(0) = \varphi^{(0)}(0), \tag{4}$$

as a condition for α. From (2), (3), and (4) it follows

$$1 + gJ(0)\alpha = \alpha \tag{5}$$

whence

$$\alpha = \frac{1}{1-J(0)g}. \tag{6}$$

In this way one obtains in the first approximation

$$\varphi^{(1)}(x) = \exp(ipx) + \frac{g}{1-J(0)g} J(x), \tag{7}$$

with $J(x)$ denoting the integral (3'). It differs from the Born approximation by the denominator involving the coupling constant g. Thus, our method is different from the usual power series expansion in the coupling constant. It becomes of a particular interest for attractive interactions (sign g = sign $J(0)$) since in this case it yields a pole which accounts for the existence of a resonance and for a possibility of a bound state. The RBA method, even in its simplest form, increases considerably the convergence radius of the iteration procedure.

Let us apply the same method to non-linear problems, e.g. to a φ^4 or φ^3 interaction term in the Lagrangian. In order to simplify the problem as much as possible let us go over from a realistic case of field theory in 3+1 dimensions to the one-dimensional case: to the anharmonic oscillator satisfying the equation

$$\ddot{x} + \omega^2 x = gx^n, \tag{8}$$

with an integer $n > 1$. The above differential equation may be replaced by an integral equation

$$x(t) = \exp(i\omega t) + g \int dt' G(t-t') \{x(t)'\}, \tag{9}$$

with the following Green function

$$G(t) = \frac{1}{2\pi} \int dp \frac{\exp(ipt)}{\omega^2 - p^2} \tag{10}$$

satisfying the equation

$$\ddot{G}(t) + \omega^2 G(t) = \delta(t). \tag{10'}$$

Putting

$$x^{(0)}(t) = \alpha \exp(i\omega t) \tag{11}$$

as the zero-order ansatz into the right hand side of (9) we get

$$x^{(1)}(t) = \exp(i\omega t) + g\alpha^n J(t). \tag{12}$$

with the integral

$$J(t) = \int dt' G(t-t') \exp(in\omega t'). \tag{12'}$$

By determining α from the requirement

$$x^{(1)}(0) = x^{(0)}(0) \tag{13}$$

we get an algebraic equation of degree n for the parameter α

$$1 + gJ(0)\alpha^n = \alpha \tag{13'}$$

possessing in general more than one root. Only one of them may be reached by the usual perturbation calculus. The problem whether the supplementary solutions starting with the non-perturbative roots of the algebraic equations like (13') converge to a really existing solution of the integral equation must be investigated in each case separately. If such non-perturbative solutions exist, the RBA method is just suitable to compute them easily. Wanelik [3] investigated recently the case of an exponential interaction exp (φ) and has demonstrated that another solution, besides the perturbative one, exists and is attainable by the RBA method.

In order to make it plausible that our method is suitable for finding non-perturbative extra solutions of non-linear problems let us consider an example even simpler than anharmonic oscillator (constituting the case of "field theory" in one dimension), viz. the algebraic equation

$$x = c + gx^2 \tag{17}$$

(constituting a "zero-dimensional case").

Assume that we were so unbelievably absent-minded that we have not noticed that (17) is nothing else but an algebraic, quadratic equation with solutions

$$x_{\pm} = \frac{1 \pm \sqrt{1-4gc}}{2g} \tag{18}$$

and try to solve it approximately by the RBA method:

$$x_{(n)} = c+gx^2_{(n-1)}, \tag{19}$$

where

$$x_{(0)} = \alpha c, \tag{20}$$

whence

$$x_{(1)} = c+g\alpha^2c^2. \tag{20'}$$

The unknown parameter α is to be determined, as usually, by equating the first with the zeroth-order approximation at the origin (which is the only point in a zero-dimensional space). The condition for α is

$$c+gc^2\alpha^2 = c\alpha. \tag{21}$$

Having noticed only now that (21) is a quadratic equation we write its roots

$$\alpha_{\pm} = \frac{1\pm\sqrt{1-4gc}}{2gc}, \tag{22}$$

whence we see, with the help of (20) or (20′), that the solution is *exactly* that given by formula (18).

It might be objected that this result is trivial, however, in this case the triviality is not a vice but a virtue of the RBA method. It shows that this method, being superior to ordinary Born method, may be also used successfully in other less trivial cases.

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