VISCOUS CAUSAL COSMOLOGIES

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(Received June 13, 1989)

We examine a set of spatially homogeneous and isotropic cosmological geometries generated by a class of non-perfect fluids. The irreversibility of this system is studied in the context of causal thermodynamics which provides a useful mechanism to conform to the non-violation of the causal principle.

PACS numbers: 98.80.Dr

1. Introduction

One of the main outstanding problems in cosmology is the so called singularity problem. The standard Big-Bang model proposes that the Universe evolved from an explosive origin, which is supposed to have happened a few billion of years ago. Although such a model became standard in our decade, there has been an increasing number of attempts to overcome this unpleasant situation, because for a physicist it is hard to deal with such uncomprehensible hypothesis as a common origin of everything in our very near past [1].

Here, we are interested only in two particular examples of alternative non-singular solutions: one, due to Murphy [2] and another one due to two of us (Salim & Oliveira) [3]. Although both these solutions do indeed lead to the avoidance of singularity (at least in a finite distance from us), the individual behavior of each of these solutions is quite distinct. One of them (M) is highly unstable, as it has been proved by Belinsky et al. [4]; the other (SO) is stable in a sense which will be made precise later on. Both solutions share another common property: they describe geometries whose sources are non-perfect fluids. In the last decade the interest in the study of gravitational processes involving non-perfect fluids has been growing considerably. Besides the property to make possible to avoid cosmical singularity there are other complementary reasons for that: the possibility to describe the interaction of fields of different types with gravity, the gravitational consequences for systems off thermodynamical equilibrium; and so on.

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The work of Belinsky et al. was considered one of the main reasons to believe in the ineficiency of non-perfect fluids in the avoidance of the cosmical singularity. Indeed, these authors showed that Murphy's solution is not stable under anisotropic perturbations. Once the Universe enters in this stage it decays almost promptly into a singular solution, which us back to the original question. However, this is not the case, in the (SO) solution. The reason for such distinct behavior is the use of causal thermodynamics, as will be seen in what follows.

2. *Non-equilibrium thermodynamics*

Although a complete theory of systems far from equilibrium and interacting gravitationally is not yet available, there are some general schemes to describe this situation which can be accepted with a reasonable degree of confidence.

Classical non-equilibrium thermodynamics needs (besides the standard variables that characterize the evolution of a general fluid) — the introduction of a four-vector current S^μ which is assumed to be a smooth well-behaved function of the universal variables that characterizes the fluid, e.g., the stress-energy tensor $T_{\mu\nu}$ and the four vector N^μ which represents the current of particles. We write

$$s^\mu = s^\mu(T_{\alpha\beta}, N_\lambda). \tag{1}$$

Let us represent by Σ the total amount of production of entropy. Then, the fundamental principle of thermodynamics implies that Σ is a non-negative quantity. Besides by the same token, Σ must depend on the same set of variables, $\Sigma = \Sigma(T_{\alpha\beta}, N_\lambda)$. This is nothing but the almost direct transposition of the postulate of the continuity equation from thermostatics to thermodynamics.

From the current of particles N^μ and of entropy S^μ we construct the quantity

$$s = \frac{1}{n^2} N^\mu S_\mu, \tag{2}$$

which defines the specific entropy per particle. In this formula the quantity n is the inverse of the specific volume $\frac{1}{v}$. If the system is in an equilibrium state we can set $N^\mu = nv^\mu$ and $T_{\mu\nu} = \varrho V_\mu V_\nu - ph_{\mu\nu}$ in which $h_{\mu\nu} = g_{\mu\nu} - V_\mu V_\nu$ is the projector into the 3 dimensional rest space of V^μ. The specific entropy $s = s(\varepsilon, v)$ is obtained as a solution of the Gibbs-Duhem equation. Note that we have introduced the internal energy per particle through the standard definition $\varepsilon = \frac{\varrho}{n} - m_0$, and m_0 is the rest mass of the constituents of the fluid. The states thus defined constitute a linear space E of finite dimension parametrized by the five quantities $\alpha = \frac{\mu}{T}$ and $\beta^\mu = \frac{1}{T} V^\mu$, in which μ is the relativistic chemical potential and T is the temperature.

In order to deal with dissipative processes we must extend such standard formalism by introducing some new dissipative variables. In this paper we restrict our considerations to the case in which any direct gravitational influence can be neglected. Besides this, we will take for granted that the dissipation phenomena occur in such a scale that we can neglect the average value of the curvature of space-time, that is $\langle R_{\alpha\beta\mu\nu}\rangle = 0$; and furthermore we can neglect any heat flux q^μ and anisotropic pressure $\pi^{\mu\nu}$. (Let us stress here that such a simplification is not dictated by any thermodynamical property but it is due only to our actual purpose here to work in spatially homogeneous and isotropic cosmological models.) Thus, within such simplified hypothesis there is no room for q^μ and $\pi^{\mu\nu}$ to appear in our present analysis.

We can then set

$$T_{\mu\nu} = \varrho V_\mu V_\nu - (p_{\text{th}} + \pi)h_{\mu\nu} \tag{3}$$

in which p_{th} is the thermodynamical pressure and π represents the isotropic viscous pressure. From the conservation of $T_{\mu\nu}$ we obtain

$$\dot{\varrho} + (\varrho + p_{\text{th}} + \pi)\theta = 0.$$

The specific entropy s depends, in the general case, on the internal energy ε, on the specific volume v and on π:

$$s = s(\varepsilon, v, \pi).$$

The Gibbs-Duhem generalized relation provides the evolution of s. We adopt the standard equations of state and set

$$\frac{\partial s}{\partial \varepsilon} = \frac{1}{T}, \qquad \frac{\partial s}{\partial v} = \frac{p_{\text{th}}}{T}, \qquad \frac{\partial s}{\partial \pi} = \frac{\alpha}{T}\pi.$$

The parameter α, which is a function of ε and v is related to the relaxation time of the dissipative processes. The quantities T and p are straightforward generalizations of the corresponding variables in the equilibrium. The Gibbs-Duhem equation yields

$$T\dot{s} = \dot{\varepsilon} + p_{\text{th}}\dot{v} + \alpha v \pi \dot{\pi}. \tag{4}$$

The phenomenological law which describes the evolution of the dissipative variable is obtained using the equation of balance of the entropy

$$S^\mu_{;\mu} = n\dot{s} + I^\mu_{;\mu} = \Sigma \geqslant 0 \cdot \tag{5}$$

in which I^μ is the flux of entropy.

We now move to the post-linear approximation and make the standard hypothesis that the flux I_μ depends on the same set of variables which guide the evolution of s. This has the direct consequence that the expansion of I^μ becomes proportional to the heat flux, yielding in the present case that I^μ vanishes. Using (4) and (5) and the form of Σ as being given by:

$$\Sigma = \frac{1}{T}(\alpha\dot{\pi} - \theta)\pi. \tag{6}$$

we obtain for the expansion, up to the first order,

$$\alpha\dot{\pi} = M_{(1)}\pi+\theta. \tag{7}$$

The Newtonian limit of this theory implies then that the parameter $M_{(1)}$ is given by

$$M_{(1)} = \frac{1}{\chi T} \geqslant 0,$$

in which χ is the bulk viscosity coefficient.

We have thus achieved our goal in the form of the equation (7). Let us now apply this formalism to the cosmical scenario.

3. *The cosmic viscous fluid*

We will take the geometry as being given by a spatially homogeneous and isotropic Universe:

$$ds^2 = dt^2-R^2(t)\,(dx^2-dy^2+dz^2). \tag{8}$$

We have chosen to work in flat (Euclidean) space section to simplify our presentation here. For the fluid velocity $V^\mu = \delta^\mu_0$ in the Gaussian system of coordinates (8), all kinematical parameters vanish identically except the expansion factor $H = \frac{\theta}{3} = \frac{\dot{R}}{R}$. Then if (8) is to be a solution of Einstein's equations of General Relativity, it follows naturally that the heat flux and the anisotropic pressure must vanish. Then

$$T_{\mu\nu} = \varrho V_\mu V_\nu-(p_{\rm th}+\pi)h_{\mu\nu}.$$

The viscous pressure must satisfy the causal requirement

$$\dot{\pi}\tau_0+\pi = -3\zeta H. \tag{9}$$

The remaining set of Einstein's equations are

$$\varrho = 3H^2-\Lambda, \tag{10a}$$

$$\pi+\lambda\varrho = -2\dot{H}-3H^2+\Lambda, \tag{10b}$$

in which $p_{\rm th} = \lambda\varrho$. It seems worth to remark that contrary to the case of the standard model (in which entropy is conserved throughout the whole history of the Universe) or like in some previous viscous models e.g. Murphy's solution (in which, although entropy is not constant, there is no evolutionary equation for the bulk viscosity), here we have introduced another dynamical variable π governed by equation (9) giving origin to a coherent causal scheme.

Instead of looking for special solutions of this set (9)–(10) of equations we decided to examine the whole set of the integral curves. This is possible due to the fact that (10) is an autonomous planar system of differential equations in the variables π and H that defines the phase plane (π, H).

We have

$$\dot{H} = F(H, \pi) = -\tfrac{3}{2}(1+\lambda)H^2 - \frac{\pi}{2} + \frac{1+\lambda}{2}\Lambda, \tag{11a}$$

$$\dot{\pi} = G(H, \pi) = -\frac{1}{\tau_0} - \frac{3\xi}{\tau_0}H, \tag{11b}$$

and Eq. (10a) is the definition of ϱ.

The existence of finite singular points (that is, the points (H_0, π_0) in the phase plane in which the functions F and G vanish simultaneous) depends on the value of the cosmological constant Λ. As we will see later on, the topological structure of the integral curves

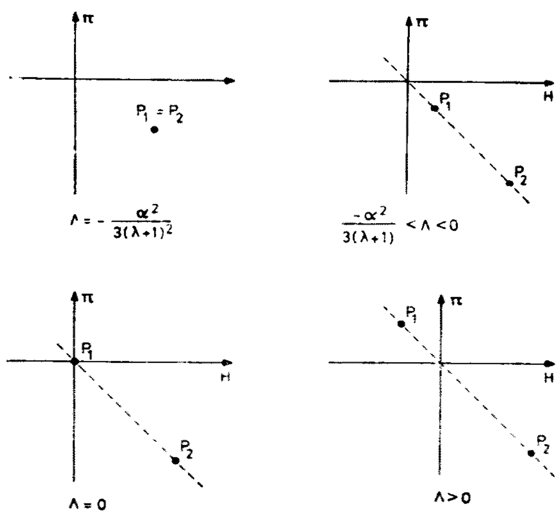

Fig. 1. P_1 and P_2 are the singular points of system (11) in the finite domain

in the neighborhood of these singular points depends on Λ too. However, the behaviour at infinity is independent of Λ. Just for simplicity we restrict our considerations here to the case in which ξ and τ are constants. We set $\xi = \frac{2}{3}\alpha = constant$. For $\Lambda < -\dfrac{\alpha^2}{3(1+\lambda)^2}$ there is no singular point in the finite region. Beyond this value, two distinct singular points appear (see Fig. 1). Let us make some comments on the general behaviour of the integral curves in the phase plane.

In the case of $\Lambda < -\dfrac{\alpha^2}{3(1+\lambda)^2}$ the non existence of singular points makes the configuration in the phase plane as given in Fig. 2. A solution which starts at the singularity, in the point A, ends at the antipodal singularity A', can have two typical behaviours. Either

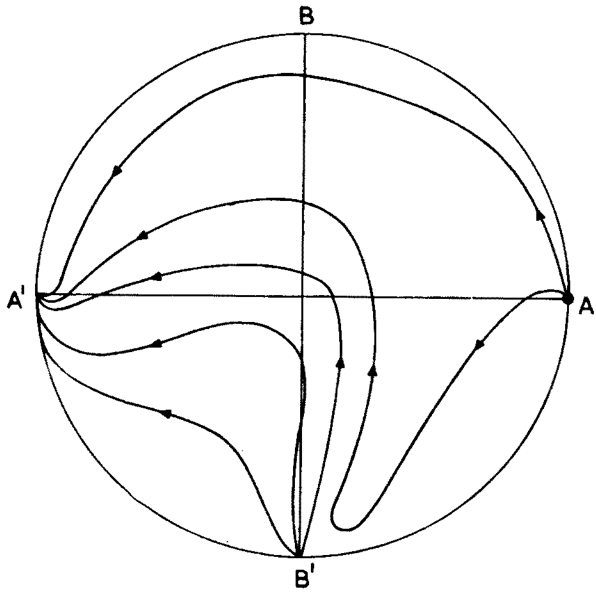

Fig. 2. Compactification of the whole plane of system (11) — Points A, A', B and B' are singular points at infinite. In this case $\Lambda < -\frac{\alpha^2}{3(1+\lambda)^2}$, and there is no finite singular point

it rests during all its history with positive viscosity ($\pi > 0$) or it enters a region which changes the sign of π. In this second case it can attain very high values of (negative) π corresponding to very small values of the expansion before the entrance in the same regime as in the first case near A'.

The configurations depicted in the graphs are almost self evident. For example let us make some comments on Fig. 3 in the case $\Lambda = 0$ and Fig. 4 for $\Lambda > 0$. There is practically no distinction between the configurations in the cases $\Lambda = 0$ and $\Lambda > 0$. In these cases there are two finite singular points: P_1 and P_2. For $\Lambda = 0$, the point P_1 is the origin 0. The origin is nothing but the unstable Minkowski space-time. The point P_2 represents a de Sitter Universe with expansion $H = \frac{2\alpha}{3(1+\lambda)}$ and constant viscous pressure $\pi = \frac{4\alpha^2}{3(1+\lambda)}$. Near the point P_2 we can approximate the generic behaviour of the Universe by $R(t) \sim \exp \frac{\alpha t}{3(1+\lambda)}$. Note that such a de Sitter solution is stable against all perturbations within the present scheme (that is, for perturbations of the system of Eq. (11)). There is a class of cosmological models that starts at the point A as a singular cosmos at past infinity and goes into the de Sitter attractor P. All these solutions have an infinite expansion at A and acquire rapidly a negative viscous pressure, which is a necessary condition to enter in the neighborhood of the de Sitter cosmos P. Note that A and B (besides the antipodals A' and B') are singular points at infinity. At the point A there exists a singular-

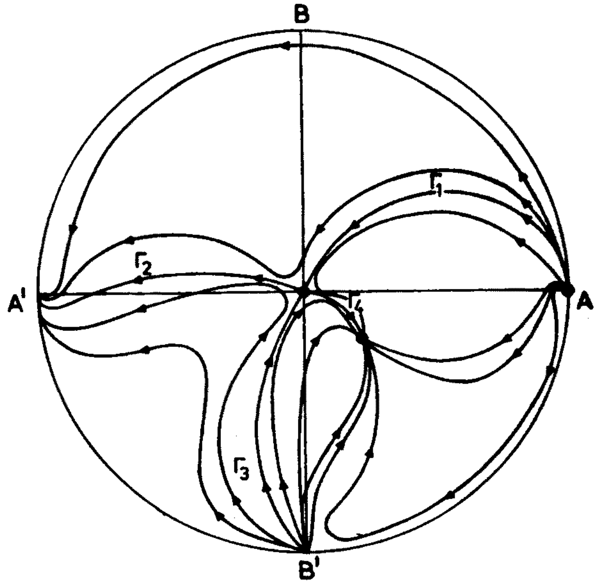

Fig. 3. The case $\Lambda = 0$. Note that besides the origin (Minkowski space-time) there is another singular point in the finite domain for $H_0 = \frac{2}{3}\frac{\alpha}{1+\lambda}$ and $\pi_0 = \frac{4\alpha^2}{3(1+\lambda)}$ which represents a de Sitter Universe without cosmological constant. The role of Λ is played by the viscosity π

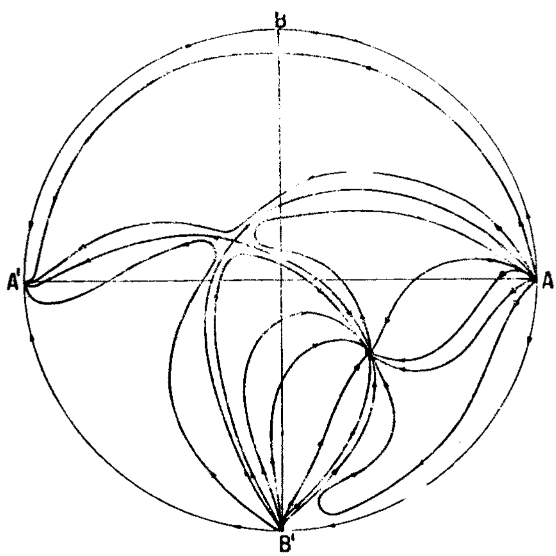

Fig. 4. The case in which $\Lambda > 0$. See the text

ity with $\varrho = \pi = \infty$. In the case $\Lambda > 0$ there are two finite singular points (see Fig. 1). Point P_2 does not represent a Minkowski space-time, but a de Sitter Universe which ever contracts by an ammount given by $H = -\frac{\alpha}{3(\lambda+1)}\left(\sqrt{1+\frac{3(\lambda+1)^2\Lambda}{\alpha^2}}-1\right)$ and a constant viscous pressure.

From the point A there is a separatrix Γ_1 which goes into the Minkowski origin O. If a curve starts at A with an initial value of viscosity π higher than that of curve Γ_1 then all these solutions penetrate the region of contraction ($H < 0$) and end at the antipodal singularity A'. There are three more curves which attain the Minkowski world at O (in the case $\Lambda = 0$). The curve called Γ_3 represents a world that starts with $\pi = -\infty$ and

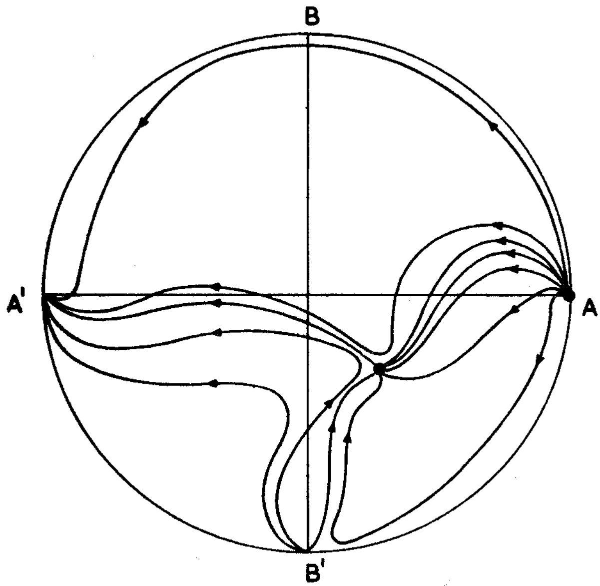

Fig. 5. The case $\Lambda = -\frac{\alpha^2}{3(1+\lambda)^2}$. See the text

an infinite density. It separates the phase plane into two regions: If at B' a curve has a value of H bigger (in absolute value) than its corresponding value at Γ_3 then it belongs to a class of integral curves which represent an infinite contracting Universe which ends at the singular point A'. The curves which near B' have smaller values of H than Γ_3, have all the same property: they end at the de Sitter model at P_2.

Finally, separatrices Γ_2 and Γ_4 have very distinct behaviour: curve Γ_2 starts at the Minkowski world at O and ends at the singularity A'; curve Γ_4 starts at the Minkowski world at O and ends at the de Sitter world P_2.

For $\Lambda > 0$ there is a particular solution that starts at the infinite point A and ends at the de Sitter world P_2. The analytical form of this case has been exhibited recently by two of us [3]. One can exhibit the analytical form of this solution, which has no physical singularity:

$$R(t) = R_0 \exp\left[2\tau_0(1+\lambda)\left\{\Lambda t - \frac{C_{(1)}}{1+\lambda}\exp\left(\frac{-t}{2\tau_0}\right)\right\}\right]$$

in which $C_{(1)}$ is a constant.

Note that any small perturbation of this geometry has the same qualitative behaviour, ending sooner or later in the de Sitter cosmos P_2. This property exhibits the main advantage of the viscous causal mechanism of avoidance of singularity: its stability behaviour.

In order to complete the analysis, we depict the case where $\Lambda = \dfrac{-\alpha^2}{3(\lambda+1)^2}$ in Fig. 5. In this case there is only one singular point in the finite region. Such point represents a de Sitter Universe, that is generically unstable although having in the phase plane a domain of stability. The analysis of the curves is, in general, similar to the precedent cases.

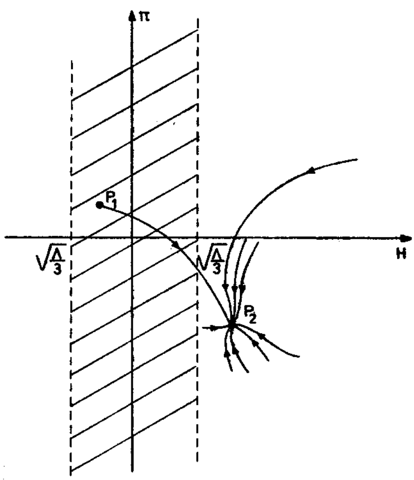

Fig. 6

Finally, let us point out that for $\Lambda > 0$, there is a possibility of the appearance of classically forbidden regions in the phase space. Such regions are characterized by $\varrho < 0$ (see Fig. 6). Thus P_1 for instance, which represents the unstable de Sitter Universe, is not a physically satisfactory solution as well as all remaining curves situated inside the region shadowed in Fig. 6.

Editorial note. This article was proofread by the editors only, not by the authors.

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