SEMICLASSICAL RESONANCE IN ROTATING MAGNETIC FIELDS

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The interaction matter-radiation is modeled by a harmonic oscillator in a rotating magnetic field. Recently discussed resonance phenomena involving the simultaneous absorption of several photons are detected in our model and described without the use of perturbative procedures. In some aspects this model is an alternative to the widely used rotating electric field model.

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1. *Introduction*

One of the fundamental subjects in Quantum Physics has been the interaction between electromagnetic waves and a bound system. A theoretical treatment of this process should basically include the quantization of both, matter and radiation. However, a lot of knowledge is achieved treating the incident field from a classical point of view (semiclassical approximation). The motion of a charged particle in the external electromagnetic field is described by the standard time dependent Hamiltonian:

$$H(t) = \frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}(\vec{x},t)\right)^2 + V(\vec{x},t), \tag{1}$$

where $V(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ stand for the atomic binding forces as well as for the external radiation. In this frame, the monochromatic radiation is usually represented by a plane wave of frequency ω. Consistently with Einstein's idea, the resonance phenomenon should occur when $\hbar\omega = E_k - E_s$ ($E_k > E_s$), with E_k, E_s corresponding to the energy levels of the bound system [1] and the atom ionizes only if $\hbar\omega \geqslant V_0$, V_0 being the ionization potential. However, all this is valid only for not too strong radiation and in the lowest order perturbation calculus.

As recently found, for high intensities of the external field the behaviour of the system can be distinct [2–7]. Here, an atypical ionization takes place with the energy of the photon lesser than the ionization potential, seemingly involving the collective absorption of several photons (multiphoton ionization [2–4]). Moreover, the energy spectrum of the ejected photoelectrons shows unexpected features [5–7]. For a wave interacting with an atomic system, if ω is not too high and the wavelength $\lambda = 2\pi c/\omega$ is much larger than the typical size of the atom, these facts have been qualitatively described using just the oscillating (or rotating) electric field to represent the linearly (or circularly) polarized electromagnetic waves (the so called "dipolar aproximation" [8–11]).

An alternative model of the external radiation has the form of a rotating magnetic field [12–13]. Such fields can be locally created close of the nodal points (the points in which the electric field is null) of some standing waves. Our present aim is to complete the sketch by considering a bound system interacting with a rotating magnetic (instead of electric [10–11, 14]) field. For simplicity, we take as the bound system a harmonic oscillator. Of course, the oscillator does not admit ionization. However, it has proved successful to describe the resonance. We are going to show that, contrasting with the oscillator in a rotating electric field, our elementary model predicts some forms of excitation, analogous to the multiphoton resonance.

2. *The physical origin of the rotating magnetic field*

To explain the physical sense of our model, we shall consider below a class of approximate electromagnetic potentials appearing in Maxwell–Faraday theory (when the retardation is neglected) and involving a homogeneous time dependent magnetic field:

$$\vec{A}(\vec{x}, t) = -\tfrac{1}{2}\,\vec{r}\times\vec{B}(t). \tag{2}$$

The subject of this note is the case of (2) when $\vec{B}(t)$ just rotates in a fixed plane with a constant angular velocity ω:

$$\vec{B}(t) = B(\vec{n}\cos\omega t + \vec{m}\sin\omega t) = B\vec{n}(t). \tag{3}$$

The fields of type (2)–(3) can be locally created by an ordinary magnet rotating with not too high frequency ω. However, there exist more interesting circumstances in which such fields arise. We refer to the physical situation produced around the nodal points of electromagnetic standing waves. Following [12–13], consider the electromagnetic standing waves (described by the vector potential $\vec{A}(\vec{x}, t)$)

$$\vec{A}_{\vec{n},\vec{s}}(\vec{x}, \omega t) = A\vec{n}\sin\frac{\omega \vec{s}\vec{x}}{c}\sin\omega t, \tag{4}$$

where $\vec{n}$, $\vec{s}$ are two orthogonal unit vectors and $A \in \mathbf{R}$ the wave amplitude. Taking the "antisymmetric combinations"

$$\vec{A}_{[\vec{n},\vec{s}]}(\vec{x}, \omega t) = \tfrac{1}{2}\left(\vec{A}_{\vec{n},\vec{s}}(\vec{x}, \omega t) - \vec{A}_{\vec{s},\vec{n}}(\vec{x}, \omega t)\right) \tag{5}$$

of (4) and superposing them we can obtain the standing wave of more complex structure:

$$\vec{A}(\vec{x}, t) = \vec{A}_{[\vec{n},\vec{s}]}(\vec{x}, \omega t) + \vec{A}_{[\vec{m},\vec{s}]}(\vec{x}, \omega t - \pi/2)$$

$$= -\tfrac{1}{2} A \left\{ \left(\vec{m} \sin \frac{\omega \vec{s}\vec{x}}{c} - \vec{s} \sin \frac{\omega \vec{m}\vec{x}}{c} \right) \cos \omega t \right.$$

$$\left. + \left(\vec{s} \sin \frac{\omega \vec{n}\vec{x}}{c} - \vec{n} \sin \frac{\omega \vec{s}\vec{x}}{c} \right) \sin \omega t \right\}. \tag{6}$$

The wave (6) contains the following subset of the nodal points:

$$\mathfrak{N} = \{\vec{x} = \lambda(l_1\vec{n}+l_2\vec{m}+l_3\vec{s}): l_1, l_2, l_3 = 0, \pm 1, \pm 2, \ldots\}. \tag{7}$$

In a neighborhood of each of them (for $|\vec{r}| \ll \lambda/2$ where $|\vec{r}|$ measures the distance from a point of $\mathfrak{N}$), the electromagnetic potential (6) can be aproximated by:

$$\vec{A}(\vec{r}, t) \cong -\tfrac{1}{2} \frac{A\omega}{c} \{(\vec{m}(\vec{s}\vec{r}) - \vec{s}(\vec{m}\vec{r})) \cos \omega t$$

$$+ (\vec{s}(\vec{n}\vec{r}) - \vec{n}(\vec{s}\vec{r})) \sin \omega t\} = -\tfrac{1}{2} \vec{r} \times \vec{B}(t) \tag{8}$$

where $\vec{B}(t)$ is given by (3) with

$$B = A\omega/c. \tag{9}$$

The electromagnetic field (2)–(3), (8)–(9) has some interesting properties in itself which have been studied elsewhere [12–13]. Below, we shall examine the influence of such a field on a bound quantum system.

3. Harmonic oscillator in the rotating magnetic field

As our bound system we choose the harmonic oscillator with the binding potential:

$$V(\vec{r}) = \tfrac{1}{2} m\omega_0^2 \vec{r}^2. \tag{10}$$

Suppose the oscillator is situated in the external radiation field (6) so that its atraction center is in one of the nodal points (7). If the wave length λ of the wave (6) is larger than the average radius $\varrho_0 = (\hbar/2m\omega_0)^{1/2}$ of the ground state of the oscillator, the evolution of the wavepackets trapped in vicinity of $r = 0$ will be well described by the Hamiltonian (1) with $\vec{A}(\vec{x}, t)$ taken from the rotating field model (8). After a brief calculation

$$H(t) = \frac{1}{2m} \left(\vec{p} + \frac{e}{2c} \vec{r} \times \vec{B}(t) \right)^2 + \tfrac{1}{2} m\omega_0^2 \vec{r}^2$$

$$= \frac{1}{2m} (\vec{p}^2 + a^2 \vec{r}_\perp^2(t)) - \frac{a}{m} \vec{n}(t)\vec{M} + \tfrac{1}{2} m\omega_0^2 \vec{r}^2, \tag{11}$$

where $a = \dfrac{eB}{2c}$, $\vec{n}(t)$ is the unit vector in (3), $\vec{M} = \vec{r} \times \vec{p}$ is the angular momentum and $\vec{r}_\perp(t)$ is the component of $\vec{r}$ orthogonal to $\vec{n}(t)$. From now on, we shall forget about the underlying heuristics and we shall treat the Hamiltonian (11) as a dynamical quantum model in itself. As can be easily seen, our model is exactly solvable.

4. *Explicit solution in the Heisenberg frame*

The evolution equation for the evolution operator $U(t)$ reads:

$$\frac{d}{dt}U(t) = -iH(t)U(t). \tag{12}$$

Without losing generality, we can suppose that the magnetic field rotates around the z-axis and the vectors $\vec{n}$, $\vec{m}$ in (3) lie along x and y axes respectively. The first step to solve the evolution problem (11)–(12), like in [12–13], will be the transition to the rotating frame. By substituting

$$U(t) = \exp(i\omega t M_z)W(t) \tag{13}$$

the evolution equation for $W(t)$ is simplified to

$$\frac{d}{dt}W(t) = -iGW(t), \tag{14}$$

where G is the time independent generator

$$G = \frac{1}{2m}\vec{p}^2 + \tfrac{1}{2}\left(\frac{a^2}{m} + m\omega_0^2\right)(y^2+z^2) + \tfrac{1}{2}m\omega_0^2 x^2 - \frac{a}{m}M_x + \omega M_z \tag{15}$$

representing the evolution "as seen in the rotating frame". Since G is time independent, the formal solution reads $W(t) = e^{-itG}$. Due to the quadratic nature of G, various mathematical formalisms are available to solve effectively (14)–(15) (e. g. the coherent state formalism [15–18]). One of the most transparent technics consists in reducing (14) to a matrix equation, which arises for the time dependent images of the 6-canonical operators $q = \begin{vmatrix} \vec{x} \\ \vec{p} \end{vmatrix}$ in the Heisenberg frame [12–13, 19–20]:

$$q_j(t) = W^*(t)q_jW(t) = e^{itG}q_je^{-itG} \quad (j = 1, \ldots, 6) \tag{16}$$

The corresponding Heisenberg motion equations

$$\frac{d}{dt}q_j(t) = [iG, q_j] \tag{17}$$

lead to the c-number matrix equation for the 6 component vector $q(t)$:

$$\frac{d}{dt}q(t) = Aq(t) \tag{18}$$

with the matrix A having the explicit form:

$$A = \begin{vmatrix} 0 & -\omega & 0 & 1/m & 0 & 0 \\ \omega & 0 & a/m & 0 & 1/m & 0 \\ 0 & -a/m & 0 & 0 & 0 & 1/m \\ -m\omega_0^2 & 0 & 0 & 0 & -\omega & 0 \\ 0 & -(a^2/m+m\omega_0^2) & 0 & \omega & 0 & a/m \\ 0 & 0 & -(a^2/m+m\omega_0^2) & 0 & -a/m & 0 \end{vmatrix}. \tag{19}$$

Therefore, the canonical trajectories (16) convert in

$$q(t) = e^{tA}q \tag{20}$$

and the corresponding trajectories in the original non rotating frame are obtained adding to (20) a permanent rotation around the z-axis with the angular velocity ω.

5. *Stability and instability domains*

The global character of the trajectories (20) depends on the algebraic type of the 6×6 matrix Λ [12, 21], determined by the roots of its characteristic polynomial $D_\Lambda(\lambda)$. A direct calculation leads to

$$D_\Lambda(\lambda) = \mathrm{Det}\,(\lambda - \Lambda) = \omega^6 \Delta(\sigma), \tag{21}$$

where $\sigma = \lambda^2/\omega^2$ and $\Delta(\sigma)$ is expressed

$$\Delta(\sigma) = \sigma^3 + (2+3w^2+4\alpha^2)\sigma^2 + (1+3\alpha^2+4\alpha^2w^2+3w^4)\sigma + (1-w^2)(\alpha^2+w^2-w^4), \tag{22}$$

with $w = \omega_0/\omega$ and $\alpha = \dfrac{a}{m\omega} = \dfrac{eB}{2mc\omega}$. The roots of $\Delta(\sigma)$ depend on two independent parameters α, w. Hence, the types of trajectories will be labelled by the points of α-w plane. The form of (21)–(22) shows that this plane splits into 5 open regions, corresponding to trajectories of definite types and to definite algebraic properties of (22). These regions are divided by threshold curves for which the character of the trajectories is unstable and undergoes a qualitative methamorphosis.

The first region Ω_1 is the one where the Cardan's discriminant $C(\alpha, w)$ of the polynomial (22) is positive:

$$\Omega_1 = \{(\alpha, w) \in \mathbf{R}^2 : C(\alpha, w) > 0\}. \tag{23}$$

In this region, two of the σ_k roots are genuine complex conjugated one to another while the third one is real negative. Henceforth, the eigenvalues of $D_\Lambda(\lambda)$ have the form $\pm i\omega|\sigma_1|^{1/2}$, $\pm\omega(\sigma_2)^{1/2}$ and $\pm\omega(\sigma_2^*)^{1/2}$. The resulting Heisenberg canonical trajectory has a diverging part (the deconfinement of the oscillator under the influence of our rotating field)[1]. Thus, Ω_1 is a resonance region. The boundary $\partial\Omega_1$ is formed by an algebraically defined curve $C(\alpha, w) = 0$ drawn in Fig. 1.

The next region of interest (Ω_2), is the one in which one of the roots of $\Delta(\sigma)$ is positive and two are negative, say

$$\sigma_1, \sigma_2 < 0 \quad \text{and} \quad \sigma_3 > 0. \tag{24}$$

The region Ω_2 can be analytically characterized by noticing that the product of the three roots of $\Delta(\sigma)$ is positive, implying:

$$\Omega_2 = \{(\alpha, w) : 1 < w < (1/2 + (1/4+\alpha^2)^{1/2})^{1/2}\}. \tag{25}$$

[1] On the level of classical theory it means that except of a subfamily of measured zero all canonical trajectories exponentially diverge.

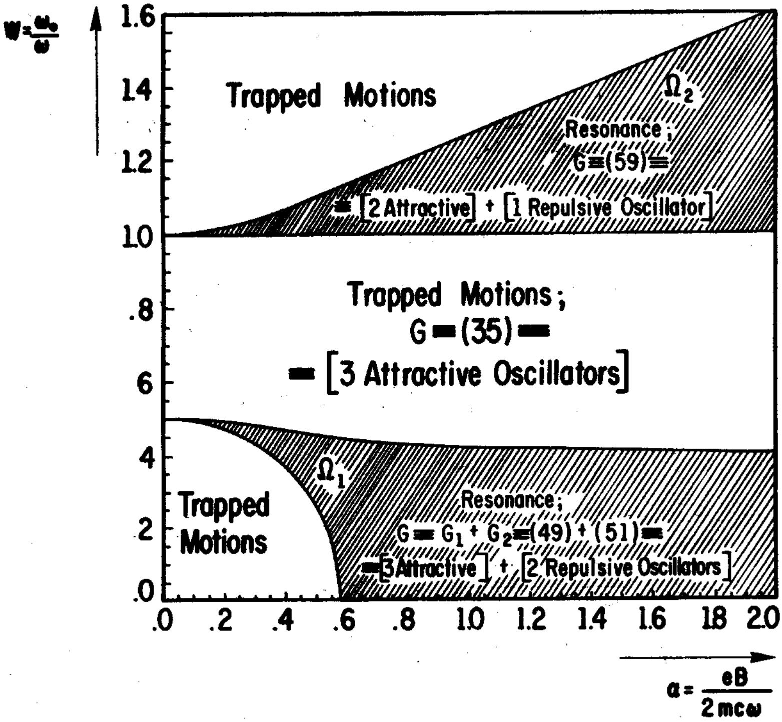

Fig. 1. Division of the α-w plane according to the kind of motions produced by the generator matrix Λ. The two shadow regions Ω_1 and Ω_2 host the deconfined motions (resonance area) while the rest of the plane is the confinement domain. The region Ω_2 possesses features resembling the multiphoton resonance (the oscillator absorbs N photons to jump M energy levels with $N > M$) while in Ω_1 the relation is apparently inverted (the point in which Ω_1 touches the w-axis suggests an excitation in which bigger quanta produce double energy jumps). The band $2\omega_0 < \omega < 2.4677\ \omega_0$ shows a curious type of resonance in which high intensities of the magnetic field produce the resonance but the further increase of the field makes it disappear

For $(\alpha, w) \in \Omega_2$, the characteristic polynomial $D_\Lambda(\lambda)$ has four purely imaginary roots $\pm\omega(\sigma_1)^{1/2}$, $\pm\omega(\sigma_2)^{1/2}$, corresponding to the circular mode of the Heisenberg canonical trajectory and two real roots $\pm\omega(\sigma_3)^{1/2}$, one of which is positive and henceforth defines a diverging mode. Thus, Ω_2 represents again a resonance.

On the threshold curves, forming the frontier of Ω_1 and Ω_2, the character of the trajectories is unstable.

The rest of α-w plane split into three disjoint open domains (see Fig.1). In each of them, $\Delta(\sigma)$ has three distinct negative real roots, i.e. $D_\Lambda(\lambda)$ has six purely imaginary roots, and the corresponding points (α, w) define the bounded (trapped) motions.

The plot of the stability and resonance regions in Fig. 1 shows some curious facts which differ our model from the traditional semiclassical theory of an oscillator driven by either a rotating or oscillating electric field e.g. $V(\vec{x}, t) = -\vec{E}_0\vec{x} \sin \omega t$. In fact, the driven oscillator can reflect to some extent the resonance phenomenon, but the excitation depends only on the parameter ω and is not sensitive to the field intensity (consistently with the spirit of the traditional Einstein's photoeffect theory).

The oscillator affected by our rotating magnetic field behaves in a different way. Here, the resonance (instability) arises along a continuous threshold in α-w plane depending on the field intensity and not on ω alone. The phenomenon's dependence on the parameter $w = \omega_0/\omega$ would correspond to the "traditional resonance effect" while the dependence on $\alpha = (e/2mc)(B/\omega)$ represents the "distorsion of the doctrine". For small α (weak field intensities), our plot still reproduces the standard resonance pattern (the resonance occurs in very narrow intervals around $\omega = \omega_0$ and $\omega = 2\omega_0$. A slight novelty here is that the model seems sensitive to the absorption of "bigger quanta", sufficient to jump exactly two energy levels). When α increases, however, the pattern changes and the resonance bands start to extend involving frequencies which should not cause the excitation according to the traditional criteria. In case of very high ω ($\omega \gg \omega_0$), our rotating model has merely an academic sense (it cannot reflect the behaviour of the oscillator in the nodal point of a standing wave because the oscillator states are "wider" than the "validity length" of our nodal aproximation). However, when ω is comparable with ω_0 (long waves affecting relatively compact oscillator states), it can describe adequately the resonance of the system in a standing wave. Thus, in vicinity of $\omega = \omega_0$ it yields a decon-

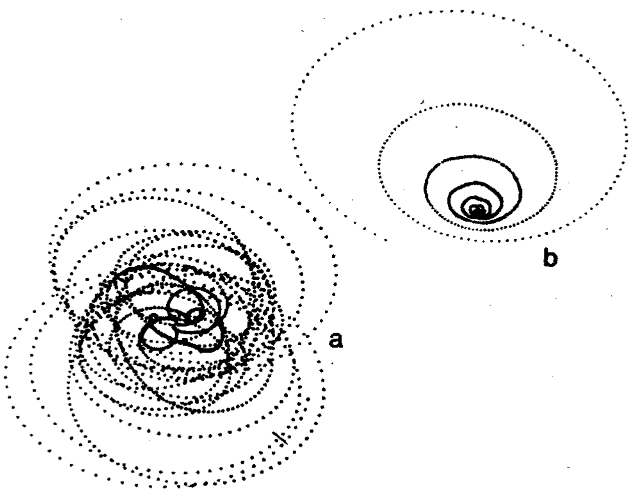

Fig. 2. Two examples of excitation for atypical frequencies. a) A slowly expanding classical trajectory illustrating the resonance in the special frecuency band $2\omega_0 < \omega < 2.4677\omega_0$. The resonance effects do not appear for either too low or too high field amplitudes. b) A quickly diverging trajectory for the oscillator with $\omega_0 = (3/2)\omega$ in the rotating field of frecuency $\omega = 10^{14}$/sec, tentatively interpretable as a resonance with the absorption of 3 photons to jump 2 oscillator levels. The electron departs from the oscillator center with the initial velocity $\simeq 10^4$ cm/sec. The applied field intensity is $B = 2.047 \times 10^7$ Gauss and the parameters α, w define a point in the resonance region Ω_2

finement (resonance) for $\omega < \omega_0$ (Fig. 2b), to be interpreted as the multiphoton resonance (the resonance produced by the absorption of N photons to jump M energy levels, where $N > M$ [2–9, 11, 14]).

The interval between $\omega = \omega_0$ and $\omega = 2\omega_0$ is "no resonance band" (curiously, no resonance appears no matter the rotating field intensity). In turn, the band $\omega > 2\omega_0$ ($w < 1/2$) hosts still more curious phenomena. In the sub-interval $2\omega_0 < \omega < 2.4677\omega_0$, too small field intensities as well as too high field intensities cannot cause the resonance. However, the medium value fields are able to excite the system in spite of $\omega \neq 2\omega_0$ (Fig. 2a) suggesting a kind of multiphoton absorption with $N < M$. This kind of resonance (i.e, which appears for high intensities but disappears again for still higher field intensities) has never been detected in theoretical models involving the rotating and/or oscillating electric forces [2] [2–11, 14]. For $\omega > 2.4677\omega_0$, the picture simplifies again: there is no resonance if α is small but it appears and persists for α crossing an instability threshold (however, in this band the adequacy of our rotating model to describe the oscillator in the radiation field is no longer granted). It is curious that for the high rotating field intensities the resonance in this band requires the frequencies in fact exceeding $2\omega_0$ (as if the "photons" of a strong standing wave could not cause the two-level oscillator jump without having some energy excess).

6. *The canonical structure of the G-generator*

Our description, until now, concerns the canonical trajectories (both, classical and quantum). It might be interesting to notice that their behaviour is related with the distinct forms of the Floquet Hamiltonian (15). This fact will be made explicit below by applying an extended version of the method used in [12], which gives different results in distinct regions of the 2-dimensional space of parameters (α, w).

Consider first any point (α, w) outside of the threshold curves of Fig. 1. The generator G (15) then defines a diagonalizable matrix Λ with 6 different eigenvalues:

$$\lambda_k, -\lambda_k; \operatorname{Re}(\lambda_k) > 0 \quad \text{or} \quad \operatorname{Re}(\lambda_k) = 0 \quad \text{and} \quad \operatorname{Im}(\lambda_k) > 0 \quad (k = 1, 2, 3). \tag{26}$$

In the 6-dimensional space of canonical variables henceforth exist the 6 eigenforms e_k^+, e_k^-

$$e_k^+ \Lambda = \lambda_k e_k^+; \quad e_k^- \Lambda = -\lambda_k e_k^- \tag{27}$$

which permit to define the 6 scalar operators A_k^+, A_k^-

$$A_k^+ = (e_k^+ q); \quad A_k^- = (e_k^- q) \tag{28}$$

obeying the following commutation relations with the generator (15):

$$[G, A_k^+] = -i\lambda_k A_k^+; \quad [G, A_k^-] = i\lambda_k A_k^- \tag{29}$$

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[2] This, in fact, is one of the most interesting aspects of the deconfinement phenomenon discussed here. It makes impossible all too simplistic interpretations of the semiclassical resonance, in a sense that the electric field associated with the rotating $\vec{B}(t)$ for high field intensities becomes so strong that it "nullifies" the original classical field of the oscillator. An impossibility of similar interpretations follows also from the existence of the no resonance band $1/2 < w < 1$.

and commuting between themselves according to:

$$[A_k^+, A_j^+] = [A_k^-, A_j^-] = 0,$$
$$[A_k^-, A_j^+] = \gamma_k \delta_{kj} \quad (\gamma_k \in C). \tag{30}$$

It turns out that the operators $A_k^\pm$ define a general canonical decomposition of the generator G in (15). Below we shall use the following extension of the lemma 2 from [12].
Lemma. If the matrix Λ is diagonalizable and its eigenvalues are non degenerate, the Floquet Hamiltonian (15) admits the representation:

$$G = \sum_{k=1}^{3} (-i)(\lambda_k/\gamma_k) A_k^+ A_k^- + g_0, \quad (g_0 \in \mathbf{R}). \tag{31}$$

Demonstration. Denote:

$$g_0 = G - \sum_{k=1}^{3} (-i)(\lambda_k/\gamma_k) A_k^+ A_k^- .$$

Due to (29), (30), g_0 commutes with all the A_k^+, A_k^-. Then, g_0 commutes with all the canonical variables $\vec{x}$, $\vec{p}$ and therefore $g_0 \in \mathrm{R}$. ■

Depending now on the region of α-w plane, some particular cases of the decomposition (31) can be distinguished.
I. Suppose first that (α, w) is in one of the three confinement regions of Fig. 1. This case is similar in many aspects to that extensively discussed in [12]. The polynomial (22) has 3 distinct negative roots σ_k $(k = 1, 2, 3)$ and the roots $\pm\lambda_k$ of (26) take purely imaginary values $\pm i\omega_k$ $(\omega_k = \omega\sqrt{|\sigma_k|})$. As a consequence, one can choose $e_k^+ = (e_k^-)^*$ and the operators $A_k^\pm$ become hermitian conjugated $A_k^+ = (A_k^-)^*$ yielding the quantities γ_k in (30) real.

Note that the eigenforms e_k^+ and the corresponding operators A_k^+ are still defined with accuracy to constant multipliers and allow the freedom of renormalization $e_k^+ \to \Gamma_k e_k^+$, $e_k^- \to \Gamma_k^* e_k^-$ and $A_k^+ \to \Gamma_k A_k^+$, $A_k^- \to \Gamma_k^* A_k^-$. By choosing the renormalization factors Γ_k so that $\Gamma_k \Gamma_k^* = 1/|\gamma_k|$ one reduces the commutation relations (30) to:

$$[A_k^+, A_j^+] = [A_k^-, A_j^-] = 0, \quad [A_k^-, A_j^+] = \operatorname{sign}(\gamma_k)\delta_{kj}. \tag{32}$$

The direct calculations show that $\operatorname{sign}(\gamma_1) = -\operatorname{sign}(\gamma_2) = \operatorname{sign}(\gamma_3) = 1$ and since $A_j^+ = (A_j^-)^*$, the Floquet Hamiltonian G in (31) takes on the simplified form:

$$G = \omega_1 A_1^* A_1 - \omega_2 A_2^* A_2 + \omega_3 A_3^* A_3 + g_0, \tag{33}$$

where we have put $A_k^- = A^k$.

The signs in (33) imply that G is not positively defined (there exist eigenstates whose eigenvalues are negative) and does not have the energy interpretation. However, it helps to construct the stationary wave packets in the form of "squeezed states", in full analogy with [12].

II. Consider in turn (α, w) in the deconfinement region Ω_1 of Fig. 1. The polynomial (22) now has one negative root σ_1 and two genuine complex roots σ_2, σ_3 $(\sigma_2 = \sigma_3^*)$.

Consequently, the matrix A has 2 pure imaginary and 4 complex roots of the form:

$$\lambda = \pm i\omega_1, \pm\zeta, \pm\zeta^*, \tag{34}$$

where $\omega_1 = \omega\sqrt{|\sigma_1|}$, $\zeta = \omega(\sigma_2)^{1/2} = \eta + i\nu$, $\eta = \mathrm{Re}\,\zeta > 0$, $\nu = \mathrm{Im}\,\zeta \neq 0$. The eigenforms $e_k^\pm$ can be now choosen as e_1, e_1^*, e_+, e_-, e_+^*, e_-^* where:

$$e_1 A = -i\omega_1 e_1, \quad e_1^* A = +i\omega_1 e_1^*, \quad e_\pm A = \pm\zeta e_\pm, \quad e_\pm^* A = \pm\zeta^* e_\pm^*. \tag{35}$$

The 6 scalar operators (28) therefore become:

$$A_1 = (e_1 q), \quad A_1^* = (e_1^* q), \quad A_\pm = (e_\pm q), \quad A_\pm^* = (e_\pm^* q) \tag{36}$$

and satisfy the commutation relations:

$$[G, A_1] = -\omega_1 A_1, \quad [G, A_1^*] = +\omega_1 A_1^*,$$
$$[G, A_\pm] = \mp i\zeta A_\pm, \quad [G, A_\pm^*] = \mp i\zeta^* A_\pm^*. \tag{37}$$

The only non vanishing commutators (30) are $[A_1, A_1^*]$, $[A_+, A_-]$, $[A_+^*, A_-^*]$ and by choosing an adequate normalization they become

$$[A_1, A_1^*] = 1, \quad [A_+, A_-] = 1, \quad [A_+^*, A_-^*] = -1 \tag{38}$$

henceforth, reducing G to:

$$G = \omega_1 A_1^* A_1 + i\zeta A_+ A_- - i\zeta^* A_+^* A_-^* + g_0 \quad (g_0 \in \mathbf{R}). \tag{39}$$

Now only the first term in (39) has an evident oscillator form $\omega_1 A_1^* A_1 \equiv (\omega_1/2)(Q_1^2 + P_1^2)$. To understand the structure of the remaining two, substitute:

$$A_+ = (A_+ + A_+^*)/2 + i(A_+ - A_+^*)/(2i) = (p_+ + ip_-)/\sqrt{2},$$
$$A_- = (A_- + A_-^*)/2 + i(A_- - A_-^*)/(2i) = (q_- + iq_+)/\sqrt{2} \text{ and c.c..} \tag{40}$$

Thus:

$$G = \omega_1 A_1^* A_1 - \eta(p_- q_- + p_+ q_+) - \nu(p_+ q_- - p_- q_+) + g_0 = G_1 + G_2, \tag{41}$$

where G_1, G_2 are two commuting terms:

$$G_1 = \omega_1 A_1^* A_1 - \nu/4((p_+ + q_-)^2 + (p_- - q_+)^2 - (p_- + q_+)^2 - (p_+ - q_-)^2), \tag{42}$$
$$G_2 = -\eta/4((p_+ + q_+)^2 - (p_+ - q_+)^2 + (p_- + q_-)^2 - (p_- - q_-)^2) + g_0. \tag{43}$$

To interprete G_1, define:

$$A_2 = (p_+ + q_-)/2 + i(p_- - q_+)/2 = (Q_2 + iP_2)/\sqrt{2},$$
$$A_3 = (p_- + q_+)/2 + i(p_+ - q_-)/2 = (Q_3 + P_3)/\sqrt{2}. \tag{44}$$

Hence:

$$G_1 = \omega_1 A_1^* A_1 - \nu A_2^* A_2 + \nu A_3^* A_3, \tag{45}$$

where the non-zero commutation relations obtained for the set A_j, A_j^*, $(j = 1, 2, 3)$ are just conventional, $[A_j, A_j^*] = 1$.

In turn, the operator G_2 is equivalent to the Hamiltonian of a 2-dimensional repulsive oscillator. This can be seen by introducing the new operators:

$$\hat{P}_2 = (p_+ + q_+)/\sqrt{2}, \quad \hat{Q}_2 = (p_+ - q_+)/\sqrt{2},$$
$$\hat{P}_3 = (p_- + q_-)/\sqrt{2}, \quad \hat{Q}_3 = (p_- - q_-)/\sqrt{2} \tag{46}$$

which reduce G_2 to the form:

$$G_2 = -\eta/2(\hat{P}_2^2 - \hat{Q}_2^2) - \eta/2(\hat{P}_3^2 - \hat{Q}_3^2). \tag{47}$$

Notice that the "attractive part" G_1 and the "repelent part" G_2 commute. Henceforth, the unitary evolution operator in the rotating frame admits an easy decomposition:

$$W(t) = e^{-itG} = (e^{-itG_2})(e^{-itG_1}). \tag{48}$$

The evolution in the Schrödinger frame is easily interpreted by selecting as the initial wavepacket one of the eigenstates of the "attractive part" G_1: $G_1 \Psi = \lambda \Psi$. Then:

$$e^{-itG}\Psi = (e^{-i\lambda t})(e^{-iG_2 t}\Psi). \tag{49}$$

As can be observed, the "attractive part" G_1 just produces a phase factor while the repelent part carries out its destructive function. The final result will be the "explosion" of the wavepacket in 4 of the 6 canonical directions, determined by G_2.

III. Finally, let (α, w) be in the deconfinement region Ω_2. Now, $\Delta(\sigma)$ has 2 negative real roots (σ_1, σ_2) and one real positive σ_3. The 6 eigenvalues of Λ (two of them are real) acquire the form:

$$\lambda = \mp i\omega_1, \mp i\omega_2, \pm\omega_3, \tag{50}$$

where $\omega_k = \omega\sqrt{|\sigma_k|}$ $k = 1, 2, 3$. The corresponding eigenforms obey

$$e_\pm \Lambda = \pm\omega_3 e_\pm, \quad e_k \Lambda = -i\omega_k e_k, \quad e_k^* \Lambda = +i\omega_k e_k^* \quad (k = 1, 2) \tag{51}$$

and the associated scalar operators

$$A_\pm = (e_\pm q), \quad A_k = (e_k q), \quad A_k^* = (e_k^* q) \quad (k = 1, 2) \tag{52}$$

satisfy the commutation rules:

$$[G, A_\pm] = \mp i\omega_3 A_\pm, \quad [G, A_k] = -\omega_k A_k, \quad [G, A_k^*] = +\omega_k A_k^* \quad (k = 1, 2). \tag{53}$$

Furthermore:

$$[A_1, A_1^*] = 1, \quad [A_2, A_2^*] = 1, \quad [A_+, A_-] = i. \tag{54}$$

Note that A_+ and A_- are hermitian. Due to (53), (54), they provide the following decomposition of the generator G:

$$G = \omega_1 A_1^* A_1 + \omega_2 A_2^* A_2 + \omega_3 A_+ A_- + g_0 \quad (g_0 \in \mathbf{R}). \tag{55}$$

The crucial term is now:

$$G_3 = \omega_3 A_+ A_-. \tag{56}$$

Denoting this time:

$$Q_3 = (A_+ - A_-)/\sqrt{2}, \quad P_3 = (A_+ - A_-)/\sqrt{2} \tag{57}$$

one sees that G_3 is a repulsive oscillator:

$$G_3 = (\omega_3/2)(P_3^2 - Q_3^2). \tag{58}$$

Consistently, in the eyes of the rotating observer the Hilbert space $\mathscr{H}$ of Schrödinger wavepackets will split into the tensor product of the Hilbert spaces of three onedimensional oscillators $\mathscr{H} = L^2(\boldsymbol{R}) \otimes L^2(\boldsymbol{R}) \otimes L^2(\boldsymbol{R})$. Two of them will be confining (atractive) while the third one will produce an explosion of the wavepacket in the two canonical directions (57) defined by the repulsive generator G_3. Since our rotating field (3) approximates the field of the electromagnetic standing wave (6) in vicinity of a nodal point, it looks that G_3 reflects a kind of coherent multiphoton resonance caused by the standing wave.

Note that a canonical form for the quadratic generators was considered by Leach, who has proved the existence of a time dependent unitary transformation reducing *any* time dependent generator $H(t)$ quadratic in x_i, p_j to a time independent oscillator Hamiltonian (called the "archtypal form" [22–24]). The construction of Leach, while more general, in a sense "is too strong" for our purposes: his final "archtypal generator" is not the Floquet Hamiltonian and is not sensitive to the stability and/or resonance aspects of the motion. In contrast, our canonical transformation is much less general, but the canonical forms obtained (31) reflect well the stability thresholds.

With all reservations concerning the semiclassical theory, it thus looks that our model describes one of rare situations when the multiphoton phenomena admit an exact description on purely quantum mechanical level.

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