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INTERMITTENCY, NEGATIVE BINOMIALS AND TWO-PARTICLE CORRELATIONS

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Recent data on factorial moments are analyzed and found to follow regularities expected from a negative binomial (NB) multiplicity distribution. A linked-pair approximation for the r-particle rapidity correlations, proposed by Carruthers and Sarcevic, is proven to lead to multiplicity distributions of NB-type in small rapidity windows. From the general theory of stochastic processes, we deduce that the random nature of hadron production in small phase space cells closely resembles that of a completely chaotic (gaussian) system. The latter is shown to be phenomenologically equivalent with the Carruthers-Sarcevic *ansatz* for two-particle correlations of exponential (Lorentzian) shape.

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1. *Introduction*

Two aspects of multiparticle production have attracted much attention in recent years: a) the widespread occurrence, over a large energy range, of the negative binomial distribution (NBD) found to describe well (charged and negative) particle multiplicity distributions (MD) in full and in restricted domains of phase space, in lepton, hadron, and nucleus induced reactions, and in e^+e^- annihilations, [1]; b) the observation of sporadic large density fluctuations in small phase space cells. Multiplicity fluctuations in a given (pseudo) rapidity interval might reveal so-called "intermittent" behaviour as put forward by Białas and Peschanski [2–4].

In a) the observable of interest is the multiplicity distribution, parametrized as

$$P_n = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)} \frac{(\langle n\rangle/k)^n}{(1+\langle n\rangle/k)^{n+k}}, \tag{1}$$

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where $\langle n \rangle$ is the average multiplicity, k is a parameter related to $D^2 = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle^2 (1/\langle n \rangle + 1/k)$. In b) attention is focussed on the dependence on δy of the normalized factorial moment

$$\langle F_r \rangle = \langle n(n-1) \dots (n-r+1) \rangle / \langle n^r \rangle, \tag{2}$$

of the multiplicity distribution, where δy is a (pseudo) rapidity interval of decreasing size, down to the experimental resolution[1]. Intermittency is characterized by a power-law

$$\langle F_r \rangle \propto (\delta y)^{-\phi_r}, \quad \phi_r > 0; \tag{3}$$

with singular behaviour for $\delta y \to 0$ and typifies a dynamical system exhibiting selfsimilarity down to some smallest scale.

It is often claimed that intermittent behaviour (3) is now firmly established in a variety of processes [5]. Experimental data indeed show an approximately linear dependence of $\ln\langle F_r \rangle$ on $(\ln 1/\delta y)$ for decreasing δy in a typical interval $1.0 > \delta y > 0.1$. However, several authors have recently pointed out that the measured effects, at presently attainable experimental resolution, can be understood from conventional short-range order, supplemented by reasonable assumptions on r-body correlations ($r > 2$), and do not necessarily reveal novel dynamics [6–8]. Interestingly, the "standard" models such as FRITIOF [9] and the Dual Parton Model [10, 11] for hadron collisions, and the Lund shower model for e^+e^- annihilations [12] cannot fully reproduce the observed δy dependence of $\langle F_r \rangle$ [13, 14].

Until recently, with the exception of Ref. [15], little attention was paid to the link between a) and b) although both merely study different aspects of the same observables: the multiplicity distribution is small regions of phase space. As we show below, closer examination of the factorial moment data from several experiments leads to a number of interesting observations.

Firstly, the powers ϕ_r extracted from the data obey regularities expected from a negative binomial distribution. Also the absolute values of $\langle F_r \rangle$ agree reasonably well with those derived from a NBD, when $\langle F_2 \rangle$ is used as input. This suggests that the multiplicity distribution remains of NB-type, at least approximately, down to the smallest δy intervals.

Secondly, inspired by the results of Refs. [7, 8] we demonstrate that the multiplicity distribution in a small domain Δ of phase space tends, under quite general conditions, to a negative binomial for $\Delta \to 0$, if the rth-order correlation functions $C_r(y_1, y_2, \dots, y_r)$ obey a simple factorization *ansatz* in terms of $C_2(y_1, y_2)$, the two-particle rapidity correlation function. This result, which rests on the importance of two-particle range correlations, could explain why the NB-phenomenology is so successful in a variety of multiparticle processes ranging from "elementary" e^+e^- annihilations to complex nuclear collisions.

Finally, we point out that the proposed factorization approximation follows naturally from the stochastic nature of particle production if the random process is, at least in small regions of phase space, of gaussian type.

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[1] In practice, and for reasons of statistics, a suitably chosen rapidity interval is subdivided in M intervals δy and (2) is calculated as a double average, first over M intervals and then over the event sample.

2. *Definitions*

Let $I_r(y_1, y_2, \ldots, y_r)$ be the inclusive probability density for producing r indistinguishable particles with rapidities $y_1, y_2, \ldots, y_r$ and consider a rapidity interval[2] δy. The *factorial moment* $\tilde{F}_r$ is defined as:

$$\begin{aligned}\tilde{F}_r &= \langle n(n-1) \ldots (n-r+1)\rangle \\ &= \int_0^{\delta y} dy_1 \int_0^{\delta y} dy_2 \ldots \int_0^{\delta y} dy_r I_r(y_1, y_2, \ldots, y_r). \end{aligned} \tag{4}$$

The $\tilde{F}_r$ are related to F_r in (2) by

$$F_r = \tilde{F}_r / \tilde{F}_1^r. \tag{5}$$

Let $C_r(y_1, y_2, \ldots, y_r)$ be the r-particle correlation function, related to I_r via the standard (Mayer and Mayer) cluster decomposition [16, 17]:

$$\begin{aligned} I_1(y) &= C_1(y), \\ I_2(y_1, y_2) &= C_1(y_1)C_1(y_2) + C_2(y_1, y_2); \\ &\ldots \end{aligned} \tag{6}$$

We assume the functions C_r to be invariant (stationary) under a shift in rapidity and thus restrict our discussion to a single interval δy centered around $y = 0$. Stationarity implies that the C_r are effectively functions of $(r-1)$ rapidity variables since the rapidity origin may be fixed. The r-fold integral of C_r over an r-cube of size δy defines the *factorial cumulants* (Mueller's moments) [17]

$$\tilde{f}_r = \int_0^{\delta y} dy_1 \int_0^{\delta y} dy_2 \ldots \int_0^{\delta y} dy_r C_r(y_1, y_2, \ldots, y_r). \tag{7}$$

Explicit relations between $\tilde{F}_r$ and $\tilde{f}_r$ are tabulated e.g. in [18] for $r \leqslant 10$. We further introduce the factorial moment generating function, $Q(s)$, of the multiplicity distribution $P_n (n = 0, 1, \ldots)$ given by:

$$Q(s) = \sum_{n=0}^{\infty} (1-s)^n P_n = 1 + \sum_{r=1}^{\infty} \frac{(-s)^r}{r!} \tilde{F}_r, \tag{8}$$

where $\tilde{F}_r$ is the factorial moment (4). The factorial cumulants $\tilde{f}_r$ are defined by the power expansion of $\ln Q(s)$:

$$\ln Q(s) = \sum_{r=1}^{\infty} \frac{(-s)^r}{r!} \tilde{f}_r. \tag{9}$$

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[2] The following expressions are valid for any set of kinematical variables; we here consider c.m. rapidities for simplicity.

For a negative binomial multiplicity distribution, we find from (1), (8) and (9)

$$Q^{NB}(s) = (1+s\langle n\rangle/k)^{-k}, \tag{10}$$

$$F_r = (1+1/k)(1+2/k)\dots(1+(r-1)/k), \tag{11}$$

$$f_r = (r-1)!/k^{r-1}, \tag{12}$$

with $F_r = \tilde{F}_r/\langle n\rangle^r$ and $f_r = \tilde{f}_r/\langle n\rangle^r$ the normalized factorial moments and cumulants, respectively. Note that $F_1 = f_1 = 1$. The second order normalized factorial moment satisfies the relation [19]:

$$F_2 = 1+\frac{\int_0^{\delta y} dy_1 \int_0^{\delta y} dy_2 C_2(y_1, y_2)}{[\int_0^{\delta y} dy I_1(y)]^2} \tag{13}$$

$$= 1+\frac{1}{k}; \tag{14}$$

where the second equation is valid for a NBD. The second term on the right-hand side of (13) is often written as $R(0, \delta y)$. This two-particle correlation function has been extensively studied in various interactions over a very wide range of c.m. energies (see e.g. [20–-24] and Refs. therein).

For symmetrical c.m. rapidity intervals centered around $y = 0$, the multiplicity distribution is known to be well described by the NBD [1]. In this case, and for small δy, we have:

$$1/k = R(0, \delta y). \tag{15}$$

This simple relation gives a natural physical meaning to the k parameter in the NBD.

3. *Factorial moments and the NBD*

The multiplicity distributions in central (pseudo) rapidity bins $|y| < y_c$ are now measured in many types of interactions. For $y_c > 0.2$ ($\delta y > 0.4$), they are well described by negative binomials [1]. Multiplicity data in still smaller intervals are available from intermittency studies in several experiments [5]. Although such data are, for fixed δy, averaged over a rapidity interval of several units, it is interesting to discuss factorial moments, and their δy dependence, under the hypothesis that the multiplicity distribution remains of NB-type down to the limit of experimental resolution. To this effect, we take the measured $\langle F_2\rangle$ (or the slopes ϕ_2) as input, and compare $\langle F_r\rangle$ ($r > 2$) with the values derived from (14) and the NB-relation (11).

The data from the KLM collaboration [25] for p-Emulsion interactions at 200 and 800 GeV/c, and for ^{16}O — Emulsion at 60 and 200 GeV/nucleon are shown in Fig. 1. The full lines are published linear fits; the dashed lines are here calculated from (11) and

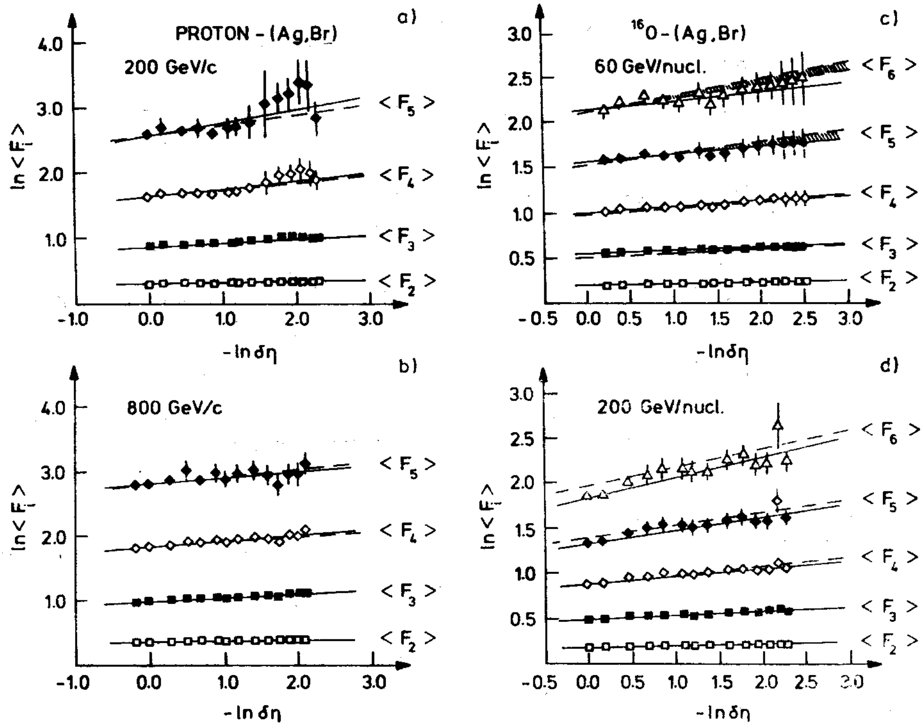

Fig. 1. Data of the KLM collaboration [25]; (a-b) p+emulsion at 200 and 800 GeV/*c*. (c-d) ^{16}O+emulsion at 60 and 200 GeV/nucleon. Dashed lines are derived from a negative binomial, using F_2 and the slopes ϕ_2 as input, solid lines are experimental fits

the published values of ϕ_2. The NBD predictions for the slopes $\phi_r(r > 2)$ essentially coincide with the experimental ones. For the ^{16}O — Emulsion data, the absolute values deviate slightly for $r = 5.6$.

The WA80 ^{16}O+C data [26] at 200 GeV/nucleon are shown in Fig. 2. Here, the $\langle F_r \rangle$ $(r > 2)$ (crosses) are calculated from $\langle F_2 \rangle$ in every δy interval. The agreement with the data is quite striking[3].

Fig. 3a shows the $\sqrt{s} = 600$ GeV data of UA1 [27]. The NBD "predictions" for the slopes ϕ_r are clearly consistent with the data. The absolute values of $\langle F_r \rangle$ are known to be less well determined, due to experimental biases, and are smaller than the predictions. We have also examined moments in azimuthal bins, where the biases are well under control[14]. They follow Eq. (11) very well (not shown).

Although we cannot derive definitive conclusions from the factorial moment data, in view of the bin-averaging involved, it is tempting to assume, as a working-hypothesis,

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[3] The Wa80 collaboration has recently announced that the published data may be biased and are being reanalyzed. Unless the agreement with the NBD is purely accidental, Fig. 2 indicates that the expected biases are probably less important than thought.

[4] private communication from B. Buschbeck.

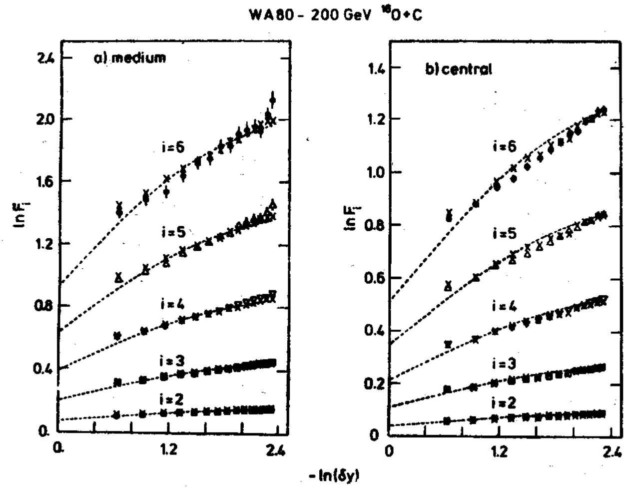

Fig. 2. Data of the WA80 collaboration [26]; crosses are derived from a negative binomial, with $\langle F_2 \rangle$ as input; dashed curves from (22) with $(\gamma_2, \xi) = (0.19, 0.4)$ (a) and $(\gamma_2, \xi) = (0.11, 0.35)$ (b); predictions based on (33) coincide with those shown

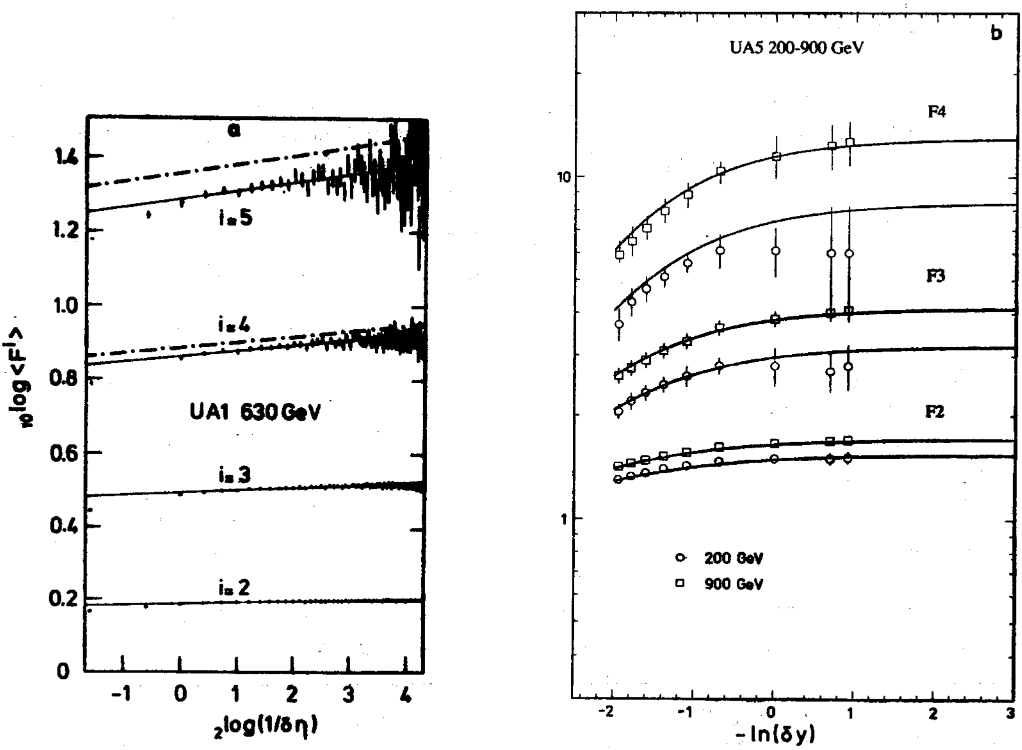

Fig. 3. (a) Data of the UA1 collaboration [27]; dashed lines as in Fig. 1; (b) Data of the UA5 collaboration [37] for symmetric pseudo-rapidity intervals around $\eta = 0$; curves as in Fig. 2

that the multiplicity distribution remains of approximate NB-type down to the smallest phase space domains presently investigated. The data certainly suggest that the hierarchy of slopes (ϕ_r) is fixed by ϕ_2 in those cases where a linear dependence of $\ln \langle F_r \rangle$ on ($-\ln \delta y$) is appropriate. Little additional information is thus gained from the higher-order factorial moments which, in any case, are very sensitive to experimental biases and statistics. More importantly, the wide occurrence of the NB, combined with the above observations, means that the multiplicity distribution in small phase space cells is effectively controlled by F_2 and, therefore, by the two-particle correlation function. Consequently, higher-order correlations must possess a well-defined structure in terms of C_2 and C_1, as we shall further argue.

With respect to NB and "intermittency" phenomenology, it is instructive to translate the well-known trends of two-particle correlations regarding energy-, charge-, and δy dependence to those of the $1/k$ parameter in the NB and of the factorial moments. This has already been emphasized in [6, 8, 28]. Another detailed investigation, combining intermittency and correlation data with QCD cascading, is presented in a recent paper by Ochs and Wosiek [29].

4. Higher order correlations and the NB

The widespread occurrence of the negative binomial implies a very specific type of multiparticle correlations[5]. In particular, two-particle correlations must play a prominent dynamical role, at least in small domains of phase space. In this, hadronization dynamics strikingly resembles many other areas of many-body physics where the dynamics is often well described by nearest-neighbour or two-body interactions only.

In hadrodynamics, little is known about correlations beyond two-, and three-particle effects. Three-particle rapidity correlations have been investigated in several hadron experiments [20, 30, 31] and found to be either absent or much weaker than those among particle pairs. While one may be tempted to assume that all C_r (and therefore the factorial cumulants) vanish for r larger than some arbitrary number, this assumption is invalid, as was shown e.g. in [32]. To reconcile the simple correlation function structure suggested by the NB-data, with the need to include higher orders, it is necessary to invoke a "closure-relation", a technique familiar e.g. from the theory of liquids [33].

This idea has been recently explored by Carruthers and Sarcevic [8], and is also implicit in the work by Capella at al. [6, 7]. It is quite remarkable that a simple closure-relation, the linked-pair structure of Ref. [8], indeed leads to approximate negative binomial multiplicity distributions in small phase space domains, as we now demonstrate.

Consider the normalized factorial cumulants in a region of phase space Δ and let $\Delta \to 0$. If we require the multiplicity distribution in Δ to be a negative binomial, then, from (12)–(15) we note that f_r has to satisfy the relation

$$f_r = (r-1)!R^{r-1}. \qquad (16)$$

[5] Cf. the "clan-concept" introduced in [19].

with R the average of the normalized two-particle correlation function in Δ. Relation (16) implies that, as $\Delta \to 0$, the higher-order correlation functions $C_r(r > 2)$ must factorize into $(r-1)$ factors C_2 and contribute $(r-1)!$ identical terms. An explicit expression with this property is proposed in [6] and [8]. Generalizing slightly the authors' arguments, we assume the following "linked-pair" structure for the reduced correlation functions:

$$c_r(y_1, y_2, \ldots, y_r) = C_r(y_1, y_2, \ldots, y_r)/C_1(y_1) \ldots C_1(y_r),$$

$$c_3(y_1, y_2, y_3) = c_2(y_1, y_2)c_2(y_2, y_3)+c_2(y_1, y_3)c_2(y_3, y_2),$$

$$c_4(y_1, y_2, y_3, y_4) = c_2(y_1, y_2)c_2(y_2, y_3)c_2(y_3, y_4)+\text{cycl. perm.} \quad (17)$$

Stationarity of all c_r implies that we may fix the position of one particle, say particle 1. The expression for c_r then has exactly $(r-1)!$ terms, equal to the number of cycles of r objects [34]. From stationarity, we further have $c_2(y_1, y_2) = c_2(y_1-y_2)$ and $c_1(y) =$ *constant*. With (17) and (7), and for $\delta y \to 0$, we recover (12) with

$$1/k = \int_0^{\delta y}\int_0^{\delta y} c_2(y_1, y_2)dy_1dy_2, \quad (18)$$

under the condition that the integral exists.

In [8], $c_2(y_1, y_2)$ is written as:

$$c_2(y_1, y_2) = \gamma_2 \exp(-|y_1-y_2|/\xi), \quad (19)$$

ξ being the inclusive two-particle correlation length. With this parametrization, one finds [8]:

$$F_2 = 1+\gamma_2[(1-\exp(-\delta y/\xi))/(\delta y/\xi)], \quad (20)$$

and

$$f_r(\delta y) = (r-1)!\,(F_2-1)^{r-1}. \quad (21)$$

This expression is of the form (16) and thus corresponds to a negative binomial. However, (21) is an approximation of the r-fold integral (7), valid for $\delta y \to 0$ only. The linked-pair *ansatz* does therefore not yield an exact negative binomial.

For finite δy and c_2 given by (19), the integral (7) is evaluated exactly in [6]. The normalized factorial cumulants take the form

$$f_r = (r-1)!\gamma_2^{r-1}B_r(x), \quad (22)$$

where the B_r are known functions of $x = \delta y/\xi$ only; for $x \to 0$, $B_r(x) \to 1$. The form of (22) is valid for any function $c_2(y_1, y_2) = \gamma_2 h(|y_1-y_2|/\xi)$ which is free of singularities at the origin. The factorial moments are easily derived from (21)–(22), using the relations between f_r and F_r given in [18].

Relations (21)–(22) adequately describe the δy dependence of F_r in several NB and

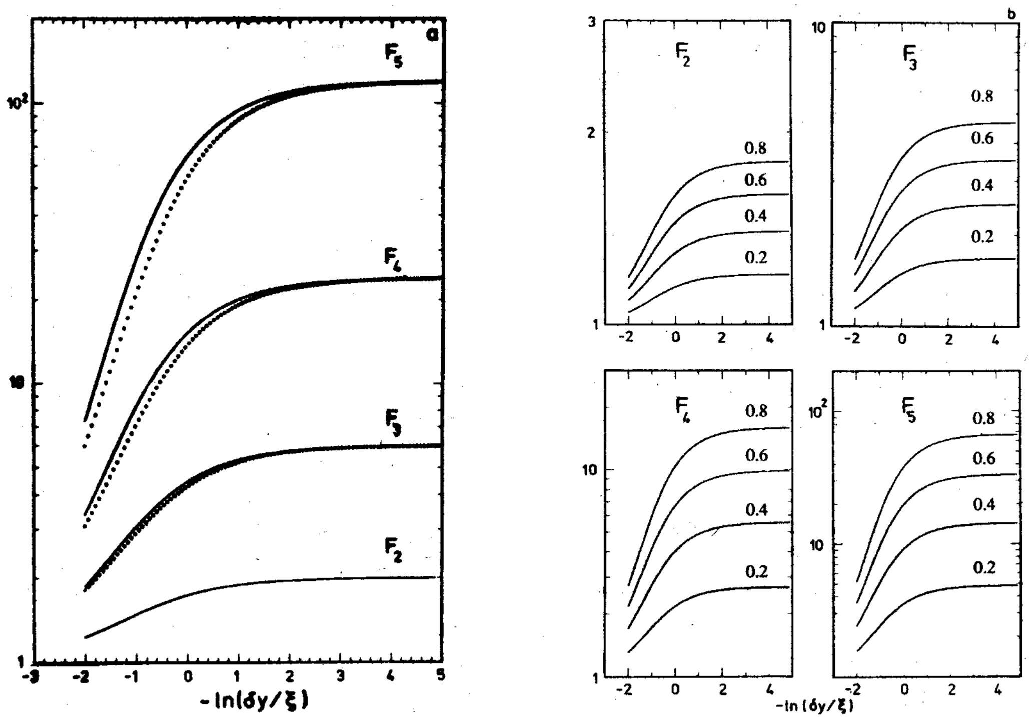

Fig. 4. (a) Normalized factorial moments derived from (22) (solid curves) and from (11) (dashed) with F_2 from (22) for $\gamma_2 = 1$; (b) the moments F_2-F_5 from (22) for indicated γ_2 values. Curves calculated from (33) coincide with those shown

intermittency data sets. This was shown, in a somewhat different way, in [6] where e^+e^-, μp and hadron data are compared to (22), and in [8] where (21) is used to analyse the UA5 (546 GeV) and NA22 (22 GeV) data[6]. The same data are discussed, and well described, in [35] in a quantum-optics context, using a mixture of coherent and chaotic fields.

Equation (22) expresses the factorial cumulants, and thus the multiplicity distribution in δy, as functions of γ_2 and $\delta y/\xi$ only. The latter property was called "β-scaling" by Fowler et al. [36]. The factorial moments F_r are plotted versus $-\ln(\delta y/\xi)$ (for $\gamma_2 = 1$) in Fig. 4a (full curves). The dotted curves correspond to the NB-approximation calculated using f_2 from (22) and expression (11). Deviations from a NB are small for $\delta y/\xi < 1$. Fig. 4b compares the $\delta y/\xi$ dependence of F_2-F_5 for various γ_2 values. The derivatives for fixed r clearly increase with γ_2 in the region $-2 < -\ln(\delta y/\xi) < 1$.

We now return to the data shown in Figs. 2–3. The dotted curves on Fig. 2, derived

[6] In [8], the factor $(r-1)!$ in (22) is not used but replaced by a_r^{r-1} and a_r is adjusted to the data. The authors find $a_3 = 1.3$, $a_4 = 1.6$, $a_5 = 2.8$ at $\sqrt{s} = 22$ and 900 GeV. From Eq. (21) we find $a_3 = \sqrt{2} = 1.41$, $a_4 = (6)^{1/3} = 1.82$, $a_5 = (24)^{1/4} = 2.21$, evidently energy independent, and close to the fitted values. In [6], the normalized factorial cumulants are written as $f_r = A_r B_r(\delta y/\xi)$; the parameters A_r are adjusted to the data at $\delta y = 1$, separately for each order r.

from Eq. (22) with $(\gamma_2, \xi) = (0.19, 0.4)$ (Fig. 2a) and $(\gamma_2, \xi) = (0.11, 0.35)$ (Fig. 2b) reproduce well the δy dependence and the absolute values of the moments[7].

In Fig. 3b we compare (22) with the UA5 central pseudo-rapidity data [37] using $(\gamma_2, \xi) = (0.54, 3.26)$ and $(\gamma_2, \xi) = (0.71, 3.63)$ at 200 and 900 GeV c.m. energy, respectively. The agreement is again excellent, not only for small δy, but also for the larger central intervals.

The successful linked-pair *ansatz* leading to (22), offers a natural and simple explanation for several trends seen in intermittency studies, as summarized in [5].

• The factorial moments are not power behaved in the δy range investigated until now, but at most in a restricted region; the "slopes" ϕ_r clearly depend on the chosen fit--interval.

• Inspection of the two-particle (pseudo) rapidity correlation data at SPS, ISR and CERN collider energies shows that the inclusive correlation length ξ is slowly increasing with $\sqrt{s}$ (see e.g. [20, 24, 38] and Refs. therein). Larger ξ at fixed δy means (compare Fig. 4) smaller "slopes" ϕ_r as $\sqrt{s}$ increases. This is observed e.g. in the UA1 analysis [27]. Similarly, selection of (semi-inclusive) data samples with larger average multiplicity $\langle n \rangle$ implies smaller ϕ_r since γ_2 is known to decrease rapidly with increasing $\langle n \rangle$. This is also seen in [27].

• Little is known about two-particle rapidity correlations in e^+e^- annihilations. The TASSO data [39, 40] show stronger correlations (larger γ_2), but of possibly shorter range (smaller ξ), than in hadron collisions at comparable energies. This results in a stronger δy dependence of the factorial moments, and larger ϕ_r, as experimentally observed [14].

• The hierarchy of the slope-values in different types of collisions (decreasing from e^+e^- to A+A) is often discussed in terms of the degree of complexity of the interactions. While this connection may seem attractive in a theoretical context, our analysis indicates that a straightforward comparison of two-particle correlations (or of γ_2 and ξ in our formalism) may be physically more transparent than a naive comparison of bin-averaged factorial moments or effective slopes. Furthermore, as (22) reveals, factorial cumulants rather than factorial moments should preferably be studied.

• The models FRITIOF, DPM and the Lund Shower model for e^+e^- fail to describe the "intermittency" data [13, 14]. This observation came as a surprise and is partly the reason for the current interest in intermittency-type analyses. From a model comparison in hadron collisions [41, 42], it is easily seen that the model failures find their origin in large discrepancies between the measured and predicted two-particle correlations. The strength (γ_2) is severely underestimated and the predicted correlation length (ξ) is too large by a factor 1.5 to 2. Recalling Fig. 4, this explains why the predicted slopes are several times smaller than measured[8]. The disagreements seem less severe, at first sight, for e^+e^-

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[7] Note that the correlation lengths $\xi \approx 0.4$ are considerably smaller than in hadron-hadron collisions at similar energy ($\xi \approx 1$–1.5). If confirmed by a direct analysis of the two-particle rapidity correlation functions, it would suggest than an ultra-short range component is present, possibly related to enhanced Bose-Einstein effects. A preliminary NA22 study of correlations in hadron-nucleus interactions, however, shows no evidence for "abnormal" short-range effects (private communication from A. M. Endler).

[8] The UA5 cluster Monte Carlo (GENCL) [43] is in much better agreement with the NA23 data correlation data [41] and may therefore describe better the intermittency data.

annihilations. On closer inspection, however, is it seen that the Lund Monte Carlo underestimates $R(y_1, y_2)$ at small δy by a factor of ≈ 1.5 and overestimates the correlation length [39]. It is likely that these shortcomings are not due to the manifestation of novel dynamics in the data, but related instead to the absence, in the present Lund string hadronization algorithm, of effects such as heavier resonance states and possible resonance-interference phenomena.

5. *Multihadron production: a gaussian stochastic process?*

The linked-pair structure of the (reduced) correlation function c_r was shown in Sect. 4 to lead, under quite general conditions, to approximate negative binomial multiplicity distributions in phase space cells of the order of, or smaller than, the typical short-range correlation length. The "particle counting" statistics in this model is fully determined by the second-order correlation function. This property a priori suggests that multihadron production may well possess characteristics similar to those of a stochastic process of gaussian type.

The theory of gaussian random processes is a well-known branch of mathematical statistics and treated in many textbooks[9]. The theory may be formulated in various different but equivalent mathematical forms. A purely classical, probabilistic treatment was first used in studies of random noise encountered in radio-physics [45, 46]. A quantum treatment was fully developed for quantum optics via the P-representation of the optical-field density matrix [47]. Here, the gaussian approximation describes thermal light and lasers below threshold. In quantum-field theory, the gaussian approximation is closely related to the so-called Random Phase Approximation [48].

The quantum optics results have been translated (see [49–51, 36] and Refs. therein) or adapted [52, 53] to hadron physics by several authors; a probabilistic treatment of stochastic point processes [54] has been successfully used by Capella and Krzywicki [55].

To elucidate the connection between the empirically derived linked-pair *ansatz* and the structure of the correlation functions in a general gaussian stochastic process, we biefly recall the main ingredients of the theory and summarize, without proof, its main result.

Let $\Pi(y)$ be a complex random field of a set of variables denoted by y, (we take y to be c.m. rapidity for simplicity) so that the average over an — as yet unspecified — ensemble, $\langle|\Pi(y)|^2\rangle$, equals the inclusive particle density at y in a collision. $\Pi(y)$ is assumed to be stationary: $\langle\Pi(y)\rangle = \langle\Pi(0)\rangle = constant$. A stochastic process is completely determined if the joint probability of the set of random variables $\{\Pi(y)\}$ for all y in an interval (0-Y) is known. Alternatively, it is fully described by the set of field correlation functions

$$G_{r,r'} = \langle\Pi^*(y_1)\Pi^*(y_2)\ldots\Pi^*(y_r)\Pi(y_{r+1})\Pi(y_{r+2})\ldots\Pi(y_{r+r'})\rangle. \tag{23}$$

The function

$$G(y_1, y_2) \equiv G_{1,1}(y_1, y_2) = \langle\Pi^*(y_1)\Pi(y_2)\rangle, \tag{24}$$

[9] A general review of the theory with many practical applications and extensive references to the original literature can be found in [44].

plays a fundamental role and is called the "coherence function". The inclusive r-particle densities are defined as the ensemble averages

$$I_r(y_1, y_2 \dots y_r) = \langle |\Pi(y_1)|^2 |\Pi(y_2)|^2 \dots |\Pi(y_r)|^2 \rangle. \tag{25}$$

They are the analogues of the r-fold intensity distributions in optics. Now, assume the complex random variable $\Pi(y)$ to be distributed according to a gaussian law. In this case, all higher-order field correlation functions can be expressed in terms of the coherence function $G(y_1, y_2) \equiv G(y_1 - y_2)$ and (25) takes the form [34]:

$$I_r(y_1, y_2, \dots, y_r) = \sum_P \prod_{j=1}^{r} G(y_j, y_{Pj}), \tag{26}$$

where the sum runs over all permutations $P: j \to Pj$ of the integers $j = 1, 2, \dots, r$. With the standard cluster expansion of I_r in terms of the correlations functions C_r, one finds [34]:

$$C_r(y_1, y_2, \dots, y_r) = \sum_C \prod_{j=1}^{r} G(y_j, y_{Cj}), \tag{27}$$

where the sum now runs over all *cyclic* permutations C of the integers $j = 1, 2, \dots, r$, and contains $(r-1)!$ terms. For $r = 2, 3, 4$ we have (in abbreviated notation):

$$C_2(y_1, y_2) = |G(1, 2)|^2, \tag{28}$$

$$C_3(y_1, y_2, y_3) = G(1, 2)G(2, 3)G(3, 1) + G(1, 3)G(3, 2)G(2, 1), \tag{29}$$

$$C_4(y_1, y_2, y_3, y_4) = G(1, 2)G(2, 3)G(3, 4)G(4, 1) + \text{cycl. perm.} \tag{30}$$

$$\dots$$

With C_2 as in (19), (24) has the form

$$G \sim \exp -(|y_1 - y_2|/2\xi). \tag{31}$$

The hadronic equivalent of the optical coherence length is thus twice the usual two-particle correlation length in rapidity. The above equations illustrate a fundamental difference between the linkedpair structure of Eq. (17) and the correlations in a gaussian stochastic process. The former are expressed in terms of C_2, the latter in terms of the coherence function G. Moreover, only "ring-graphs" are present in the gaussian model[10]. Unlinked pairs (e.g. of the type $G(1, 2)\, G(3, 4)$ are absent by the nature of factorial cumulants. With (28)–(30), the normalized factorial cumulants of the multiplicity distribution in the interval (0-Y) are derived in [34] and found to be:

$$f_r = \sum_C \int_0^Y dy'_1 \int_0^Y dy'_2 \dots \int_0^Y dy'_r \prod_{j=1}^{r} G(y'_j - y'_{Cj}). \tag{32}$$

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[10] An identical result holds for the Random Phase Approximation in many-body theory [48].

For a Lorentzian shape as in (31), they have been calculated by several authors [56, 46] and take the form:

$$f_r = (r-1)!B'_r(\beta), \tag{33}$$

where the B'_r are functions of $\beta = \delta y/2\xi$. They are tabulated e.g. in [56].

The resemblance of equations (33) and (22) is striking. It is easily verified that the δy dependence of f_r, for G of the form (31), is identical in both models for $r = 2, 3$. Differences appear in the expressions for higher orders but are numerically small ($\leqslant$ a few % for $\beta \geqslant 1$ and negligible for $\beta \leqslant 1$). Consequently, for two-particle correlations of Lorentzian form, we find that the gaussian approximation and the linked-pair structure lead to an essentially identical form of the correlation functions in phase space cells of size of the order of, or smaller than the hadronic correlation length.

As could be expected from the outset, the simple, one-component gaussian process yields a Bose-Einstein multiplicity distribution in the limit $\delta y \to 0$. Indeed, the factor γ_2^{s-1}, present in (22) is absent from (33). To recover the successful NB-phenomenology with $k > 1$ and stay as close as possible within the present framework, one may follow the approach of Ref. [55] (see also [49]) and assume that the "observable" stochastic field $\Pi(y)$ is the sum of a (random) number of m "elementary" chaotic fields. If these fields are statistically independent and have identical average properties, the resulting multiplicity distribution is a convolution of $\langle m \rangle$ identical Bose-Einstein distributions i.e. a negative binomial with $k = 1/\gamma_2 = \langle m \rangle$[11].

Although the validity of the last arguments would merit further discussion, we believe that the main result of this section has more general validity than the specific model from which it is derived. Indeed, from the particular structure of higher order correlations in multihadron production processes, as revealed by the data, we are led to conclude that the hadronic field strongly resembles a gaussian, i.e. completely chaotic (complex) field. This is likely to be *the* basic common ingredient, shared by the plethora of models, which aim at a description of the multiplicity distribution is small phase space domains.

6. *Summary and final comments*

Many experimental studies of a large variety of high energy multihadron processes have revealed that the multiplicity distribution in restricted domains of phase space, is quite well parametrized by a negative binomial distribution. The recent "intermittency" data on factorial moments also show regularities consistent with a NB down to the (present) limits of resolution in rapidity.

The fact that the NB occurs, either approximately or exactly, in a large variety of models, including QCD parton cascades at high energies [58–61] a priori raises the suspicion, as often in many-body physics, that its origin is rooted in quite general, and probably not too discriminative properties of hadronization.

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[11] A closed form for the generating function, valid for finite δy, and a recurrence relation for the probabilities P_n can easily be derived from the formulae given in [57]

Combining experimental observations with ideas borrowed from recent phenomenological analyses, we have searched for a structure of multi-particle correlations which yields approximate NB-behaviour in small phase space cells. The linked-pair approximation, proposed in [8] satisfies this condition. Moreover, it suggests a generalization of the NB, valid in a finite but still small phase space domain, where the conditions of stationarity are valid. This result further defines a convenient framework for a critical examination of "intermittency effects", or lack thereof, in terms of standard short-range order phenomenology. Two-particle correlations are shown to play a crucial role and thus merit further detailed experimental study, particularly in the interesting small δy region. Equally interesting are analyses of higher-order correlations, which could directly prove or invalidate the proposed factorization scheme.

Presently popular models such as FRITIOF or the Dual Parton Model for hadron collisions, and the Lund shower model for e^+e^- annihilations do not fully reproduce the intermittency data. We have indicated that the model failures are direct consequences of rather serious discrepancies between the measured and the predicted short-range correlations, especially in hadron collisions. This fact has largely been overlooked in the past. It is not clear, at present, if these shortcomings are easily curable, or hint instead to a more fundamental flaw in our understanding of hadronization.

Dynamics dominated by two-particle interactions is quite common in almost any branch of many-body physics. In Sect. 5 we have examined the general class of gaussian stochastic processes, the prototype of models where all correlations are expressible via a single two-point "coherence" function. We consider it highly non-trivial that the structure of the correlations in this model is practically equivalent to, although fundamentally different from that of the linked-pair approximation which was proposed on empirical grounds. The gaussian model therefore provides a general and satisfactory theoretical foundation for the phenomenological description of the data, discussed in Sect. 4.

Although the formalism leading to these results is most simply formulated in classical, probabilistic terms, and does nowhere invoke quantum physics, there exists, of course, a strong analogy with the quantum treatment of a thermal electromagnetic field (the theory of photo-electron statistics) [44] which has been repeatedly applied to hadron physics. We suspect that the success of this approach is less due to a strict similarity between hadron and optical fields, than to the fundamentally chaotic nature of the two processes, which allows a common mathematical description.

We have already mentioned the similarity between the classical gaussian process and the Random Phase Approximation in quantum field theory. This analogy has been extensively studied by Karczmarczuk [62, 48], who showed quite generally how the method leads, under certain conditions, to a negative binomial multiplicity distribution.

In a historical context, it should be mentioned that the exact result (33) was first obtained by D. Slepian [46]. It describes the statistics of the average power dissipated, in a finite time-interval, by a simple RC-circuit driven by gaussian noise.

It would obviously be naive — and not too encouraging — to conjecture that the dynamical properties of hadronic matter are effectively equivalent to that of gaussian noise. The dynamics of the hadronization of multi-quark and gluon systems involves complicated non-linear phenomena, and the linearization implied by the models discussed is likely to be an extremely crude approximation to reality. But, the statistical properties of the hadron field, reflected in simple counting distributions, cannot be expected to reveal fine details of the dynamics of the sources emitting these fields, a fact well-known from optics.

In experiments where smaller and smaller regions of phase space are scrutinized, and all other information is discarded, it may well happen that, unless strong short-range dynamical excitations indeed occur, a "minimal-information" regime — in the information-theory sense — is ultimately reached where nearly all sensitivity to the "emitting sources" is lost, and where particle fluctuations indeed exhibit characteristics resembling that of gaussian noise.

Note added: After completion of this work, we received a paper by M. Biyajima et al. (Vienna preprint UWThPh-1989-44), where the NA22, KLM and UA1 data are analyzed by means of pure-birth stochastic equations, and of the negative binomial. The conclusions of this paper and those of our Sect. 3, are essentially identical.

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