CONSTRUCTION OF SPACE-TIME FROM LINEAR CONNECTIONS

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It is proposed that the existence of the space-time manifold with its pseudo-Riemannian geometry is a result of a connection in the bundle of frames of a larger fundamental manifold together with choice of a cross-section in such a bundle. No theory is formulated, but various aspects of the construction are discussed for two different local structures of the fundamental manifold: $\mathbf{R}^5$ and $\mathbf{C}^4$.

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Introduction

A fundamental aim of contemporary theoretical physics is to unify the forces of nature in a single account. Most theoretical physicists working towards such a goal would agree, that a proper understanding of the nature of space and time is most likely the all important key to a successful completion of such a program. Rosen writes [1]: "*The space and time of immediate experience are no longer taken as composing the fundamental abstraction. Instead they are regarded as themselves merely extrinsic, topological manifestation of a higher-dimensional continuum unknowable by direct intuition.*" Indeed, hardly anyone still believes today that the space-time continuum as we know it from the general theory of relativity can be used also to describe the interior of the elementary particles. A search for the fundamental continuum and its relationship to the observable space-time constitutes one of the most interesting and challenging directions in the contemporary theoretical physics.

Historically, the first step in that direction was the original Kaluza-Klein theory [2, 3] with its fifth dimension suitable for a unified description of gravitation and electromagnetism. Since then, the dimension of the fundamental continuum grew significantly in order to accommodate other interactions within the unified system. The dimensions which are not macroscopically observable are usually hidden by means of compactification, compressing them at an extremely small scale. In some theories the compactification occurs spontaneously as a result of solving the fundamental equations [4]. Hundreds of papers were written about the subject and a considerable progress made. From time to time a completion of the unification program is announced. In fact, there are members of public who actually believe that a final unified theory of everything already exists. Within the scientific community, however, most people would say that we are still a long way from an obviously successful conclusion of this particular step in the development of our knowledge of the world. Hence it is justifiable to continue the search for the fundamental continuum and its relationship to the observable space-time.

In most theories in which the fundamental continuum is mathematically a manifold, space-time is its submanifold. In other words, the space-time manifold with various properties like linear or affine connection, metric etc. is used as a foundation onto which the extra dimensions and supplementary structures are added. It might be, however, that by approaching the problem in such a particular way we are missing a chance to find the true unified geometric picture of the world, because the space-time with its dimension, metric and other properties might exist only due to the special way in which we carry out the macroscopic observations. Take away the observer, and the fundamental continuum may not contain anything like the space-time manifold. The present paper explores possible ways how to implement such a particular philosophy. The fundamental geometric system is assumed to be a manifold with a linear connection. The observer is identified with a choice of a cross-section in the bundle of frames of the fundamental manifold, i.e. a smooth choice of a frame for each point of the manifold. How the macroscopic properties of the space-time are deduced from such a picture is described in Section 1. A precise mathematical formulation of a similar construction has been presented in Ref. [5], but here the stress is on the basic philosophy of the idea rather than on an abstract formulation. In Sections 2 and 3 which go beyond the content of Ref. [5] concrete examples of the fundamental manifolds are considered. In Section 2 the fundamental manifold is assumed to be locally $\mathbf{R}^5$. It appears that Einstein's vaccuum equations have a rather natural place in such a geometry. Section 3 discusses some implications of the fundamental manifold being locally $\mathbf{C}^4$. Such a model has an obvious advantage of containing fields with transformation properties of Dirac spinor fields as an integral part of its geometry.

1. An extension of the role of the observer

The role of the observer in the classical theory of relativity shows a way to its possible extension. Although the fundamental continuum of the general relativity is a four-dimensional real space-time manifold, it is not directly observable. General coordinates of the manifold do not even distinguish between space and time, while in any direct observation such a distinction is clearly made. In other words, to describe the space-time manifold in terms of the general coordinates x^μ, one needs to set up a system of Lorentz frames described by the tetrads $h^i_\mu(x)$[1]. A particular choice of the frames depends on the observer, with frames at the same point of the space-time manifold connected by the action of the

[1] Greek indices are used for general coordinates, while the directions in the tangent Minkowski space are denoted by Latin indices.

Lorentz group. Although theoretically all frames are accessible, there exists a clear practical difference between spacial rotations and space-time "rotations", i.e. Lorentz boosts. While there is no practical limit imposed on the spacial rotations, only a rather narrow range of the Lorentz boosts parameter v/c can be reached in the normal practice of space and time observations. It is conceivable that there exist further rotations whose range is so narrowly limited in macroscopic observations that for all practical purposes local frames at every point may be regarded as fixed as far as the additional rotations are concerned. Yet, these rotations may play an important role. Namely, they may be observable when one shifts from one point of the fundamental manifold to another, measuring the discrepancy between the actual frames and parallelly displaced frames. In other words, they might be our only key to register translations. In more mathematical terms this may be described as follows.

Consider a bundle which is locally $M \times G$, where M is a manifold of dimension $n \geqslant 4$ serving as the base manifold of the bundle and G is de Sitter group. A connection in such a bundle can be defined in terms of the horizontal lift [5].

$$X_\mu^{(h)} = \frac{\partial}{\partial x^\mu} - \tfrac{1}{2} A_\mu^{ij}(x) W_{ji}, \tag{1}$$

where x^μ are local coordinates in M, and $W_{ij} = -W_{ji}$, $i, j = 1, \ldots, 5$, are the right invariant vector fields in G. For $i, j = 1, \ldots, 4$ the fields W_{ij} span the Lorentz algebra, while W_{i5}, $i = 1, \ldots, 4$, corresponds to the additional rotations. A gauge transformation[2] characterized by matrices $b^i_j(x)$ belonging to the de Sitter group leads to the transformation of the connection components according to

$$\tilde{A}_\mu^{ij} = b^i_k A_\mu^{kl} b^j_l + (\partial_\mu b^j_k) b^i_l g^{kl}, \tag{2}$$

where $g^{kl} = \mathrm{diag}(1, 1, 1, -1, \pm 1)$.

Restricting the gauge transformation to the Lorentz subgroup leads to the transformation

$$\tilde{A}_\mu^{i5} = b^i_k A_\mu^{k5}, \tag{3}$$

which is identical to the transformation of tetrads $h^i_\mu(x)$ in the usual general relativistic description of space-time. Hence when it is possible to choose a coordinate system in M in such a way that

$$A_\mu^{k5}, \mu = 1, \ldots, 4, \text{ form an invertible matrix and}$$
$$A_\mu^{k5} = 0, \quad \mu = 5, \ldots, n, \tag{4}$$

functions A_μ^{i5}, $\mu = 1, \ldots, 4$, can be up to a multiplicative constant used as the tetrads on the submanifold of M defined by fixing all corrdinates $x^5, \ldots, x^n$. Functions A_μ^{ij}, $\mu = 1, \ldots, 4$, $i, j = 1, \ldots, 4$, serve as the components of the connection. The Christoffel

[2] In other words a change of the reference cross-section identifying the identity group element on each fibre of the bundle.

symbols can be obtained by the usual relation

$$\Gamma^{\sigma}_{\mu\nu} = h^{\sigma}_{i} A^{ij}_{\mu} h^{k}_{\nu} g_{jk} - (\partial_{\mu} h^{\sigma}_{k}) h^{k}_{\nu}, \tag{5}$$

where h^{σ}_{i} is the inverse of matrix h^{i}_{σ} and the summations are from 1 to 4. Eq. (5) ensures that the connection defined on the submanifold in this way preserves the metric $g_{\mu\nu} = h^{i}_{\mu} h^{j}_{\nu} g_{ij}$.

The construction uses the fact that the non-commutativity of generators W_{i5} has no effect on the interpretation of A^{ij}_{μ} and A^{i5}_{μ}. Only the Poincaré-like commutation rules

$$[W_{ij}, W_{k5}] = g_{jk} W_{i5} - g_{ik} W_{j5} \tag{6}$$

are needed. Thus even a flat Minkowski space is described by a non-flat connection in the original bundle. When the bundle is in fact the bundle of frames of the base manifold, the curvature has a limiting influence on possible global structures the base manifold may assume. This, in its turn, affects the global structure of the constructed space-time. So even though the construction works in an abstractly defined de Sitter structured fibre bundle [5], the interplay between the geometry of the base manifold and that of the constructed space-time is promising to lead to many interesting questions. That is why in this paper the starting point will always be a linear connection in the base manifold and not just a connection in a de Sitter structured principal fibre bundle as in Ref. [5]. Global questions, however, will not be discussed here.

2. *Real five-dimensional base manifold*

Consider a five-dimensional manifold with general coordinates x^{μ}, $\mu = 1, \ldots, 5$, and a connection in its bundle of frames reducible to a de Sitter group. Let h^{i}_{μ}, $\mu = 1, \ldots, 5$, $i = 1, \ldots, 5$, be the five-frames with respect to which the connection has de Sitter components A^{ij}_{μ}. This case will be used mainly as an illustration of the construction described in Section 1, hence only the simplest way of constructing a pseudo-Riemannian four-dimensional space-time will be considered. Nevertheless, it is interesting to see Einstein's vacuum equations to emerge naturally in the process.

Let x^{5} be the extra unobservable coordinate and so put

$$A^{ij}_{5} = 0, \quad i, j = 1, \ldots, 5. \tag{7}$$

The induced tetrads of the space-time submanifold are given by

$$A^{i5}_{\mu} = \frac{1}{l} h^{i}_{\mu}, \tag{8}$$

where l is a fundamental length.

The five-dimensional frames will be chosen in such a way that the fifth coordinate separates from the rest and their 4×4 restriction is equal to the tetrads of the space-time submanifold:

$$h^{5}_{\mu} = 0, \quad \mu = 1, \ldots, 4, \quad h^{i}_{5} = 0, \quad i = 1, \ldots, 4, \quad h^{5}_{5} = 1, \quad \text{and}$$

$$h^i_\mu = \text{same as in Eq. (8)},$$

$$i, \mu = 1, \ldots, 4. \tag{9}$$

Assuming that the linear connection in the five-dimensional manifold is torsion free,

$$\Gamma^\sigma_{5\nu} = (\partial_5 h^k_\nu) h^\sigma_k = \Gamma^\sigma_{\nu 5} = h^\sigma_k A^{k5}_\nu g_{55}, \quad \text{or} \quad \partial_5 h^k_\nu = \frac{1}{l} g_{55} h^k_\nu = \frac{1}{l} h^k_\nu \tag{10}$$

selecting $g_{55} = 1$. Eq. (10) leads to an exponential dependence of the space-time metric $g_{\mu\nu} = h^i_\mu h^j_\nu g_{ij}$ (all indices only 1 to 4) on x^5. In this way the original manifold is foliated into a family of 4-dimensional space-times differing from each other only by a scaling factor. We might live in one such space while the triviality of the horizontal lift of $\partial/\partial x^5$ (Eq. (7)) prevents us from crossing into the other spaces. From Eqs (5) and (10) we can get the following relations for some Christoffel symbols

$$\Gamma^5_{\mu\nu} = -\frac{1}{l} g_{\mu\nu}, \quad \Gamma^\sigma_{5\nu} = \Gamma^\sigma_{\nu 5} = \frac{1}{l} \delta^\sigma_\nu, \quad \Gamma^\sigma_{55} = \Gamma^5_{5\nu} = \Gamma^5_{\nu 5} = \Gamma^5_{55} = 0 \tag{11}$$

(the indices μ, ν, σ range 1 to 4).

Then using the usual formula for the Ricci tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^\sigma_{\mu\sigma}}{\partial x^\nu} - \frac{\partial \Gamma^\sigma_{\mu\nu}}{\partial x^\sigma} + \Gamma^\sigma_{\mu\varrho} \Gamma^\varrho_{\nu\sigma} - \Gamma^\sigma_{\mu\nu} \Gamma^\varrho_{\sigma\varrho} \tag{12}$$

we obtain

$$R_{55} = \frac{4}{l^2}, \quad R_{\mu 5} = R_{5\mu} = 0, \tag{13}$$

$$R_{\mu\nu} = R^{(4)}_{\mu\nu} + \frac{4}{l^2} g_{\mu\nu}, \quad \mu, \nu = 1, \ldots, 4, \tag{14}$$

where $R^{(4)}_{\mu\nu}$ is the Ricci tensor for the space-time submanifold. Eq. (13) is a part of a tensor equation

$$R_{\mu\nu} = \frac{4}{l^2} g_{\mu\nu}, \quad \mu, \nu = 1, \ldots, 5. \tag{15}$$

If Eq. (14) represents remaining components of the same equations, then $R^{(4)}_{\mu\nu} = 0$.

3. Complex four-dimensional base manifold

To describe quantum behaviour of fermions one has to use complex vectors with SL(2, C) acting on them rather than real vectors with the usual Lorentz SO(3, 1). SL(2, C) is a larger group with a natural homomorphism map from SL(2, C) onto SO(3, 1). Since SL(2, C) is connected with micro-phenomena while macroscopic observations use SO(3, 1), one might expect the fundamental continuum to have a direct relationship with SL(2, C), while the introduction of an observer should lead naturally to the loss of the doublevalued character of SL(2, C) in macroscopic observations.

Let us assume that the base manifold is a four-dimensional complex manifold with a connection in its bundle of frames. Complex manifolds with linear connections are described in a book by Yano [6], but the use of non-coordinate bases, general components of the connection and their gauge transformations are not discussed to a large extent. The basic structure will be, therefore, described here.

An n-dimensional complex manifold can be described using $2n$ real coordinates x^μ and y^μ, $\mu = 1, \ldots, n$, or, alternatively, complex coordinates $z^\mu = x^\mu + iy^\mu$ and $\overline{z^\mu} = x^\mu - iy^\mu$. In general, the structure group of a linear connection in such a manifold is Gl(n, C), the group of $n \times n$ invertible complex matrices. The right-invariant vector fields in Gl(n, C) can be expressed as

$$W_b^a = w_c^a \frac{\partial}{\partial w_c^b} \quad \text{and} \quad \overline{W_b^a} = \overline{w_c^a} \frac{\partial}{\partial \overline{w_c^b}}, \tag{16}$$

where w_b^a are the complex matrix elements and $\overline{w_b^a}$ are their complex conjugates.

Given a reference cross-section, a connection in the bundle of frames can be defined by a horizonatal lift of $\partial/\partial z^\mu$

$$Z_\mu^{(h)} = \frac{\partial}{\partial z^\mu} - A_{\mu b}^a(z) W_a^b - \overline{B_{\mu b}^a}(z) \overline{W_a^b} \tag{17}$$

together with the complex conjugate

$$\overline{Z}_\mu^{(h)} = \frac{\partial}{\partial \overline{z^\mu}} - B_{\mu b}^a(z) W_a^b - \overline{A_{\mu b}^a}(z) \overline{W_a^b}. \tag{18}$$

If a gauge transformation characterized by a z-dependent complex invertible matrix $w_b^a(z)$ is performed, the components of the connection transform according to

$$\tilde{A}_{\mu b}^a = u_c^a A_{\mu d}^c w_b^d + u_c^a (\partial_\mu w_b^c), \quad \tilde{B}_{\mu b}^a = u_c^a B_{\mu d}^c w_b^d + u_c^a (\partial_{\bar{\mu}} w_b^c), \tag{19}$$

where $u_b^a(z)$ is the inverse matrix of $w_b^a(z)$, and $\partial_{\bar{\mu}}$ denotes $\partial/\partial \overline{z^\mu}$.

When the connection is reducible to a pseudo-unitary subgroup characterized by a real diagonal metric g_{ab} it is possible to write the horizontal lift as

$$Z_\mu^{(h)} = \frac{\partial}{\partial z^\mu} - A_\mu^{ab}(z) W_{ba}, \tag{20}$$

where

$$W_{ba} = g_{bc} W_a^c - g_{ac} \overline{W_b^c},$$

and the correspondence with Eq. (17) is by

$$A_{\mu b}^a = A_\mu^{ac} g_{cb}, \quad \overline{B_{\mu b}^a} = -A_\mu^{ca} g_{cb}. \tag{21}$$

The real representation of the manifold with its connection in terms of coordinates x^μ and y^μ is obtained by

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial z^\mu} + \frac{\partial}{\partial \bar{z}^\mu}, \qquad \frac{\partial}{\partial y^\mu} = i\left(\frac{\partial}{\partial z^\mu} - \frac{\partial}{\partial \bar{z}^\mu}\right) \tag{22}$$

together with the corresponding relations for the horizontal lifts.

If the connection components satisfy

$$A^{ab}_\mu = -\overline{A^{ba}_\mu}, \tag{23}$$

the horizontal lift in the imaginary direction becomes trivial

$$Y^{(h)}_\mu = i\,(Z^{(h)}_\mu - \bar{Z}^{(h)}_\mu) = \frac{\partial}{\partial y^\mu}. \tag{24}$$

According to the basic proposition of the present paper only the real n-dimensional submanifold is then observable. It is thus of particular interest to study four-dimensional complex manifolds. Moreover, the obvious choice of the metric is

$$g_{ab} = \text{diag}\,(1, 1, -1, -1), \tag{25}$$

since group U(2, 2) contains both de Sitter groups. The condition (23) can be also written as

$$A^a_{\mu c} g^{cb} + \overline{A^b_{\mu c}} g^{ca} = 0. \tag{26}$$

In the matrix form with $A^a_{\mu b}$ being elements of matrix A_μ and g^{ab} elements of the Dirac matrix γ_0, Eq. (26) implies that $A_\mu \gamma_0$ is a skew-hermitian matrix. Such a condition is satisfied by the sixteen matrices $i\gamma_k$, $\gamma_i\gamma_j$, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, $\gamma_5\gamma_i$, and $i1$

The four-dimensional real submanifold of the complex base manifold is given a local Riemann-Cartan structure by the connection components of the form

$$A_\mu = \tfrac{1}{2} A^{ij}_\mu (\gamma_i\gamma_j) + \frac{1}{l}\, h^k_\mu (i\gamma_k), \tag{27}$$

where $A^{ij}_\mu = -A^{ji}_\mu$ and h^k_μ are the Lorentz components and the tetrads respectively.

Under a Lorentz gauge transformation A^{ij}_μ and h^k_μ transform in the expected way. At the same time, the original complex tetrads of the base manifold transform as the Dirac spinor fields. Thus the original geometry is based on the so-called double valued representation of the Lorentz group, while in the macroscopically observed space and time this extra structure is lost as the real representation is obtained from the adjoint action of the group elements on the generators of de Sitter "translations".

There is a curious point of contact between the present construction and quantum mechanics. One of the accepted peculiarities of quantum mechanics is the fact that even an electron at rest, presumably a structureless particle, needs a fast oscillating function of time for its description. Could this be connected with the macroscopic observation of time? Consider a connection that in some gauge describes a flat space-time using Minkowski coordinates. It means that

$$A_\mu = \frac{i}{l}\gamma_\mu, \quad \mu = 0, 1, 2, 3. \tag{28}$$

Let us assume that in some other gauge the time dimension becomes unobservable, i.e. $\tilde{A}_0 = 0$. What gauge transformation connects the observational gauge with such a special "timeless" gauge From Eq. (19) it can be seen that in the matrix form the gauge transformation satisfies

$$\frac{i}{l}\gamma_0 w + \partial_0 w = 0. \tag{29}$$

Thus the four columns of w happen to be the Dirac fields describing the four states of a spin 1/2 particle (and antiparticle) with mass $1/l$ at rest.

4. Conclusion

The purpose of the present paper was to show a possibility and to some extent also a plausibility of a construction which allows the properties of the observable space-time to result from the use of a specific gauge in a bundle of frames of a larger manifold. If such mathematical model corresponds to the reality, then any physical theory should be constructed as a fully gauge invariant geometric theory in the fundamental manifold, while its spacial and temporal characteristics reveal themselves only when the specific gauge can be, and is, selected. The theory of gravitation could especially benefit from such an approach, since many problems of general relativity, like the difficulties concerning energy-momentum tensor, singularities, quantization etc., are closely connected with the fact that the physical fields and space-time geometry are too intertwined. This prompted the rise of numerous theories which are trying to disentangle the two aspects, e.g. by using the background and the effective metrics [7]. The proposition presented here could eventually serve a similar purpose. It goes, however, one step further, since what is classified as "effective" is not only the metric, but the whole space-time manifold.

The case of the base manifold being locally $\mathbf{C}^4$ is especially inviting further investigation. There is a chance that the very existence of fundamental spin 1/2 particles could be related to the existence of regions in the base manifold with a geometry that does not allow the existence of the observational space-time gauge.

REFERENCES

[1] S. M. Rosen, *Found. Phys.* **18**, 1093 (1988).
[2] T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Phys. Math.* **K1**, 966 (1921).
[3] O. Klein, *Z. Phys.* **37**, 895 (1926).
[4] E. Cremmer, J. Scherk, *Nucl. Phys.* **B118**, 61 (1977).
[5] P. K. Smrz, *J. Math. Phys.* **28**, 2824 (1987).
[6] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon 1965.
[7] A. A. Logunov, M. A. Mestvirishvili, *Prog. Theor. Phys.* **74**, 31 (1985).