A CLASS OF TYPE D METRICS

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We study a class of solutions to the Einstein-Maxwell equations that includes the Kinnersley-Walker solutions, but without using a contraction procedure. Type D metric postulated in this paper is endowed with seven arbitrary parameters.

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1. Introduction

In the paper done by one of us (J. F. P.) and Demiański [1] (see also [2]) a sevenparameter type D solution to the Einstein-Maxwell equations with cosmological constant has been found.

Then it has been shown that all known type D solutions of E-M equations can be obtained from this sevenparameter solution by the so-called "contraction procedure" [1]. In particular in this way one finds the Kinnersley-Walker (K-W) solution [3].

In the present work we consider the sevenparameter metric that includes as its special cases both the K-W metric and the metric of Ref. [1], but without the contraction procedure.

Let $x^\mu = (p, q, \sigma, \tau)$ be real coordinates. We consider the metric of the following form

$$ds^2 = \sin^{-2}(p+q)\left[\frac{\Delta}{P}dp^2+\frac{P}{\Delta}[\cos\phi\sin^2 q\,d\tau+\sin\phi\cos^2 q\,d\sigma]^2 + \frac{\Delta}{Q}dq^2-\frac{Q}{\Delta}[-\sin\phi\cos^2 p\,d\tau+\cos\phi\sin^2 p\,d\sigma]^2\right] \tag{1.1}$$

where ϕ is a constant parameter, Δ is defined by

$$\Delta \equiv (\cos\phi\sin p\sin q)^2+(\sin\phi\cos p\cos q)^2, \tag{1.2}$$

$P = P(p)$ and $Q = Q(q)$ are arbitrary structural functions.

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2. Metric, tetrads and connections

We study a class of metrics given in real coordinates $x^\mu = (p, q, \sigma, \tau)$ by (1.1). The signature is $(+++-)$ and the units are such that $c = 1 = G$.

Select the real orthonormal tetrad $(e^{a'})$ and the complex null tetrad (e^a) according to

$$e^{1'} = \frac{1}{\sqrt{2}}(e^1+e^2) = \frac{1}{\sin(p+q)}\sqrt{\frac{\Delta}{P}}\,dp,$$

$$e^{2'} = \frac{1}{i\cdot\sqrt{2}}(e^1-e^2) = \frac{1}{\sin(p+q)}\sqrt{\frac{P}{\Delta}}\,[\cos\phi\sin^2 q\,d\tau+\sin\phi\cos^2 q\,d\sigma],$$

$$e^{3'} = \frac{1}{\sqrt{2}}(e^3+e^4) = \frac{1}{\sin(p+q)}\sqrt{\frac{\Delta}{Q}}\,dq,$$

$$e^{4'} = \frac{1}{\sqrt{2}}(e^3-e^4) = \frac{1}{\sin(p+q)}\sqrt{\frac{Q}{\Delta}}\,[-\sin\phi\cos^2 p\,d\tau+\cos\phi\sin^2 p\,d\sigma]. \tag{2.1}$$

Then

$$ds^2 = (e^{1'})^2+(e^{2'})^2+(e^{3'})^2-(e^{4'})^2 = g_{a'b'}e^{a'}e^{b'} = 2e^1e^2+2e^3e^4 = g_{ab}e^{a'}e^b \tag{2.2}$$

We have also

$$e' \equiv \det\|e^{a'}_\mu\| = \frac{\Delta}{\sin^4(p+q)}. \tag{2.3}$$

The connection forms with respect to the null tetrad given by (2.1) can be found from the 1st Cartan structure equations $(de^a+\Gamma^a{}_b\wedge e^b = 0)$:

$$\begin{matrix}\Gamma_{42}\\ \Gamma_{31}\end{matrix} = A\begin{cases}e^1\\ e^2\end{cases} - B\begin{cases}e^3\\ e^4\end{cases} \tag{2.4}$$

$$\Gamma_{12}+\Gamma_{34} = C(e^1-e^2)+D(e^3-e^4),$$

where

$$\mathrm{Im}\,(A+D) = 0 = \mathrm{Im}\,(B+C). \tag{2.5}$$

The functions A, B, C and D are given by

$$A = \sqrt{\frac{Q}{2\Delta}}\,\frac{\bar{\mu}}{\zeta},\qquad B = \sqrt{\frac{P}{2\Delta}}\,\frac{\bar{\nu}}{\zeta},$$

$$C = \sin(p+q)\cdot\sqrt{\frac{P}{2\Delta}}\cdot(\ln\Omega)_p,\qquad D = \sin(p+q)\cdot\sqrt{\frac{Q}{2\Delta}}\cdot(\ln\Omega)_q, \tag{2.6}$$

where

$$\zeta \equiv \sin\phi\cos p\cos q+i\cos\phi\sin p\sin q,$$

$$\mu \equiv \sin\phi\cos^2 p+i\cos\phi\sin^2 p,$$

$$\nu \equiv \sin\phi\cos^2 q+i\cos\phi\sin^2 q, \tag{2.7}$$

and the function Ω is given by

$$\Omega = \frac{\zeta \sin(p+q)}{(PQ)^{1/2}}. \quad (.8)$$

Moreover, we have

$$d \ln \Omega = C(e^1 + e^2) + D(e^3 + e^4).$$

We conclude this Section with the remark that as $\Gamma_{424} = 0 = \Gamma_{422}$ and $\Gamma_{313} = 0 = \Gamma_{311}$ the real null directions e^3 and e^4 are geodesic and shear-free. Changing them by a factor:

$$k_\mu^{(\pm)} dx^\mu \equiv \sin(p+q) \cdot \sqrt{\frac{2\Delta}{Q}} \begin{cases} e^3 \\ e^4 \end{cases} \quad (2.9)$$

we find that these new geodesic and shearless null directions have the common complex expansion:

$$Z = \frac{\bar{\mu} \sin(p+q)}{\zeta}. \quad (2.10)$$

3. Curvature

Having the connection (2.4) and (2.6) we can compute the curvature from the second Cartan structure equations

$$d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b = R^a{}_b \equiv 1/2 R^a{}_{bcd} e^c \wedge e^d.$$

The result reads

$$R_{12} = \frac{1}{2\Delta}(\sin^2(p+q) \cdot \ddot{P} - 3\sin 2(p+q) \cdot \dot{P} + 4[3 - 2\sin^2(p+q)] \cdot P)$$

$$+ \frac{1}{\Delta^2}(\tfrac{1}{2}(\nu\zeta + \bar{\nu}\bar{\zeta}) \sin(p+q) \cdot \dot{P} + [\nu\bar{\nu} - 2\cos(p+q) \cdot [\nu\zeta + \bar{\nu}\bar{\zeta}]]P)$$

$$- \frac{1}{\Delta^2}(\tfrac{1}{2}(\mu\zeta + \bar{\mu}\bar{\zeta}) \sin(p+q) \cdot \dot{Q} + [\mu\bar{\mu} - 2\cos(p+q) \cdot [\mu\zeta + \bar{\mu}\bar{\zeta}]]Q),$$

$$R_{34} = \frac{1}{2\Delta}(\sin^2(p+q) \cdot \ddot{O} - 3\sin 2(p+q) \cdot \dot{Q} + 4[3 - 2\sin^2(p+q)] \cdot Q)$$

$$- \frac{1}{\Delta^2}(\tfrac{1}{2}(\nu\zeta + \bar{\nu}\bar{\zeta}) \sin(p+q) \cdot \dot{P} + [\nu\bar{\nu} - 2\cos(p+q) \cdot [\nu\zeta + \bar{\nu}\bar{\zeta}]]P)$$

$$+ \frac{1}{\Delta^2}(\tfrac{1}{2}(\mu\zeta + \bar{\mu}\bar{\zeta}) \sin(p+q) \cdot \dot{Q} + [\mu\bar{\mu} - 2\cos(p+q) \cdot [\mu\zeta + \bar{\mu}\bar{\zeta}]]Q),$$

$$C^{(3)} = \frac{R}{6} + \frac{1}{\zeta^2\bar{\zeta}}\left(\bar{\mu}\left(\sin(p+q)\cdot\dot{Q} + \left[\frac{2\bar{\mu}}{\xi} - 4\cos(p+q)\right]\cdot Q\right)\right.$$
$$\left. -\bar{\nu}\left(\sin(p+q)\cdot\dot{P} + \left[\frac{2\bar{\nu}}{\zeta} - 4\cos(p+q)\right]\cdot P\right)\right). \tag{3.1}$$

In these formulae $R_{ab} \equiv R^c{}_{abc}$ are the tetrad components of the Ricci tensor, $R = R^a$ is the scalar curvature. The five complex scalars $C^{(a)}$ are the objects used to describe the conformal curvature.

In addition we obtain

$$C^{(5)} = C^{(4)} = 0 = C^{(2)} = C^{(1)} \tag{3.2}$$

and all components of the Ricci tensor, except R_{12} and R_{34}, vanish.

The result described by the formulae (3.1)–(3.2) exhibits the algebraic structure of the curvature tensor of our metric.

The case of

$$R = -4\lambda, \tag{3.3}$$

where λ is the cosmological constant, deserves attention.

As $R = 2(R_{12}+R_{34})$, the equations (3.3) and (3.1) yield

$$\sin^2(p+q)\cdot(\ddot{P}+\ddot{Q}) - 3\sin 2(p+q)\cdot(\dot{P}+\dot{Q}) + 4[3-2\sin^2(p+q)\cdot(P+Q)] = -4\lambda\Delta. \tag{3.4}$$

We intend to integrate this equation. Define $P_0 \equiv P - C + \frac{\lambda}{16}$ and $Q_0 \equiv Q + C + \frac{\lambda}{16}$. Then

$$\left(\frac{\partial^2}{\partial p^2}+16\right)\cdot\left(\frac{\partial^2}{\partial p^2}+4\right)P_0 = 0, \quad \left(\frac{\partial^2}{\partial q^2}+16\right)\cdot\left(\frac{\partial^2}{\partial q^2}+4\right)Q_0 = 0 \tag{3.5}$$

and one finds that

$$P = -\frac{\lambda}{16} + c + \alpha\sin 2p + \beta\sin 4p + \left(\frac{\lambda\cos 2\phi}{12} + b\right)\cos 2p + \left(-\frac{\lambda}{48} + a\right)\cos 4p,$$
$$Q = -\frac{\lambda}{16} - c + \alpha\sin 2q + \beta\sin 4q + \left(\frac{\lambda\cos 2\phi}{12} - b\right)\cos 2q + \left(-\frac{\lambda}{48} - a\right)\cos 4q, \tag{3.6}$$

where a, b, c, α, β and λ (cosmological constant) are free parameters of the solution.

Assuming that the structural functions have their special form (3.6), we compute the curvature quantities R_{12}, R_{34}:

$$R_{12} = -\lambda + (a + b\cos 2\phi + c)\cdot\frac{\sin^4(p+q)}{\Delta^2},$$
$$R_{34} = -\lambda - (a + b\cos 2\phi + c)\cdot\frac{\sin^4(p+q)}{\Delta^2}. \tag{3.7}$$

4. *Electromagnetic field*

Having P and Q given by (3.6) we construct a solution of the Maxwell-Einstein equations with cosmological constant. We apply the Rainich-Misner-Wheeler procedure [4, 5].

The Einstein-Maxwell equations written in tensorial notation are

$$f^{\mu\nu}{}_{;\nu} = 0, \quad \check{f}^{\mu\nu}{}_{;\nu} = 0, \tag{4.1}$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu} + \lambda g_{\mu\nu}, \tag{4.2a}$$

where

$$4\pi T_{\mu\nu} \equiv -f_{\mu\sigma} f_{\nu}{}^{\sigma} + g_{\mu\nu} F \tag{4.2b}$$

$f_{\mu\nu}$ is the tensor of electromagnetic field and $T_{\mu\nu}$ is the energy-momentum tensor of electromagnetic field. The invariants of the electromagnetic field are:

$$F = \tfrac{1}{4} f_{\mu\nu} f^{\mu\nu}, \quad \check{G} = \tfrac{1}{4} f_{\mu\nu} \check{f}^{\mu\nu}, \tag{4.3}$$

where the duality operation is defined by

$$\check{f}^{\mu\nu} \equiv \frac{i}{2 \cdot \sqrt{-g}} \varepsilon^{\mu\nu\varrho\sigma} f_{\varrho\sigma}. \tag{4.4}$$

We now assume that the metric (1.1) with the specific structural functions P and Q given by (3.6) fulfills the dynamical equations (4.1) and (4.2)

We determine the corresponding electromagnetic field by working with its orthonormal tetrad components

$$f_{a'b'} = f_{\mu\nu} e_{a'}{}^{\mu} e_{b'}{}^{\nu}. \tag{4.5}$$

Now, $g = \det ||g_{\mu\nu}|| = \det ||g_{a'b'} e^{a'}{}_{\mu} e^{b'}{}_{\nu}|| = -(e')^2$, where according to (2.3) $e' \equiv \det ||e^{a'}{}_{\mu}|| = \Delta \sin^{-4}(p+q)$. We can understand $\sqrt{-g}$ in (4.4) as given by $\sqrt{-g} = e'$. Then, the tetrad components of (4.4) are

$$\check{f}^{a'b'} = \frac{i}{2} \varepsilon^{a'b'c'd'} f_{c'd'}, \tag{4.6}$$

where $\varepsilon^{a'b'c'd'}$ is the numerical Levi-Civita symbol. Of course, indices a', b' are to be manipulated by $||g_{a'b'}|| = ||\mathrm{diag}\,(1, 1, 1, -1)|| = ||g^{a'b'}||$.

It is well known that the equations (4.1) are equivalent to the statement that the complex 2-form

$$\omega \equiv f + \check{f} = \tfrac{1}{2} (f_{a'b'} + \check{f}_{a'b'}) e^{a'} \wedge e^{b'} \tag{4.7}$$

is closed, i.e., we can replace the homogeneous Maxwell equations by the equivalent condition

$$d\omega = 0. \tag{4.8}$$

Now, the tetrad components of the Einstein equations are:

$$G_{a'b'} = \mathscr{E}_{a'b'} + \lambda g_{a'b'}, \tag{4.9}$$

where:

$$\mathscr{E}_{a'b'} = 2(-f_{a'n'}f_{b'}{}^{n'} + g_{a'b'}F). \tag{4.10}$$

In the case of our metric where all components of R_{ab}, except of R_{12} and R_{34}, vanish one finds that

$$\|G^{a'}{}_{b'}\| = \|\mathrm{diag}\,(-R_{34}, -R_{34}, -R_{12}, -R_{12})\|. \tag{4.11}$$

Therefore, having at our disposal (3.7)

$$\|G^{a'}{}_{b'}\| = \lambda\|\delta^{a'}{}_{b'}\| - (a+b\cos 2\phi + c)\frac{\sin^4(p+q)}{\Delta^2}\,\|\mathrm{diag}\,(-1, -1, 1, 1)\|. \tag{4.12}$$

Thus, in order to have this condition fulfilled

$$\|\mathscr{E}^{a'}{}_{b'}\| = -(a+b\cos 2\phi + c)\frac{\sin^4(p+q)}{\Delta^2}\,\|\mathrm{diag}\,(-1, -1, 1, 1)\|. \tag{4.13}$$

This condition holds if and only if the nonvanishing components of $f_{a'b'}$ are:

$$f_{1'2'} \equiv -\mathscr{H}, \quad f_{3'4'} \equiv \mathscr{E}, \tag{4.14}$$

If so, the only nonvanishing components of $\check{f}_{a'b'}$ are

$$\check{f}_{1'2'} \equiv i\mathscr{E}, \quad \check{f}_{3'4'} \equiv i\mathscr{H}. \tag{4.15}$$

Hence

$$\mathscr{F} \equiv F + \check{G} = -\tfrac{1}{2}(\mathscr{E} + i\mathscr{H})^2. \tag{4.16}$$

We have thus according to (4.10) and (4.14) that the components of $\mathscr{E}^{a'}{}_{b'}$ are

$$\|\mathscr{E}^{a'}{}_{b'}\| = (\mathscr{E}^2 + \mathscr{H}^2)\cdot\|\mathrm{diag}\,(-1, -1, 1, 1)\|. \tag{4.17}$$

Then, by comparing with (4.13)

$$\mathscr{E}^2 + \mathscr{H}^2 = -(a+b\cos 2\phi + c)\cdot\frac{\sin^4(p+q)}{\Delta^2}. \tag{4.18}$$

This valid, we have

$$\mathscr{E} + i\mathscr{H} = \sqrt{-(a+b\cos 2\phi + c)}\cdot e^{i\psi}\cdot\frac{\sin^2(p+q)}{\zeta^2}, \tag{4.19}$$

where ζ is given by (2.7) and real function ψ is to be determined. The phase factor $e^{i\psi}$ is related to the ambiguity of the duality rotations with precision to which the energy-momentum tensor determines the electromagnetic field in general algebraic case ($\mathscr{F} \neq 0$); we will determine ψ from Maxwell equations. From (4.7), (4.14) and (4.15) one gets

$$\omega = (\mathscr{E} + i\mathscr{H})\cdot(e^{3'} \wedge e^{4'} + ie^{1'} \wedge e^{2'}), \tag{4.20}$$

Substituting here $e^{a'}$ from (2.1) and $\mathscr{E}+i\mathscr{H}$ from (4.19) we obtain

$$\omega = \sqrt{-(a+b\cos 2\phi+c)}\cdot e^{i\psi}\cdot d\left(\frac{\sin q\cos p\, d\tau - i\sin p\cos q\, d\sigma}{\sin\phi\cos q\cos p+i\cos\phi\sin q\sin p}\right). \tag{4.21}$$

Finally, defining

$$e+ig \equiv -\sqrt{-(a+b\cos 2\phi+c)}\, e^{i\psi} \tag{4.22}$$

we have

$$\omega = (e+ig)\cdot d\left(\frac{\sin q\cos p\, d\tau - i\sin p\cos q\, d\sigma}{\sin\phi\cos q\cos p+i\cos\phi\sin q\sin p}\right). \tag{4.23}$$

The electromagnetic field is non-trivial if $e+ig \neq 0$. Having ω, we easily see that the condition $d\omega = 0$ is equivalent to $d\psi\wedge\omega = 0$. The latter condition yields (see (4.20)).

$$d\psi = 0. \tag{4.24}$$

Therefore ψ must be a constant and, consequently, e and g are real constants which characterize the electromagnetic field.

Notice that

$$\mathscr{F} = F+\check{G} = -\tfrac{1}{2}\left(\frac{\sin(p+q)}{\sin\phi\cos q\cos p+i\cos\phi\sin q\sin p}\right)^4\cdot(e+ig)^2, \tag{4.25}$$

and the following relation holds

$$a+b\cos\phi+c+e^2+g^2 = 0. \tag{4.26}$$

Observe also that if we write

$$\omega = (e+ig)\cdot d\Theta, \tag{4.27}$$

with Θ defined by

$$\Theta \equiv \frac{\sin q\cos p\, d\tau - i\sin p\cos q\, d\sigma}{\sin\phi\cos q\cos p+i\cos\phi\sin q\sin p}, \tag{4.28}$$

then using (2.1) we obtain

$$\Theta = -\frac{\sin(p+q)}{2\Delta^{1/2}}\cdot\left[\frac{i\sin 2p}{P^{1/2}}e^{2'}+\frac{\sin 2q}{Q^{1/2}}e^{4'}\right]$$

$$= -\frac{\sin(p+q)}{2(2\Delta)^{1/2}}\left[\frac{\sin 2p}{P^{1/2}}(e^1-e^2)+\frac{\sin 2q}{Q^{1/2}}(e^3-e^4)\right]. \tag{4.29}$$

These formulae determine the tetrad components of the electric and magnetic potentials in the present gauge.

Then

$$C^{(3)} = -2i\left[\alpha e^{i\phi}+2\beta e^{+i\phi}+i(e^2+g^2)\frac{\sin(q-p)}{\zeta}\right]\left(\frac{\sin(p+q)}{\zeta}\right)^3, \tag{4.30}$$

where $\zeta = \sin\phi\cos p\cos q+i\cos\phi\sin p\sin q$.

5. The sevenparameter solution and the Kinnersley-Walker metric

The metric given by (1.1) is invariant under the change $\phi \to \phi+\pi$ ($\Rightarrow \omega \to -\omega$, $\mathscr{F} \to \mathscr{F}$ and $C^{(3)} \to C^{(3)}$). Therefore, one can assume that $\phi \in (-\pi/2, \pi/2]$.

Starting with the metric of the form (1.1) with the structure functions P and Q given by (3.6), we perform the transformation of coordinates given by

$$p' = (\mathrm{tg}\,\phi)^{1/2}\,\mathrm{ctg}\,p, \quad \sigma' = -(\sin\phi\cos\phi)^{1/2}\tau,$$
$$q' = (\mathrm{tg}\,\phi)^{1/2}\,\mathrm{ctg}\,q, \quad \tau' = -(\sin\phi\cos\phi)^{1/2}\sigma, \tag{5.1}$$

where $\phi \in (0, \pi/2)$. Substituting (5.1) into (1.1) we obtain

$$ds^2 = \frac{1}{(p'+q')^2}\cdot\left(\frac{1+(p'q')^2}{P'}\,dp'^2 + \frac{P'}{1+(p'q')^2}\,[d\sigma'+q'^2 d\tau']^2\right.$$
$$\left. + \frac{1+(p'q')^2}{Q}\,dq'^2 - \frac{Q'}{1+(p'q')^2}\,[d\tau'-p'^2 d\sigma']^2\right), \tag{5.2}$$

where

$$P' = \left(-\frac{\lambda}{6} - g_0^2 + \gamma\right) + 2np' - \varepsilon p'^2 + 2mp'^3 + \left(-\frac{\lambda}{6} - e_0^2 - \gamma\right)p'^4,$$
$$Q' = \left(-\frac{\lambda}{6} + g_0^2 - \gamma\right) + 2nq' + \varepsilon q'^2 + 2mq'^3 + \left(-\frac{\lambda}{6} + e_0^2 + \gamma\right)q'^4, \tag{5.3}$$

and the arbitrary parameters from both metrics are related as follows

$$n = \left(\frac{\cos\phi}{\sin\phi}\right)^{1/2}\frac{\alpha-2\beta}{\cos^2\phi}, \quad \varepsilon = \left(\frac{\cos\phi}{\sin\phi}\right)^{2/2}\frac{6a-2c}{\cos^2\phi},$$
$$m = \left(\frac{\cos\phi}{\sin\phi}\right)^{3/2}\frac{\alpha+2\beta}{\cos^2\phi},$$
$$e_0+ig_0 = \left(\frac{\cos\phi}{\sin\phi}\right)^{2/2}\frac{e+ig}{\cos^2\phi},$$
$$\gamma = \left(\frac{\cos\phi}{\sin\phi}\right)^{4/2}\frac{g^2}{\cos^4\phi} + \frac{a-b+c}{\cos^2\phi}. \tag{5.4}$$

The corresponding expressions for the electromagnetic field, its complex invariant and the only nonvanishing component of the conformal curvature are obtained by substituting (5.1) into (4.23), (4.25), and (4.30)

$$\omega = (e_0+ig_0)\,d\left(\frac{q'd\tau'+ip'd\sigma'}{1-ip'q'}\right), \tag{5.5}$$

$$\mathscr{F} = -\tfrac{1}{2}(e_0+ig_0)^2\left[\frac{p'+q'}{1-ip'q'}\right]^4, \tag{5.6}$$

$$C^{(3)} = 2\left[(m+in)-(e_0^2+g_0^2)\frac{p'-q'}{1+ip'q'}\right]\left(\frac{p'+q'}{1-ip'q'}\right)^3. \tag{5.7}$$

This is the well known solution given in [1] (see also [2]). Here the parameters n, m, a, b, e_0, g_0 correspond to mass, NUT parameter, angular momentum, acceleration, and the electric and magnetic charge, respectively.

We obtain another class of solutions starting again with (1.1) and assuming of ϕ to be either zero or $\pi/2$. With $\phi = \pi/2$ and the transformation of corrdinates

$$p' = \operatorname{tg} p, \quad \sigma' = -\sigma,$$

$$q' = \operatorname{tg} q, \quad \tau' = \tau, \tag{5.8}$$

we obtain

$$ds^2 = (p'+q')^{-2}\left(\frac{dp'^2}{P'} + P'd\sigma'^2 + \frac{dq'^2}{Q'} - Q'd\tau'^2\right), \tag{5.9}$$

where

$$P' = \left(-\frac{\lambda}{6}+a+b+c\right)+2n_1p'-\varepsilon_1p'^2+2m_1p'^3+(a-b+c)p'^4,$$

$$Q' = \left(-\frac{\lambda}{6}-a-b-c\right)+2n_1q'+\varepsilon_1q'^2+2m_1q'^3+(-a+b-c)q'^4. \tag{5.10}$$

From (4.26) with $\phi = \pi/2$ we have $a-b+c = -(e^2+g^2)$. Then (5.10) can be rewritten as follows

$$P' = \left(-\frac{\lambda}{6}+\gamma_1\right)+2n_1p'-\varepsilon_1p'^2+2m_1p'^3-(e^2+g^2)p'^4,$$

$$Q' = \left(-\frac{\lambda}{6}-\gamma_1\right)+2n_1q'+\varepsilon_1q'^2+2mq_1'^3+(e^2+g^2)q'^4, \tag{5.11}$$

where

$$n_1 = \alpha+2\beta, \quad \varepsilon_1 = 6a-2c, \quad m_1 = \alpha-2\beta, \quad \gamma_1 = a+b+c. \tag{5.12}$$

The electromagnetic field is given by

$$\omega = (e+ig)d(q'd\tau'+ip'd\sigma'). \tag{5.13}$$

Then

$$\mathscr{F} = -\tfrac{1}{2}(e+ig)^2(p'+q')^4, \tag{5.14}$$

and the only nonvanishing conformal curvature invariant is

$$C^{(3)} = 2[m_1-(e^2+g^2)(p'-q')][p'+q']^3. \tag{5.15}$$

Now, assume $\phi = 0$. The following coordinate transformation

$$p' = \operatorname{ctg} p, \quad \tau' = -\sigma, \quad q' = \operatorname{ctg} q, \quad \sigma' = -\tau \tag{5.16}$$

leads to the metric (5.9) with the structural functions

$$\begin{aligned} P' &= \left(-\frac{\lambda}{6}+\gamma_0\right)+2n_0p'-\varepsilon_0p'^2+2m_0p'^3-(e^2+g^2)p'^4, \\ Q' &= \left(-\frac{\lambda}{6}-\gamma_0\right)+2n_0q'+\varepsilon_0q'^2+2m_0q'^3+(e^2+g^2)q'^4, \end{aligned} \tag{5.17}$$

where

$$\gamma_0 = a-b+c; \quad a+b+c = -(e^2+g^2),$$

$$n_0 = \alpha-2\beta; \quad m_0 = \alpha+2\beta; \quad \varepsilon_0 = \varepsilon_1 = 6a-2c.$$

The electromagnetic field and the invariant conformal curvature are given by (5.13)–(5.15), (the latter with $m_1 \to m_0$). The solution obtained above ($\phi = 0$ and $\phi = \pi/2$) was studied extensively by Kinnersley and Walker.

As it is well known this solution can be also obtained from the sevenparameter solution (5.2), (5.5) by the following contraction procedure [1].

$$(p', q', \tau', \sigma') \to l^{-1}(p', q', \tau', \sigma')$$

$$n \to ln_0$$

$$\varepsilon \to l^2\varepsilon_0$$

$$m \to l^3m_0$$

$$\gamma \to l^4g_0^2+\gamma_0$$

$$e_0+ig_0 \to l^2(e+ig)$$

$$\lambda \to \lambda$$

with $l \to \infty$.

Assume now that the parameter ϕ of (1.1) belongs to the interval $(-\left(\frac{\pi}{2}, 0\right)$. Equivalently, one can perform the transformation $\phi \to -\phi$ and then assume that $\phi \in \left(0, \frac{\pi}{2}\right)$. Consequently, the following transformation

$$p' = (\operatorname{ctg}\phi)^{\frac{1}{2}} \operatorname{tg} p, \quad \sigma' = (\sin\cos\phi)^{\frac{1}{2}}\sigma, \quad q' = (\operatorname{ctg}\phi)^{\frac{1}{2}} \operatorname{tg} q, \quad \tau' = -(\sin\phi\cos\phi)^{\frac{1}{2}}\tau$$

leads to (5.2) and (5.5) again.

Discussions with Dr. M. Przanowski are appreciated.

Editorial note. This article was proofread by the editors only, not by the authors.

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