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CHAOS IN YANG-MILLS MECHANICS*

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Classical Yang-Mills mechanics is studied numerically in detail. Several typical trajectories corresponding to different values of energy were found. The Lyapunov characteristic functions together with the general qualitative analysis suggest that the system for sufficiently big energy exhibits weak chaotic behaviour.

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Introduction

Weak and strong interactions are described — as is widely believed — by the Yang–Mills (Y–M) theory. This theory is based on partial differential equations of strongly nonlinear type. The methods in use for such mathematical problems are rather poor up to now. Therefore only partial solutions of the whole Y–M theory are known. From the other side the mathematical difficulties encountered in the so-called Y–M mechanics — the dynamical system closely related to the Y–M theory — are not so great. That is why it seems worth examining.

The equations of the Y–M mechanics arise as the Y–M potentials in the whole Y–M theory are assumed to be time dependent only. The actual form of these equations depends of course on a fixed gauge. In our considerations we limit ourselves to the "classical" version of the Y–M mechanics corresponding to the simplest possible gauge choice. This dynamical system may be viewed as a one-parameter conservative Hamiltonian system in four dimensional phase space [1, 2]. However the authors of the papers [1, 2] consider Y–M mechanics for a very special value of the parameter only. The theory arising then is claimed to be chaotic. From the other side the existence of the non-chaotic region is proved for a slightly different version of the Y–M mechanics [3]. Therefore in the present paper we continue this subject by accomplishing — in our opinion — a more representative choice of the above mentioned parameter. In Sect. 1, after some technical manipulations

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we get finally the base of our study: the system of two first order, time dependent differential equations. Although this system is non-Hamiltonian it is very well suited for numerical calculations. Sect. 2 involves two definitions of chaos adopted for our purposes. Sect. 3 — the main part of our paper — presents several numerical orbits of the system under study. They correspond to different values of the initial conditions and energy. Along each trajectory the linearized equations are analyzed and the appropriate Lyapunov functions are obtained. All results are presented graphically to give a better qualititative insight into considered problems. In Sect. 4 we resume our paper. Some general conclusions and comments are provided to stress that the Y–M mechanics is a chaotic system in a weak sense only.

1. Yang-Mills Mechanics

We regard Y–M mechanics as the dynamical system corresponding to SU(2) gauge theory in the way described in [1]. This system is governed by the following set of the Lagrangian type differential equations:

$$\ddot{s} = -w(sw-uz), \quad \ddot{z} = u(sw-uz), \quad \ddot{u} = z(sw-uz), \quad \ddot{w} = -s(sw-uz), \tag{1.1}$$

subject to the kinematical constraint:

$$s\dot{z}-\dot{s}z+u\dot{w}-\dot{u}w = 0. \tag{1.2}$$

(A dot over a letter denotes the differentiation with respect to time.) Introducing new variables $r_1, \theta_1, r_2, \theta_2$:

$$r_1 \cos\theta_1 = \frac{1}{\sqrt{2}}(s+w), \quad r_1 \sin\theta_1 = \frac{1}{\sqrt{2}}(u-z),$$

$$r_2 \cos\theta_2 = \frac{1}{\sqrt{2}}(s-w), \quad r_2 \sin\theta_2 = \frac{1}{\sqrt{2}}(u+z) \tag{1.3}$$

it easily turns out that the quantities $L_1 = r_1^2\dot{\theta}_1$ and $L_2 = r_2^2\dot{\theta}_2$ are constants of motion of (1.1) and the constraint (1.2) requires the equality $L_1 = L_2$. Taking this into account we may reduce (1.1) and (1.2) to the system of two equations with one parameter $L \equiv L_1 = L_2$:

$$\ddot{r}_1 = \frac{L^2}{r_1^3} - \tfrac{1}{2} r_1(r_1^2-r_2^2), \quad \ddot{r}_2 = \frac{L^2}{r_2^3} + \tfrac{1}{2} r_2(r_1^2-r_2^2). \tag{1.4}$$

The above equations can be obtained from the following Lagrangian:

$$\mathscr{L} = \tfrac{1}{2}(\dot{r}_1^2+\dot{r}_2^2) - \tfrac{1}{8}(r_1^2-r_2^2)^2 - \frac{L^2}{2}\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right). \tag{1.5}$$

This system with $L = 0$ was studied extensively in [1, 2]. It is claimed there to be chaotic. However $L \neq 0$ implies quite different behaviour of (1.5): $r_1 = 0$ and $r_2 = 0$

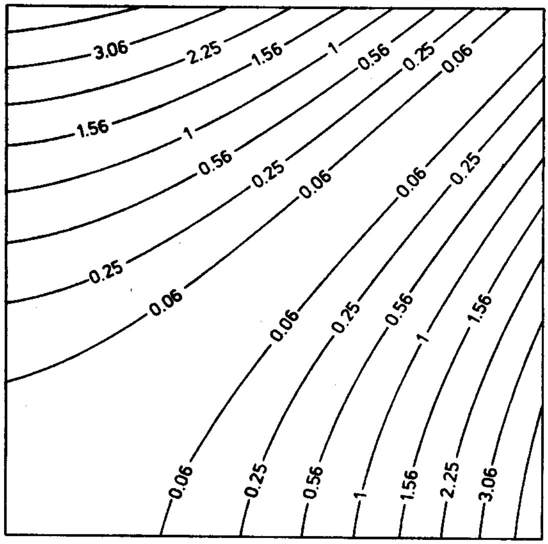

Fig. 1. The equipotential lines of $V(r_1, r_2) = \frac{1}{8}(r_1^2 - r_2^2)^2$

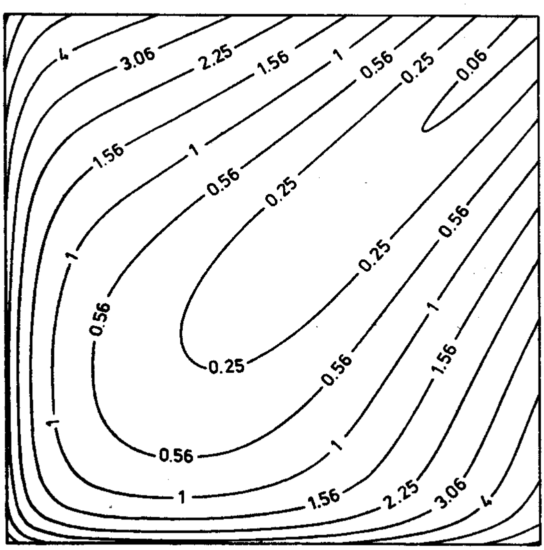

Fig. 2. The equipotential lines of $V(r_1, r_2) = \frac{1}{8}(r_1^1 - r_2^2)^2 + \frac{1}{4}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)$

straight lines become then singular. (Compare Fig. 1 and Fig. 2 where the equipotential lines are presented for $L = 0$ and $L \neq 0$.) In the present paper we analyze the equations (1.3) in the region $r_1, r_2 > 0$ with $L \neq 0$. The behaviour of the potential for big values of r_1, r_2 is similar to the appropriate case with $L = 0$ and only one trajectory of (1.4) reaching infinity is expected: namely that one with $r_1 \equiv r_2$ [2]. All other orbits are anticipated to have oscillating character both in $r_1 = r_2$ and $r_1+r_2 = const.$ directions. The total energy E of (1.5) is of course the constant of motion and (1.5) can (in principle) be reduced to the time-dependent Hamiltonian system with two-dimensional phase space [4]. In our case, however, it can be done only locally but this is useless for the numerical analysis. We reduce (1.4) to the system of the two time-dependent first order differential equations in a little bit different way. First four new variables x, p, I, ϕ are introduced:

$$x = \frac{1}{\sqrt{2}}(r_1+r_2), \quad p = \frac{1}{\sqrt{2}}(\dot{r}_1+\dot{r}_2),$$

$$\sqrt{\frac{2I}{x}}\sin\phi = \frac{1}{\sqrt{2}}(r_1-r_2), \quad \sqrt{2Ix}\cos\phi = \frac{1}{\sqrt{2}}(\dot{r}_1-\dot{r}_2) \tag{1.6}$$

(I, ϕ are the action-angle type variables). Then it can easily be checked that $\dot{\phi} > 0$; so ϕ may be chosen (instead of time) as the new independent variable. The energy formula allows us to calculate I as a function of x, p, ϕ and the parameters L, E. Finally we obtain the following system of differential equations:

$$\frac{dx}{d\phi} = f(\phi, x, p; L, E), \quad \frac{dp}{d\phi} = g(\phi, x, p; L, E), \tag{1.7}$$

where the explicit forms of f and g are rather complicated; to present them some additional functions must be introduced earlier:

$$v = \frac{1}{x}, \quad a = (2E-p^2)v^4\sin^2\phi, \quad b = 2L^2v^6\sin^2\phi,$$

$$\Delta = \tfrac{1}{3}b^3+b^2[\tfrac{1}{4}(5+4a)^2-\tfrac{1}{3}(1-a)^2]+\tfrac{2}{3}b(1-a)^3,$$

$$d_1 = \sqrt[3]{(1-a)^3+\tfrac{9}{2}b(5+4a)+9\sqrt{\Delta}},$$

$$d_2 = \sqrt[3]{(1-a)^3+\tfrac{9}{2}b(5+4a)-9\sqrt{\Delta}}, \tag{1.8a}$$

and

$$\psi = \arccos\left(\frac{-(1-a)^3-\frac{9}{2}ba^2}{(\sqrt{(1-a)^2-3b})^3}\right),$$

$$\psi_1 = \frac{\psi}{3}; \quad \psi_2 = \frac{\psi}{3}+\frac{2\pi}{3}; \quad \psi_3 = \frac{\psi}{3}+\frac{4\pi}{3},$$

$$c = \min(\cos\psi_1, \cos\psi_2, \cos\psi_3),$$

$$z = \begin{cases} \frac{1}{3}a+\frac{1}{3}(2-d_1-d_2), & \text{if} \quad \Delta \geqslant 0, \\ \frac{1}{3}a+\frac{2}{3}(1+c\sqrt{(1-a)^2-3b}), & \text{if} \quad \Delta < 0. \end{cases} \tag{1.8b}$$

Using this notation we find that the functions f, g from the formula (1.7) have the form:

$$f = \frac{pv}{m(\phi, x, p; L, E)}, \quad g = \frac{l(\phi, x, p; L, E)}{m(\phi, x, p; L, E)}, \tag{1.9a}$$

where

$$l(\phi, x, p; L, E) = -x^2 z + 2L^2 v^4 \frac{1+3z}{(1-z)^3},$$

$$m(\phi, x, p; L, E) = 1 + pv^2 \sin\phi \cos\phi + b\frac{3+z}{(1-z)^3}. \tag{1.9b}$$

Although the obtained system of differential equations (1.7) is non-Hamiltonian it is well suited for a numerical study. We will examine it for several different values of energy E and one fixed value of the L parameter: namely $L^2 = 0.5$ (the influence of a value of L on the qualitative behaviour of our system is very weak).

2. *Chaos in dynamical systems*

The notion of chaos is not unique. Let us briefly recall two simple definitions of chaos and adopt them for our purposes.

Let $\mathcal{M}$ be a compact symplectic manifold and F_{t_2,t_1} the evolution operator corresponding to some Hamiltonian H on $\mathcal{M}$ (that is $F_{t_2,t_1}(q_1, p_1) = (q_2, p_2)$ where $(q_1, p_1) = (q(t_1), p(t_1))$, $(q_2, p_2) = (q(t_2), p(t_2))$; $(q(t), p(t))$ is the solution of the Hamiltonian equations generated by H and (q, p) denotes of course a point of $\mathcal{M}$). We say that the dynamical system $(\mathcal{M}, H)$ is chaotic if the following condition is valid [4]:

$$\lim_{t_2\to\infty} \frac{\mathrm{vol}\,(F_{t_2,t_1}(\mathcal{F}) \cap \mathcal{G})}{\mathrm{vol}\,(\mathcal{F})} = \frac{\mathrm{vol}\,(\mathcal{G})}{\mathrm{vol}\,(\mathcal{M})}, \tag{2.1}$$

where $\mathcal{F}$, $\mathcal{G}$ are arbitrary subareas of $\mathcal{M}$ and vol ( ) denotes the volume of the appropriate area. We propose to call the dynamical system strongly chaotic if it satisfies the above definition. Let us stress that in practice the condition (2.1) is very difficult to check. Although our system of differential equations (1.7) is non-Hamiltonian and the space of (x, p) is not compact it seems worth asking if some mixing property similar to (2.1) is fulfilled. We will return to this question in Sect. 2.

The second notion of chaos we want to adopt is concerned with the so-called Lyapunov exponents [6, 7]. Let us consider a set of differential equations:

$$\frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}), \tag{2.2}$$

where $\vec{x}$, $\vec{f}$ are some n-dimensional vectors. Let $\vec{x}(t)$ be the solution of (2.2) satisfying the initial condition $\vec{x}(t_0) = \vec{x}_0$. The first variation $\Delta\vec{x}$ of the trajectory $\vec{x}(t)$ obeys the set of linear non-autonomous differential equations:

$$\frac{d(\Delta\vec{x})}{dt} = ((\Delta\vec{x}) \circ \vec{\nabla}_{\vec{x}})\vec{f}(t, \vec{x}(t)). \tag{2.3}$$

We define (the one- and two-dimensional) Lyapunov functions in the following manner:

$$\lambda_1(t, \vec{x}(t); t_0, \Delta\vec{x}_0) = \frac{\|U(t, t_0; \vec{x}(t))\Delta\vec{x}_0\|}{\|\Delta\vec{x}_0\|},$$

$$\lambda_2(t, \vec{x}(t); t_0, \Delta\vec{x}_0, \Delta\vec{y}_0) = \frac{\|U(t, t_0; \vec{x}(t))\Delta\vec{x}_0 \wedge U(t, t_0; \vec{x}(t))\Delta\vec{y}_0\|}{\|\Delta\vec{x}_0 \wedge \Delta\vec{y}_0\|}, \tag{2.4}$$

where $U(t, \vec{x}(t); t_0)$ is the time evolution operator of (2.3) and $\Delta\vec{x}_0$, $\Delta\vec{y}_0$ are the n-dimensional vectors playing the role of the initial conditions at $t = t_0$. The Lyapunov exponents are simply the limits of the appropriate Lyapunov functions for $t \to \infty$. We call the trajectory $\vec{x}(t)$ chaotic if

$$\lambda_{max} = \max_{\Delta\vec{x}_0} (\lim_{t\to\infty} \lambda_1(t, \vec{x}(t); t_0, \Delta\vec{x}_0)) > 0. \tag{2.5}$$

The system (2.2) is chaotic if all its orbits are chaotic in the above sense. Of course λ_{max} cannot be obtained in numerical calculations and we will only examine the Lyapunov functions for possibly big values of time. Let us note that λ_2 gives some information about non-Hamiltonian character of the system under study; it is strictly zero for Hamiltonian systems.

3. *Numerical results*

Let us begin this Section with some important remarks. We have tried to find some typical orbits of the system (1.7) for different values of E. Several numerical algorithms have been applied: the fifth order Runge-Kutta method with adaptive stepsize control [8], the Bulirsch-Stoer method with the Richardson extrapolation [8] and the Gear's algorithm for stiff differential equations [9]. All trajectories we have found start with $\phi_0 = 0$, $p(\phi_0) = 0$ and different values of $x(\phi_0)$. Each orbit was calculed forward in inde-

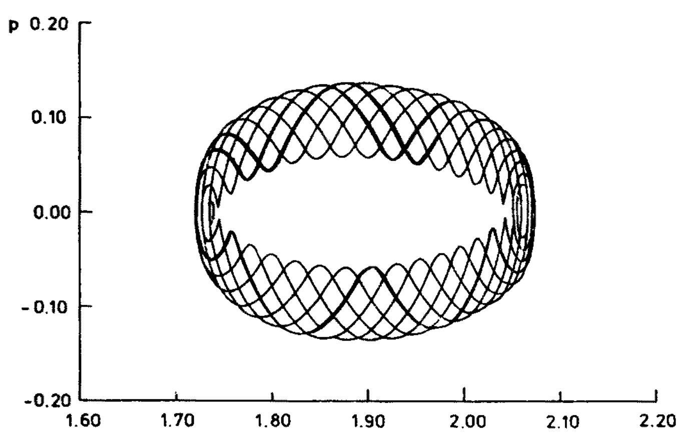

Fig. 3. An example of a trajectory with a small amplitude ($E = 0.5$, $x_0 = 1.72$)

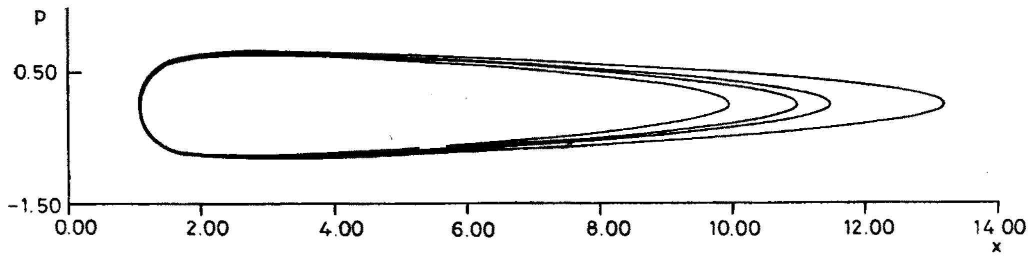

Fig. 4. An example of a trajectory with a big amplitude ($E = 0.5$, $x_0 = 1.1$)

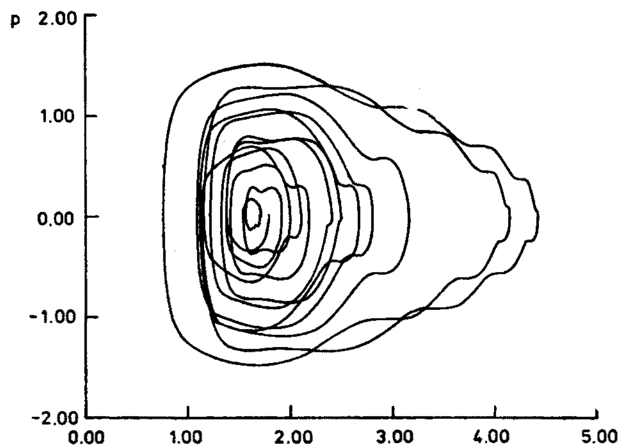

Fig. 5. Another example of a trajectory with a small amplitude ($E = 2.0$, $x_0 = 1.785$)

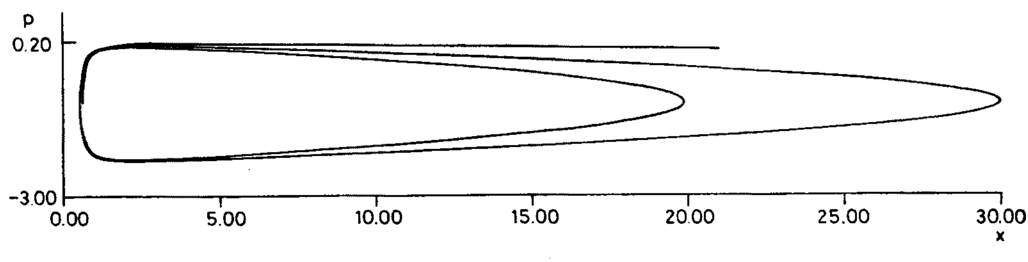

Fig. 6. Another example of a trajectory with a big amplitude ($E = 2.0$, $x_0 = 0.6$)

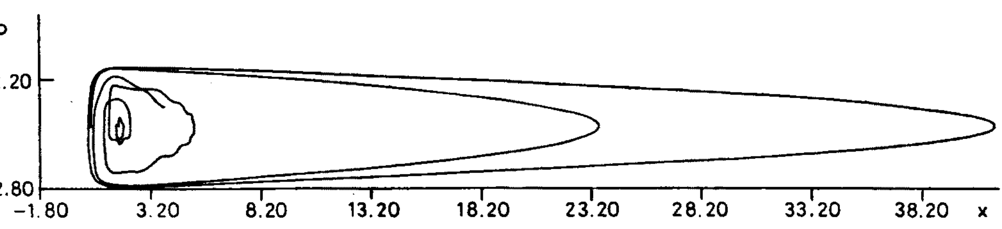

Fig. 7. An example of an ergodic trajectory ($E = 4.0$, $x_0 = 0.55$)

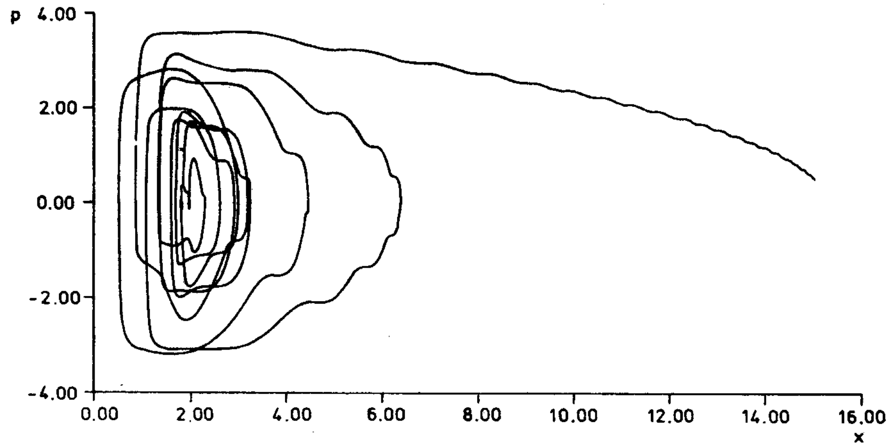

Fig. 8. Another example of an ergodic trajectory ($E = 8.0$, $x_0 = 1.85$)

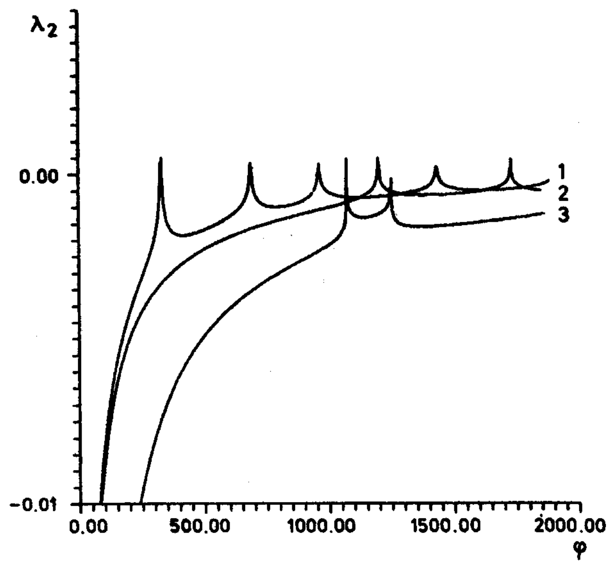

Fig. 9. The two dimensional Lyapunov functions for trajectories with big amplitudes (*1*: $E = 0.5$, $x_0 = 1.1$; *2*: $E = 2.0$, $x_0 = 0.6$; *3*: $E = 8.0$, $x_0 = 0.525$)

pendent variable ϕ to some $\phi_1 > \phi_0$ and then backward to ϕ_0 to restore the appropriate initial conditions. This was a test of the accuracy of the chosen numerical algorithm. The maximal possible value ϕ_1 was reached in each case. Along the fixed orbit the Lyapunov functions were calculated for different $\Delta\vec{x}_0$, $\Delta\vec{y}_0$. The following results have been obtained:

As was anticipated earlier in Sect. 1 all trajectories have oscillating character but the amplitude in the x-direction e.g. $x_{max} - x_{min}$ is different for each oscillation. It is impossible to foresee its changes. For small values of E (for instance $E = 0.5$) two types of orbits should be distinguished: those with $x_{max} - x_{min}$ less than ~ 1.0 (Fig. 3) and

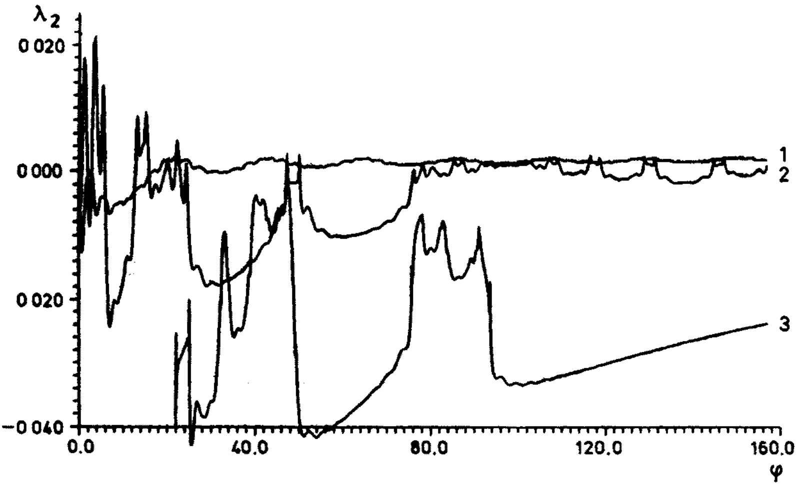

Fig. 10. The two dimensional Lyapunov functions for trajectories with small amplitudes (*1*: $E = 0.5$, $x_0 = 1.72$; *2*: $E = 2.0$, $x_0 = 1.785$; *3*: $E = 8.0$, $x_0 = 1.85$)

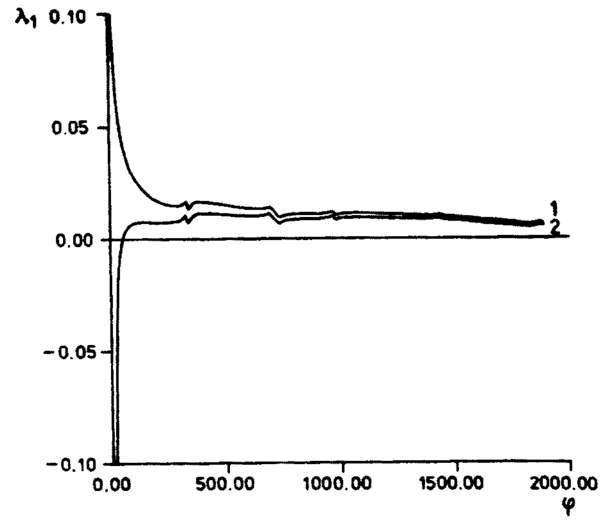

Fig. 11. Temporal convergence of the one dimensional Lyapunov function ($E = 0.5, x_0 = 1.1$; *1*: $\Delta\vec{x}_0 = (1.0, 0.0)$; *2*: $\Delta\vec{x}_0 = (0.0, 1.0)$)

those with $x_{max} - x_{min}$ greater than $\sim$ 10.0 (Fig. 4). This feature seems to remain valid for $E \simeq 2.0$ (Fig. 5 and Fig. 6) but it fails for $E \geqslant 4.0$. Let us compare two trajectories (Fig. 7 and Fig. 8). The oscillations of the first ($E = 4.0$) are at the beginning quite large but then become very small. The amplitude of the second ($E = 8.0$) increases rapidly after some small oscillations. Thus using the terminology of Sect. 2 we can say that our system is rather regular for small E and becomes chaotic for greater values of E indicating the mixing property. Of course this statement is based on pure qualitative analysis only

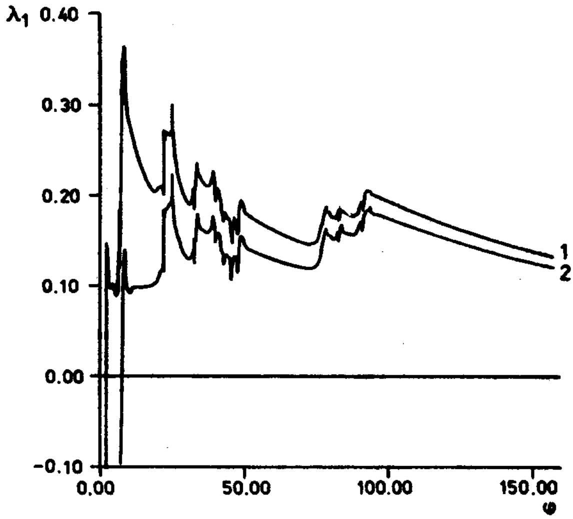

Fig. 12. Temporal convergence of the one dimensional Lyapunow function ($E = 8.0$, $x_0 = 1.85$; *1*: $\Delta\vec{x}_0 = (1.0, 0.0)$; *2*: $\Delta\vec{x}_0 = (0.0, 1.0)$)

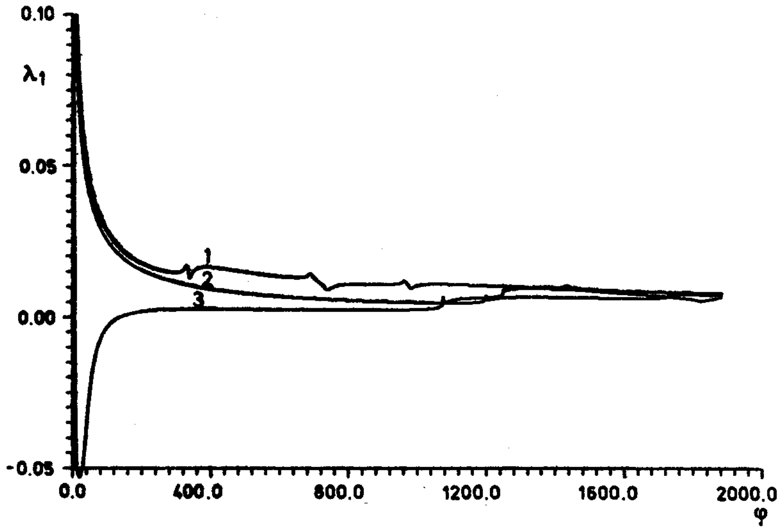

Fig. 13. The one dimensional Lyapunov functions for the trajectories with big amplitudes (*1*: $E = 0.5$, $x_0 = 1.1$; *2*: $E = 2.0$, $x_0 = 0.6$; *3*: $E = 8.0$, $x_0 = 0.525$)

and the strong condition (2.1) cannot be applied here. It should also be noted that the system under study is close to Hamiltonian system because the Lyapunov function λ_2 remains very small all the time (Fig. 9 and Fig. 10).

Let us now apply the second definition of chaos from Sect. 2. We have examined the Lyapunov function λ_1 for all orbits presented earlier. Their shapes and magnitudes —

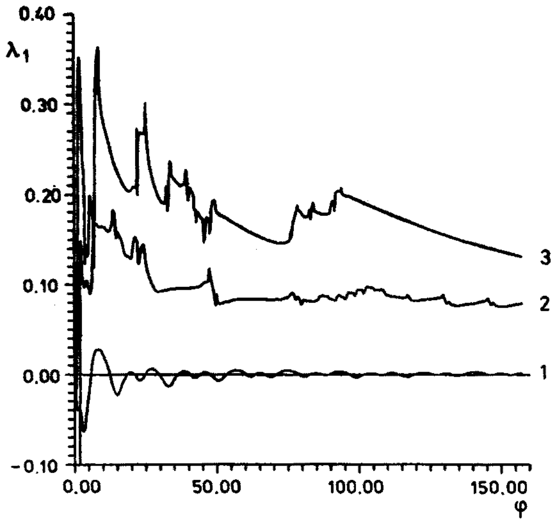

Fig. 14. The one dimensional Lyapunov functions for the trajectories with small amplitudes (*1*: $E = 0.5$, $x_0 = 1.72$; *2*: $E = 2.0$ $x_0 = 1.785$; *3*: $E = 8.0$, $x_0 = 1.85$)

as is expected for chaotic systems [7] — do not depend on $\Delta\vec{x}_0$ (at least for sufficiently big values of ϕ — Fig. 11 and Fig. 12 illustrate this fact for $E = 0.5$ and $E = 8.0$ correspondingly). But the main result of our analysis is the following one: the Lyapunov function λ_1 does not depend in practice on energy E for trajectories with big amplitudes (Fig. 13) and rather strongly increases with E for orbits with small amplitudes (Fig. 14). More generally: the region of small x, p is more chaotic than the rest of the (x, p)-space.

4. Conclusions

Let us briefly resume our considerations. First of all it should be noted that our numerical results of Sect. 3 do not depend on the chosen algorithm. So these results are believable. However they give mainly the qualitative description only. This description shows that our system is chaotic for sufficiently big energy in the area of small values of x and p. Although we did not examine the system for different values of the L it is not expected to depend strongly on this parameter. It should be stressed that the greatest value of the Lyapunov function λ_1 we have obtained does not exceed the value ~ 0.25. It is not too much comparing to ~ 1.45 of the Lorentz system which is regarded as the "classical" chaotic system [6]. Also the mixing property in our system seems to be limited because the orbits with extremely big amplitudes remain such for all oscillations. However these trajectories are very difficult to examine. Thus generally speaking the Y–M mechanics is the dynamical system exhibiting a weak chaotic behaviour only (at least in (x, p) surface) and the attempts to interpret the main features of the whole Y–M theory on the base of chaos seem to be exaggerated.

REFERENCES

[1] S. G. Matinyan, G. K. Savvidi, N. G. Ter-Arutyunyan, *Zh. Eksp. Teor. Fiz.* **80**, 830 (1981).
[2] I. Frøland, *Phys. Rev.* **D27**, 943 (1983).
[3] U. Kursawe, E. Malec, *J. Math. Phys.* **26**, 2643 (1985).
[4] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, MIR, Moskva 1975, Sect. 45.2.
[5] V. I. Arnold, *Additional Chapters in the Theory of Ordinary Differential Equations*, Nauka, Moskva 1978, Sect. 13. B.
[6] I. Shimada, T. Nagashima, *Prog. Theor. Phys.* **61**, 1605 (1979).
[7] P. W. Cleary, *J. Math. Phys.* **30**, 689 (1989).
[8] W. H. Press, B. P. Flannery, S. A. Teukolsky, W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press 1985, Chap. 15.
[9] C. W. Gear, *Commun. ACM* **14**, 185 (1971).