MULTIDIMENSIONAL COSMOLOGICAL MODELS WITH HYDRODYNAMICAL ENERGY-MOMENTUM TENSOR. PART II. ANALYSIS OF DYNAMICAL SYSTEMS AT INFINITY*

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The dynamics of the full class of multidimensional cosmological models with topology FRW×T^D, (where T^D is a D-dimensional torus) near the singularity is investigated. Phase portraits show possible evolutions of FRW×T^D models with hydrodynamical energy-momentum tensor. The problem of stability of solutions with a "crack-of-doom" singularity is also discussed.

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0. Introduction

This paper is a continuation of our previous work [4]. In the present part we disscussed trajectories at infinity.

In Section 1 analysis of the dynamics near singularities is performed with the help of the method of dynamical systems. It has been possible to reduce the problem to a 2-dimensional phase space [Part I]. The conclusions are summarized in Section 2.

1. Dynamics of cosmological models near singularity

To have a complete analysis of the phase portrait of a dynamical system one must know how the system's trajectories behave at infinity. To this purpose one should perform the transformation to projective variables ($z = x^{-1}$, $u = yx^{-1}$); ($v = y^{-1}$, $w = xy^{-1}$)

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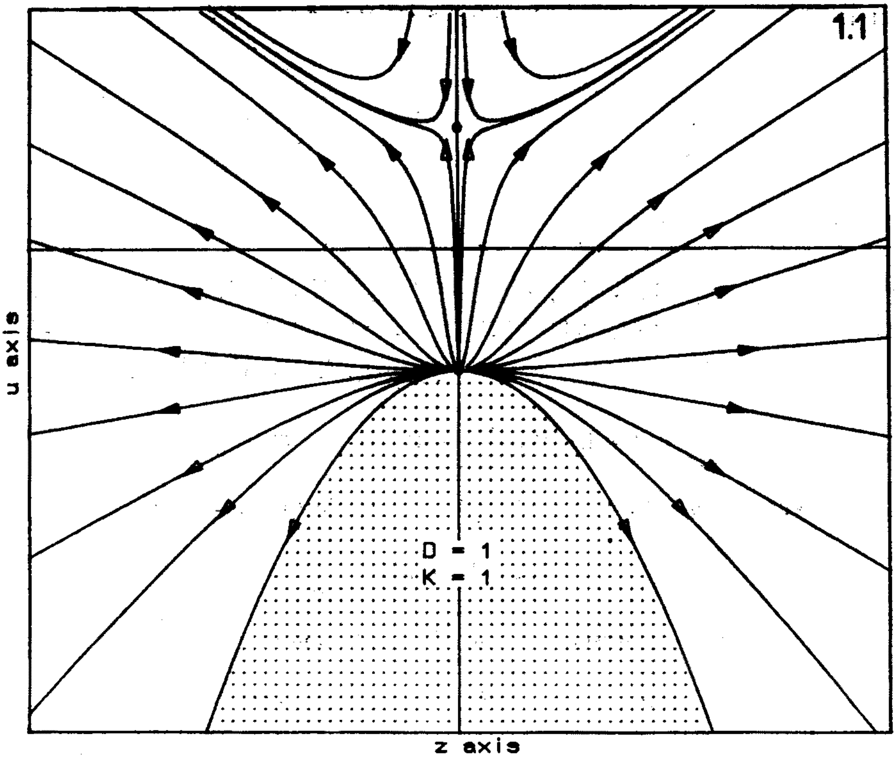
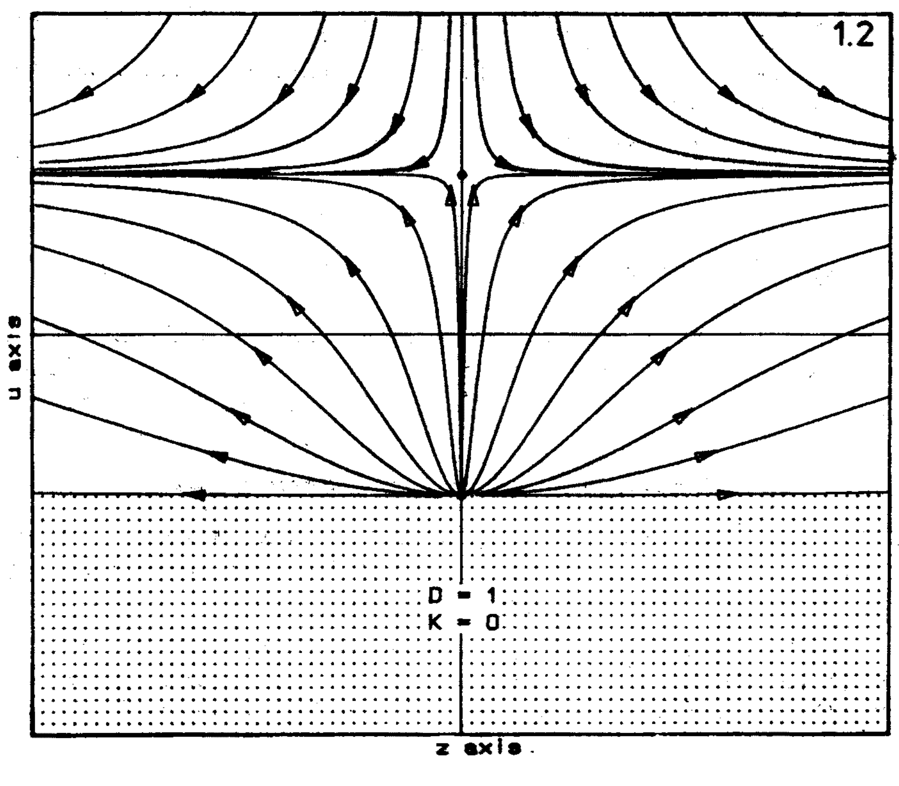

1.1
u axis
z axis
D = 1
K = 1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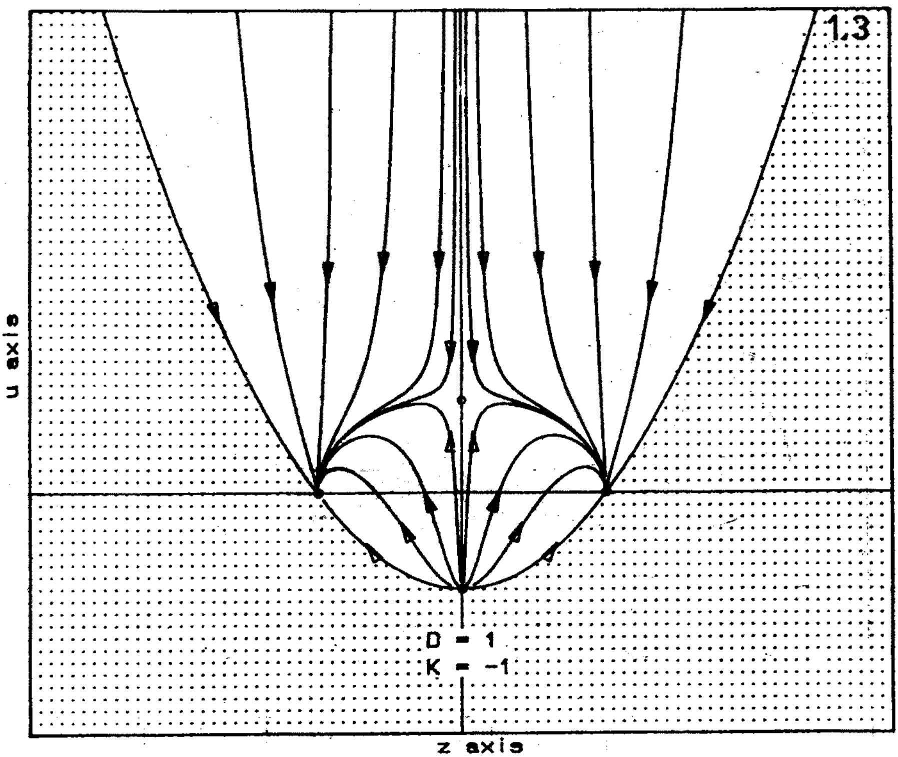


Fig. 1.1–1.3. Phase portraits for model FRW$\times$T^D with dust matter in projective coordinates (z, u), for $D = 1$, and $K = 1, 0 -1$

and conformal time τ: $d\tau = x\,dt$ (for z, u) and $d\tau = y\,dt$ (for v, w). In these variables the straight lines $z = 0$ $-\infty < u < \infty$ and $v = 0$ $-\infty < w < \infty$ correspond to infinitely distant points of the plane (x, y), and studying of this system in projective variables is analogical to investigating the system on finite regions. As previously, we will consider five cases.

A. Vacuum model

System of equations (see Eq. (15)-Part I) in projective coordinates (z, u) assumes the form:

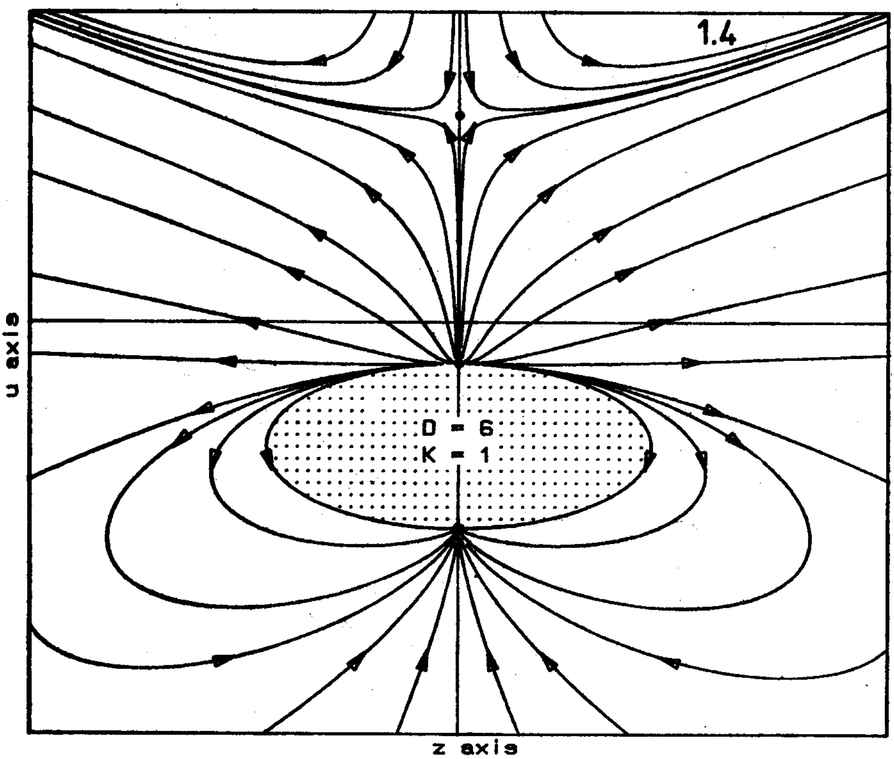
$$\frac{dz}{d\tau} = z - Duz - \frac{D(D-1)}{3}u^2 z, \qquad \frac{du}{d\tau} = -2u - 2Du^2 - \frac{D(D-1)}{3}u^3, \tag{1}$$

where $d\tau_1 = H\,dt$, with a condition:

$$K(-6 - 6Du - D(D-1)u^2) \geqslant 0. \tag{2}$$

In projective variables (v, w) we have

$$\frac{dv}{d\tau_1} = Dv + 3vw, \qquad \frac{dw}{d\tau_1} = 2Dw + 2w + \frac{D(D-1)}{3}, \tag{3}$$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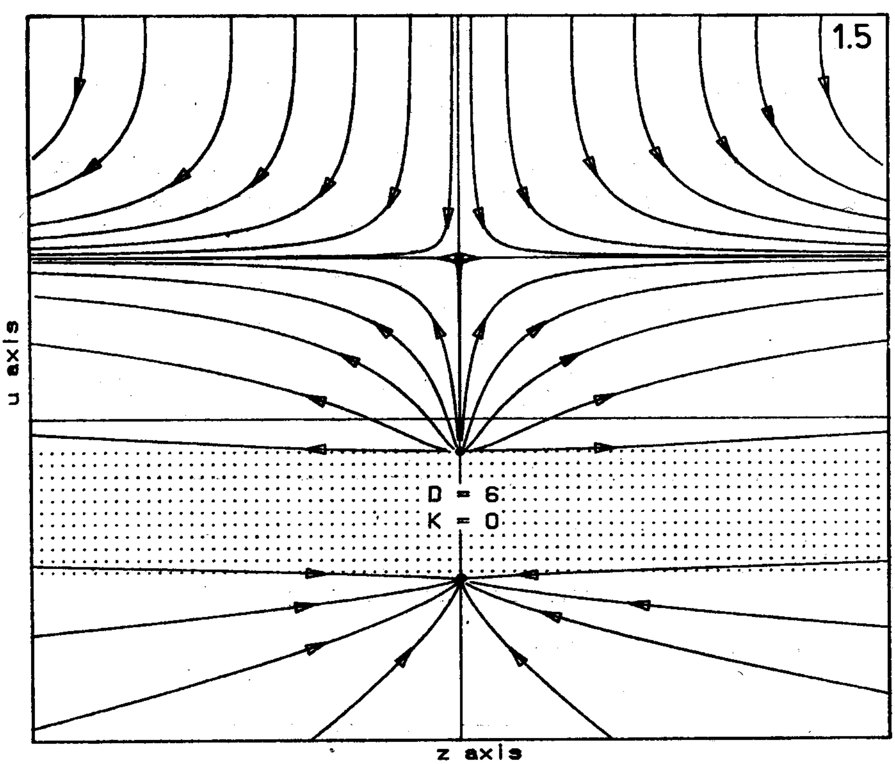
K = 1
u axis
z axis


1.5
D = 6
K = 0
u axis
z axis

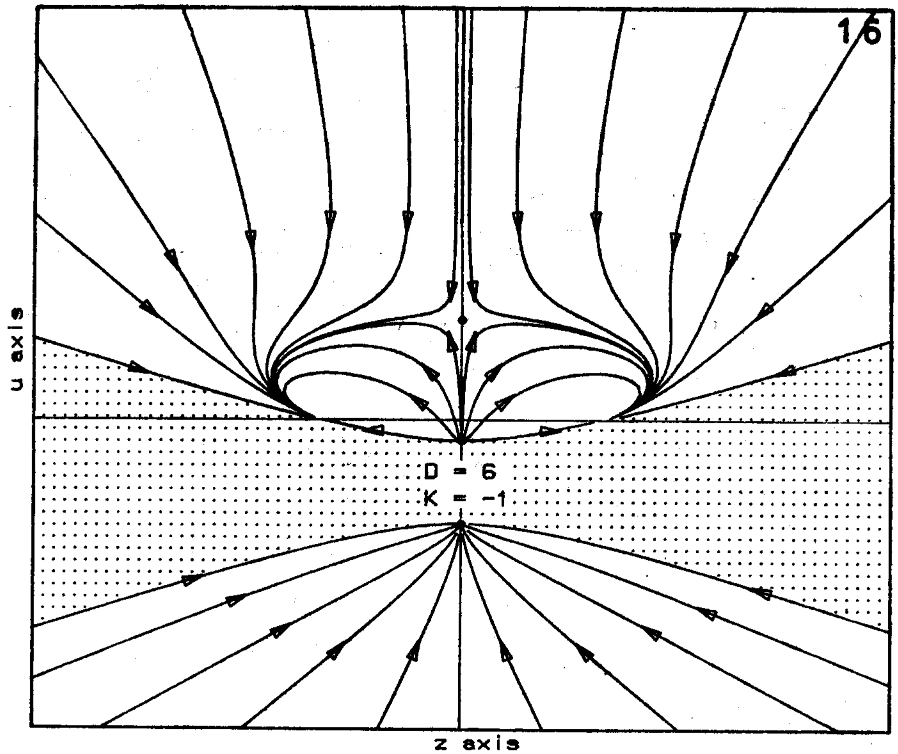


Fig. 1.4–1.6. Phase portraits for model FRW×T^D with dust matter in projective coordinates (z, u), for $D = 6$, and $K = 1, 0\ -1$

with the condition:

$$K\left(-\frac{D(D-1)}{2}-3Dw-3w^2\right)\geqslant 0, \tag{4}$$

where $d\tau_2 = h\,dt$.

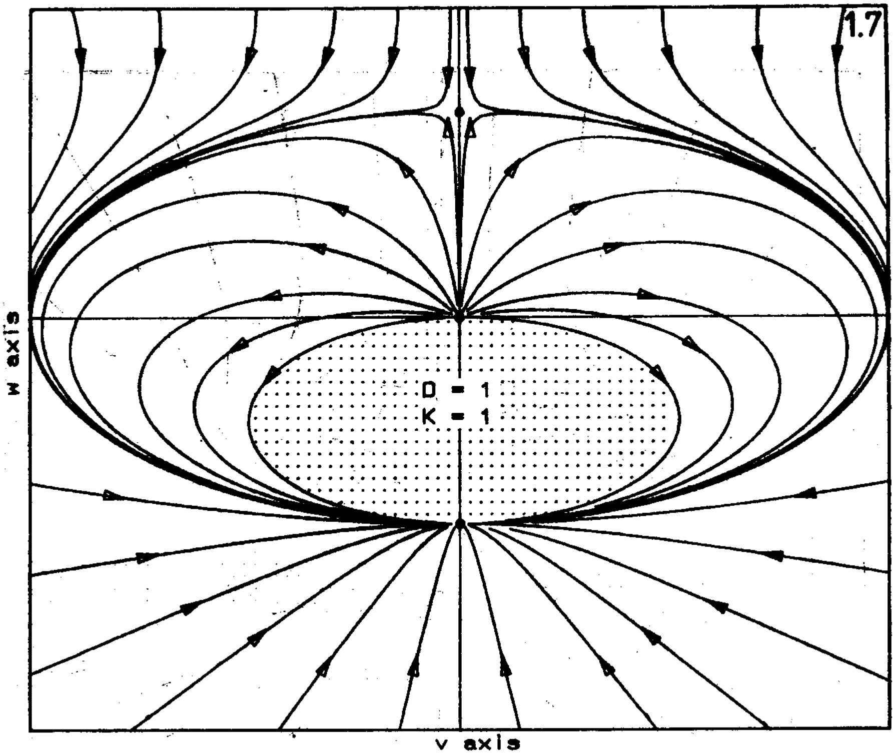
B. Model with dust matter

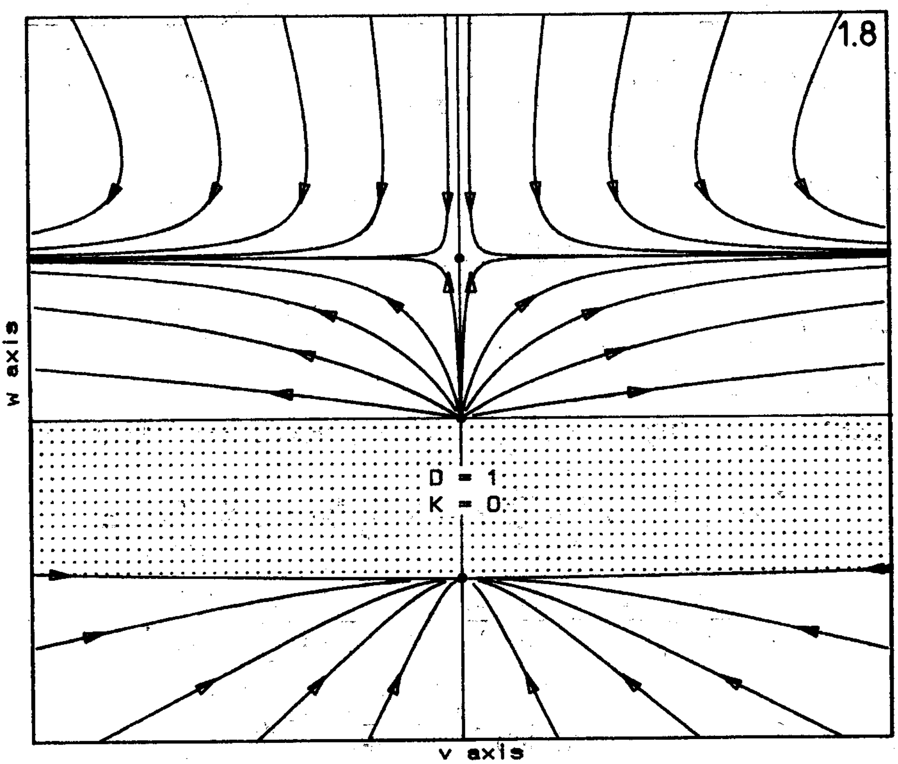
Dynamical system (19) (see Part I) in variables (z, u) assumes the form:

$$\frac{dz}{d\tau_1}=\frac{2D+1}{(D+2)}z+\frac{D(D-1)}{(D+2)}zu-\frac{D(D-1)}{2(D+2)}u^2z+\frac{(2D+1)}{D+2}Kz,$$

$$\frac{du}{d\tau_1}=\frac{3}{D+2}+\frac{3(D-1)}{D+2}u+\frac{D(D-7)}{2(D+2)}u^2+\frac{2K}{(D+2)}z$$

$$-\frac{D(D-1)}{2(D+2)}u^3+\frac{(2D+1)}{D+2}Kz^2u, \tag{5}$$


1.7
D = 1
K = 1
w axis
v axis


1.8
D = 1
K = 0
w axis
v axis

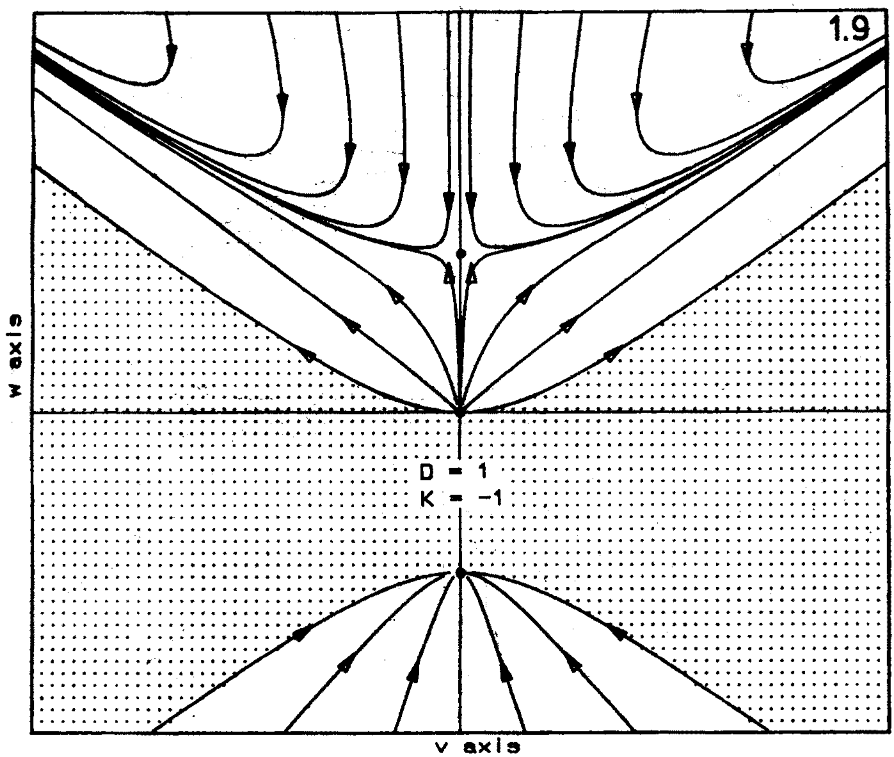


Fig. 1.7–1.9. Phase portraits for model FRW×T^D with dust matter in projective coordinates (v, w), for $D = 1$, and $K = 1, 0 \; -1$

in the domain

$$3+3Du+\frac{D(D-1)}{2}u^2+3Kz^2 \geqslant 0. \tag{6}$$

For coordinates (v, w) we have:

$$\frac{dv}{d\tau_2}=\frac{D(D+5)}{2(D+2)}v-\frac{D-4}{D+2}vw-\frac{3}{D+2}vw^2-\frac{3K}{D+2}v^3,$$

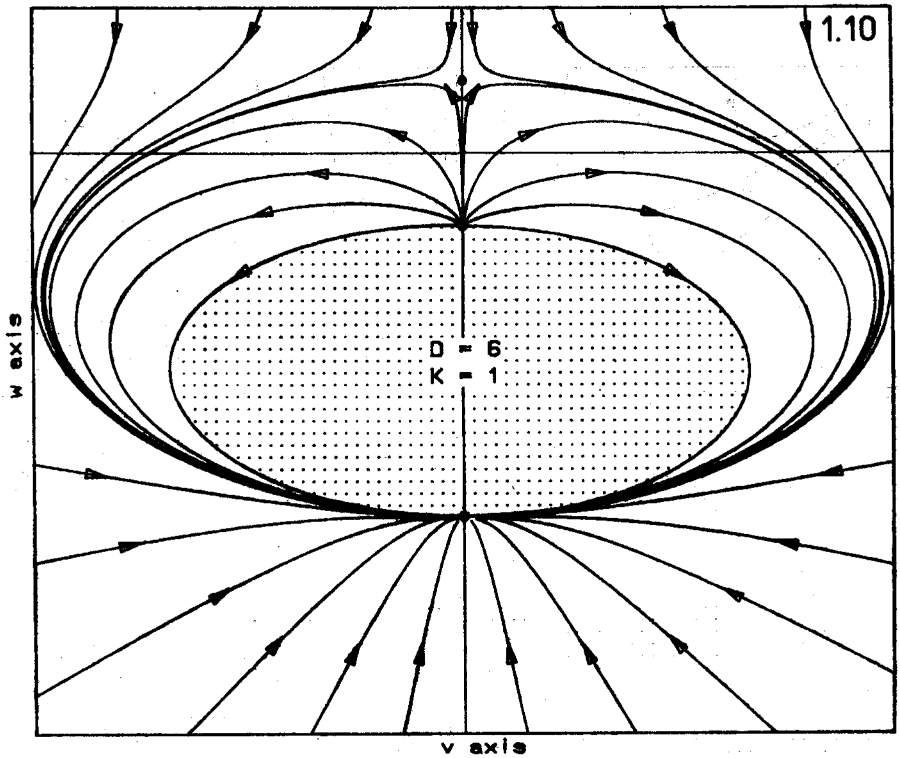
$$\frac{dw}{d\tau_2}=\frac{D(D-1)}{2(D+2)}-\frac{D(D-7)}{2(D+2)}w-\frac{3(D-1)}{D+2}w^2-\frac{3}{D+2}w^3$$

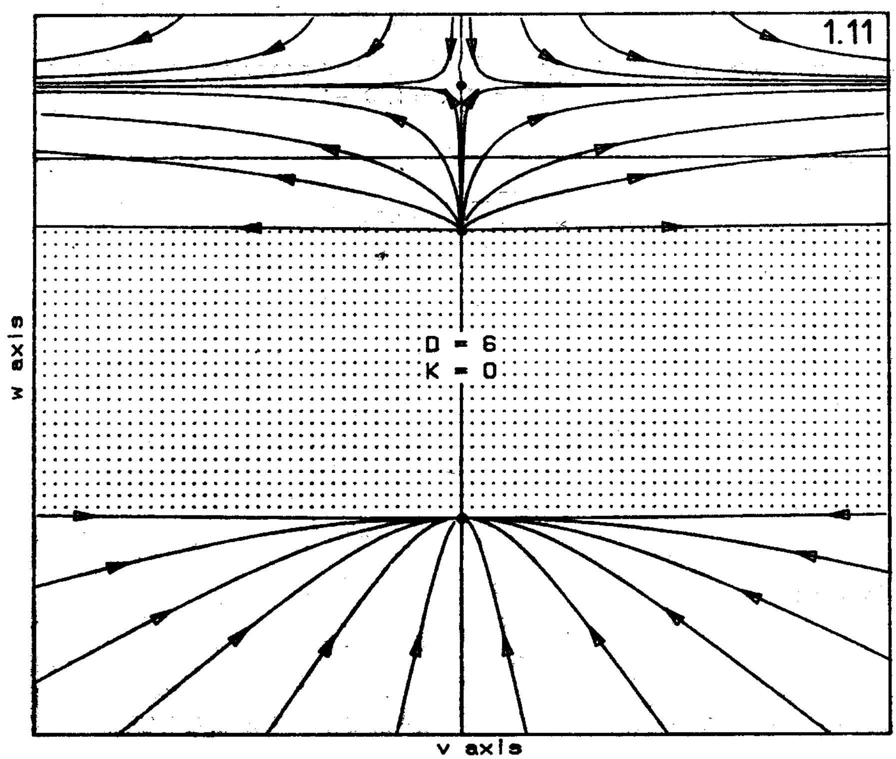
$$-\frac{2D+1}{D+2}Kv^2-\frac{3K}{D+2}wv^2, \tag{7}$$

in the domain:

$$3w^2+3Dw+\frac{D(D-1)}{2}+3Kv^2 \geqslant 0. \tag{8}$$

The phase portraits of systems (5) and (7) presented in Figs. 1.1–1.12.


1.10
D = 6
K = 1
w axis
v axis


1.11
D = 6
K = 0
w axis
v axis

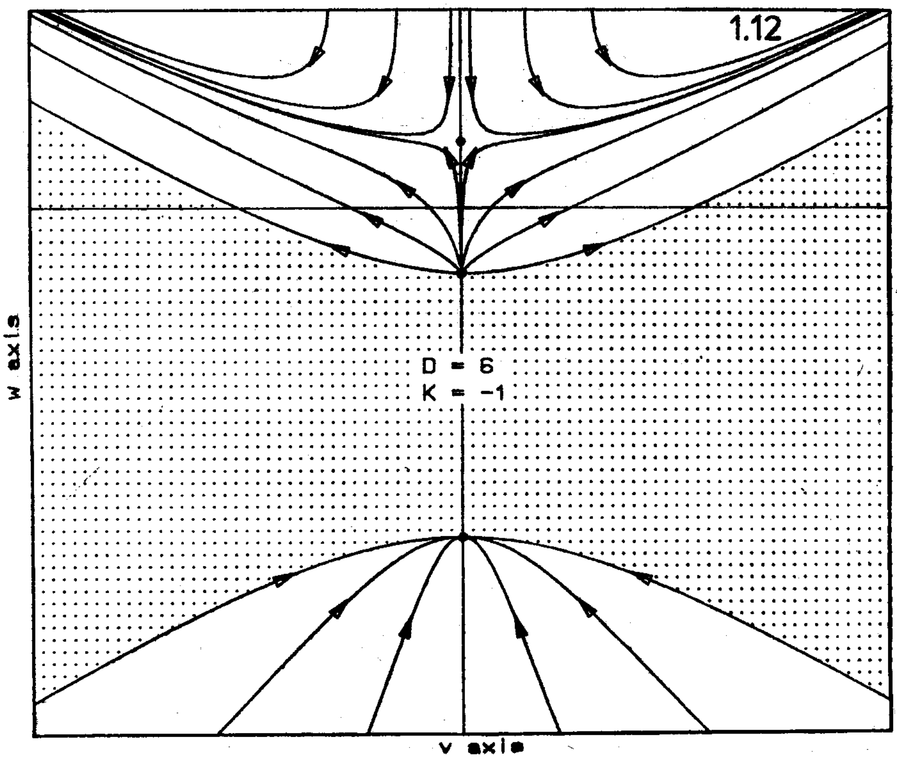


Fig. 1.10–1.12. Phase portraits for model FRW×T^D with dust matter in projective coordinates (v, w), for $D = 6$, and $K = 1,\ 0\ -1$

C. Model with a massless scalar field

After transformation to variables (z, u) and time τ_1: $d\tau_1 = x\, d\tau$, system (23) (see Part I) assumes the form:

$$\frac{dz}{d\tau_1} = 2z + Dzu + 2Kz^3, \qquad \frac{du}{d\tau_1} = -2Kz^2u, \tag{9}$$

with the condition:

$$3 + 3Du + \frac{D(D-1)}{2}\, u^2 + 3Kz^2 \geqslant 0, \tag{10}$$

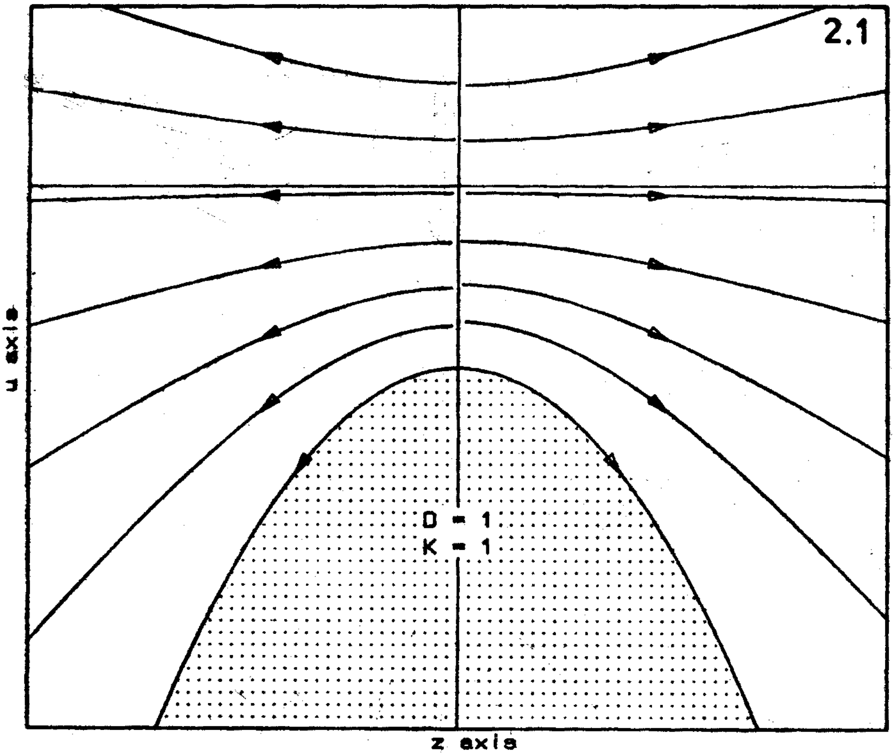
while for variables (v, w) and time τ_2; $d\tau_2 = y\, d\tau$ we obtain:

$$\frac{dv}{d\tau_2} = 2vw + Dv, \qquad \frac{dw}{d\tau_2} = -2Kv^2, \tag{11}$$

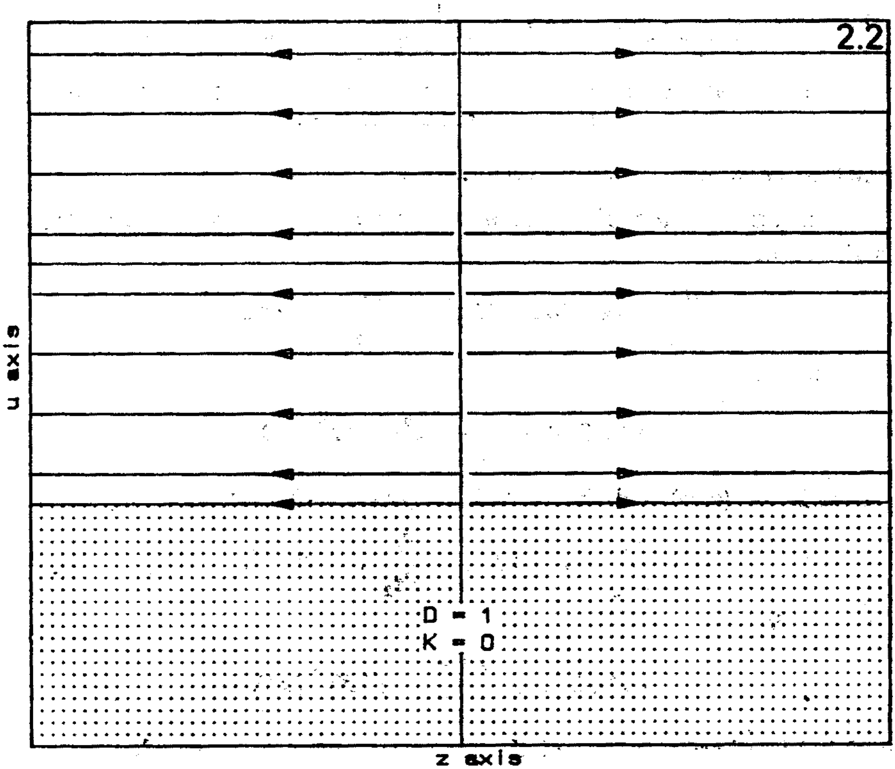
with the condition:

$$3w^2 + 3Dw + \frac{D(D-1)}{2} + 3Kv^2 \geqslant 0. \tag{12}$$

In Figs. 2.1–2.12 phase portraits of the above systems are presented.



2.1
D = 1
K = 1
u axis
z axis



2.2
D = 1
K = 0
u axis
z axis

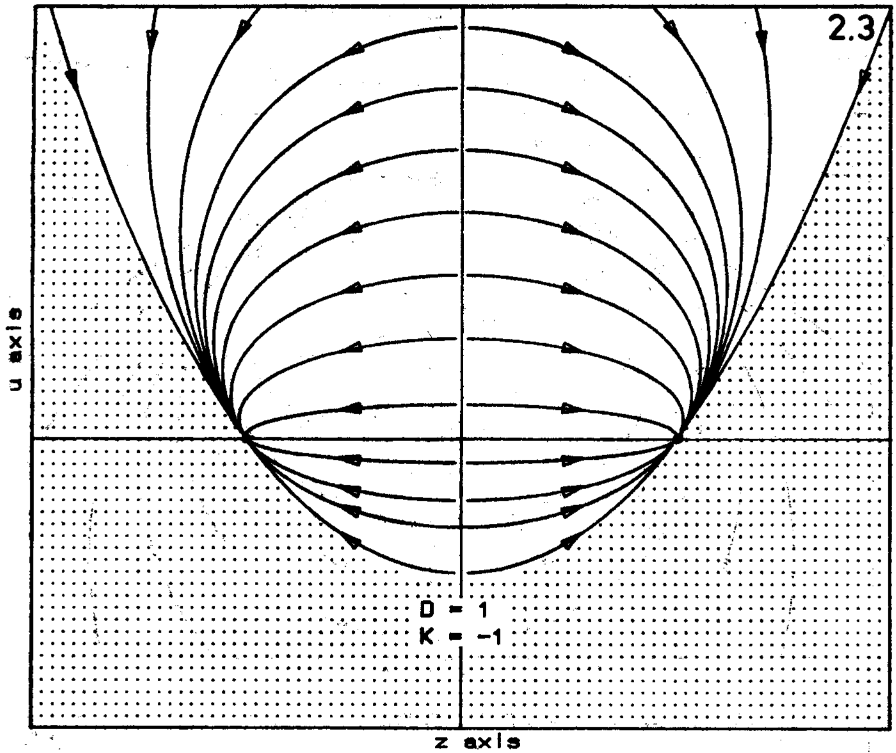


Fig. 2.1–2.3. Phase portraits for model FRW×T^D with massless scalar field in projective coordinates (z, u), for $D = 1$, and $K = 1, 0\ -1$

D. Model with radiative matter

Dynamical system (26) (see Part I) in projective variables (z, u) assumes the form:

$$\frac{dz}{d\tau_1} = \frac{3(D+2)}{D+3}z + \frac{D^2}{D+3}uz - \frac{D(D-1)}{2(D+3)}u^2 z + \frac{(2D+3)K}{D+3}z^3, \tag{13}$$

$$\frac{du}{d\tau_1} = \frac{3(D-1)}{D+3}u + \frac{3}{D+3} + \frac{D(D-7)}{2(D+3)}u - \frac{D(D-1)}{2(D+3)}u^3$$

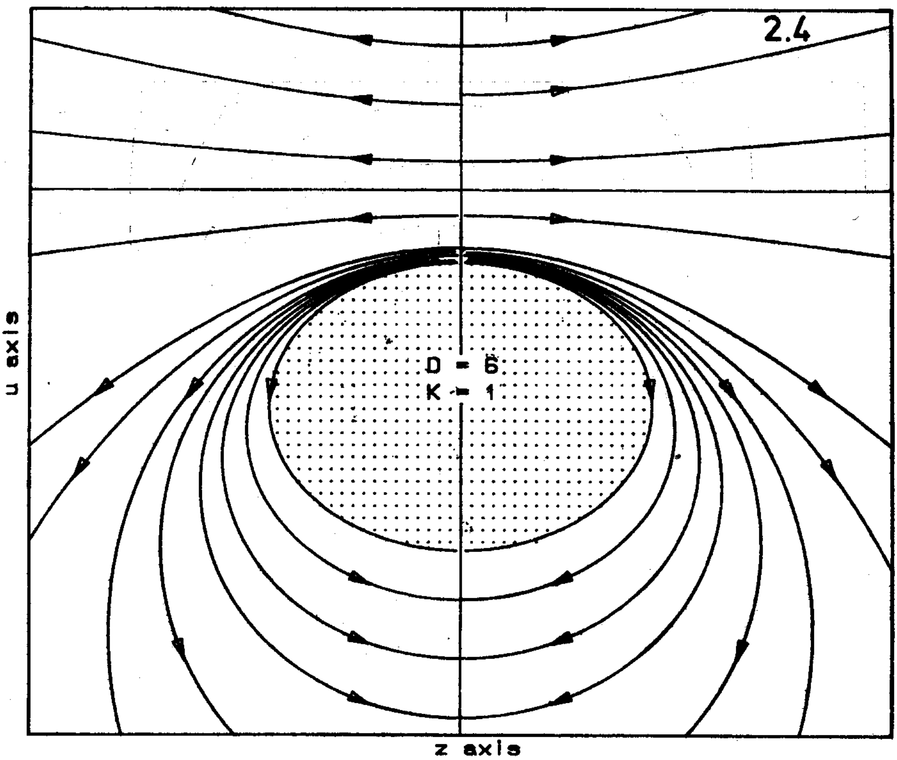
$$+\frac{3K}{D+3}z^2 + \frac{(2D+3)K}{D+3}z^2 u.$$

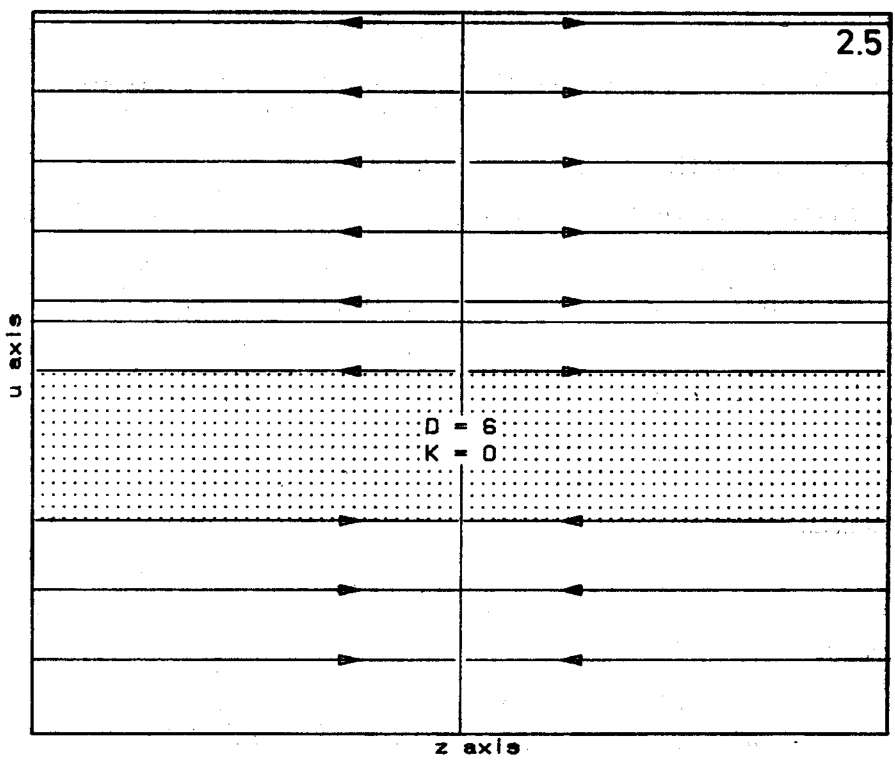
in the domain

$$3+3Du+\frac{D(D-1)}{2}u^2+3Kz^2 \geqslant 0. \tag{14}$$

For variables (v, w) we have:

$$\frac{dv}{d\tau} = \frac{D(D+7)}{2(D+3)}v + \frac{9}{D+3}vw - \frac{3}{D+3}vw^2 - \frac{3K}{D+3}v^3,$$


2.4
D = 6
K = 1
u axis
z axis


2.5
D = 6
K = 0
u axis
z axis

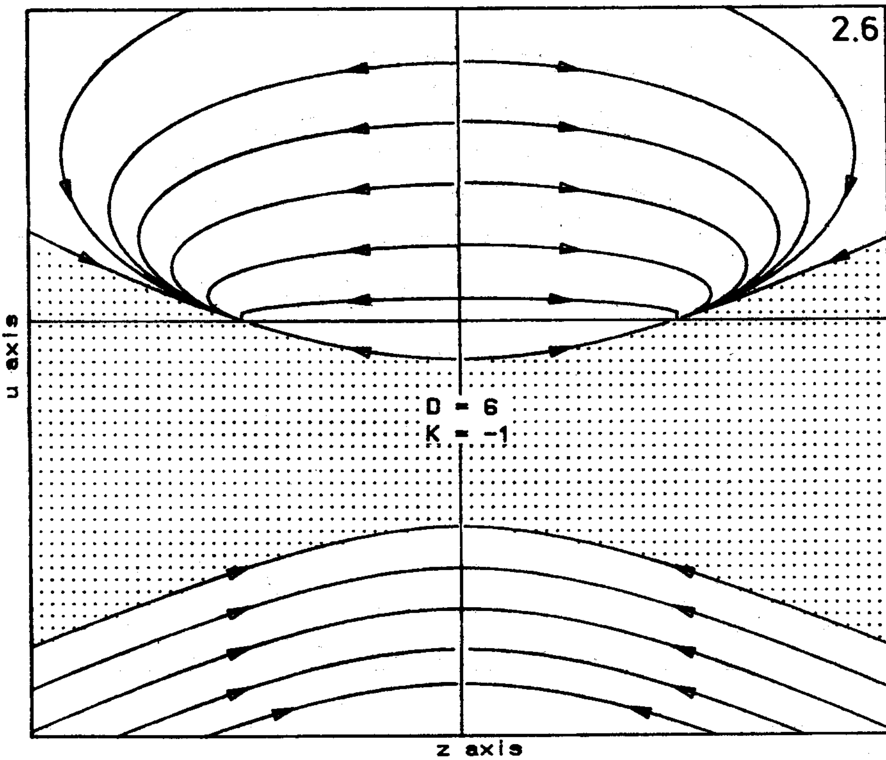


Fig. 2.4–2.6. Phase portraits for model FRW$\times$T^D with massless scalar field in projective coordinates (z, u), for $D = 6$, and $K = 1, 0 \; -1$

$$\frac{dw}{d\tau_1} = \frac{D(D-1)}{2(D+3)} - \frac{D(D-7)}{2(D+3)} w - \frac{3(D-1)}{D+3} w^2 - \frac{3}{D+3} w^3 - \frac{3K}{D+3} v^2 w - \frac{2(D+3)K}{D+3} v^2, \tag{15}$$

in the domain:

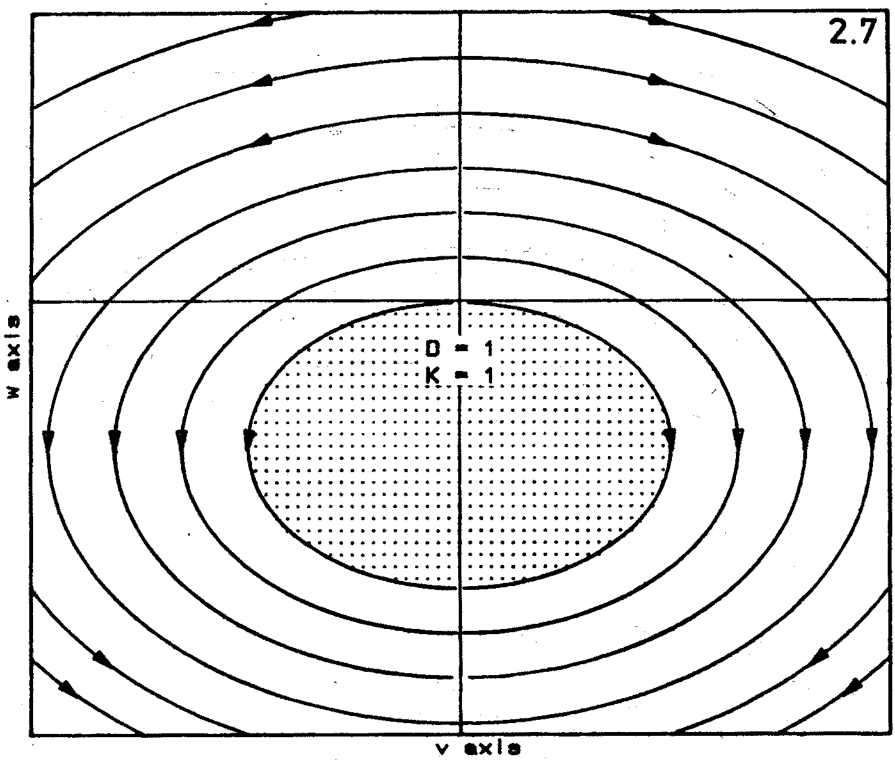
$$3w^2 + 3Dw + \frac{D(D-1)}{2} + 3Kv^2 \geqslant 0. \tag{16}$$

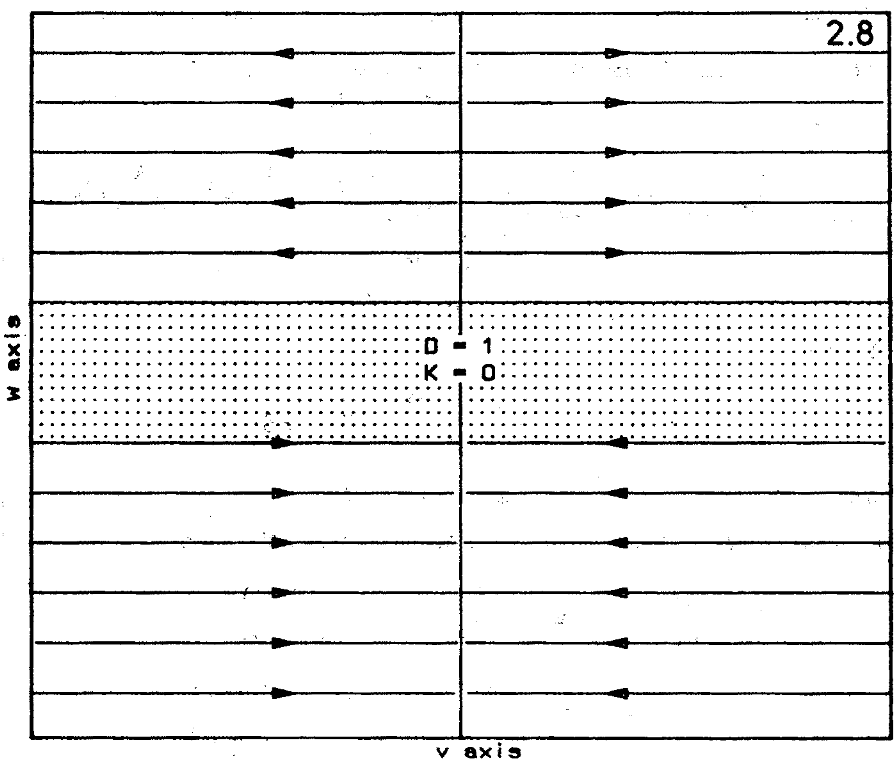
E. Model with the cosmological constant

For variables (z, u) system (28) (see Part I) assumes the form:

$$\frac{dz}{d\tau} = z - Duz - \frac{D(D-1)}{3} u^2 z + \frac{2(D-1)}{3(D+2)} \Lambda z^3, \tag{17}$$

$$\frac{du}{d\tau} = -2u - 2Du^2 + \frac{2\Lambda}{D+2} - \frac{D(D-1)}{3} u^3 + \frac{2(D-1)}{3(D+2)} \Lambda u z^2,$$


2.7
D = 1
K = 1
w axis
v axis


2.8
D = 1
K = 0
w axis
v axis

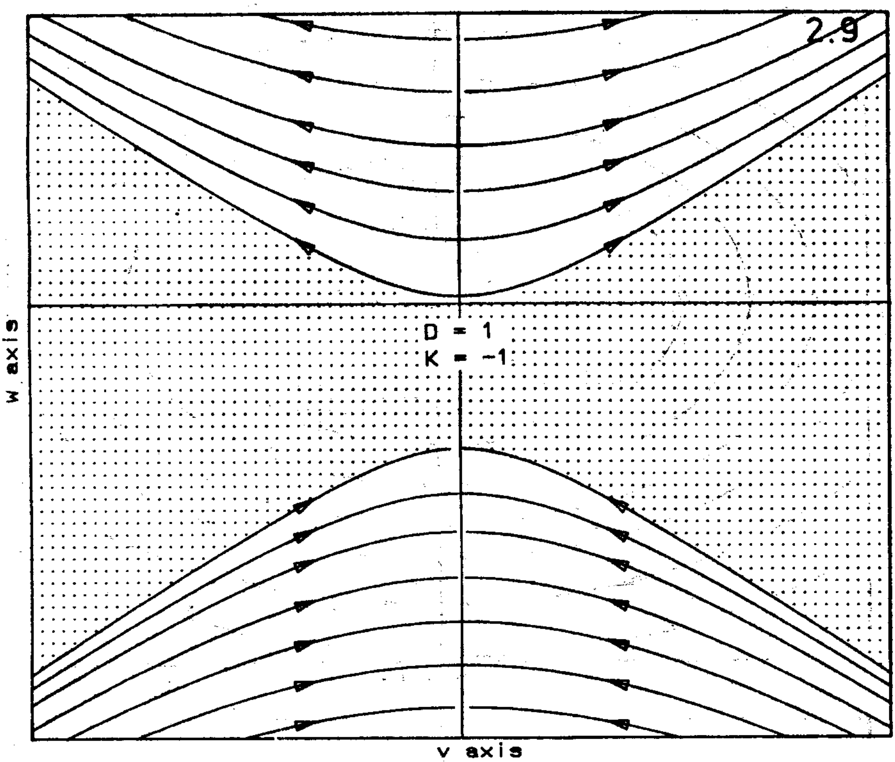


Fig. 2.7–2.9. Phase portraits for model FRW$\times T^D$ with massless scalar field in projective coordinates (v, w), for $D = 1$, and $K = 1, 0\ -1$

in the domain:

$$K\left(-3-3Du-\frac{D(D-1)}{2}u^2+\Lambda z^2\right) > 0. \tag{18}$$

In variables (v, w) the system is:

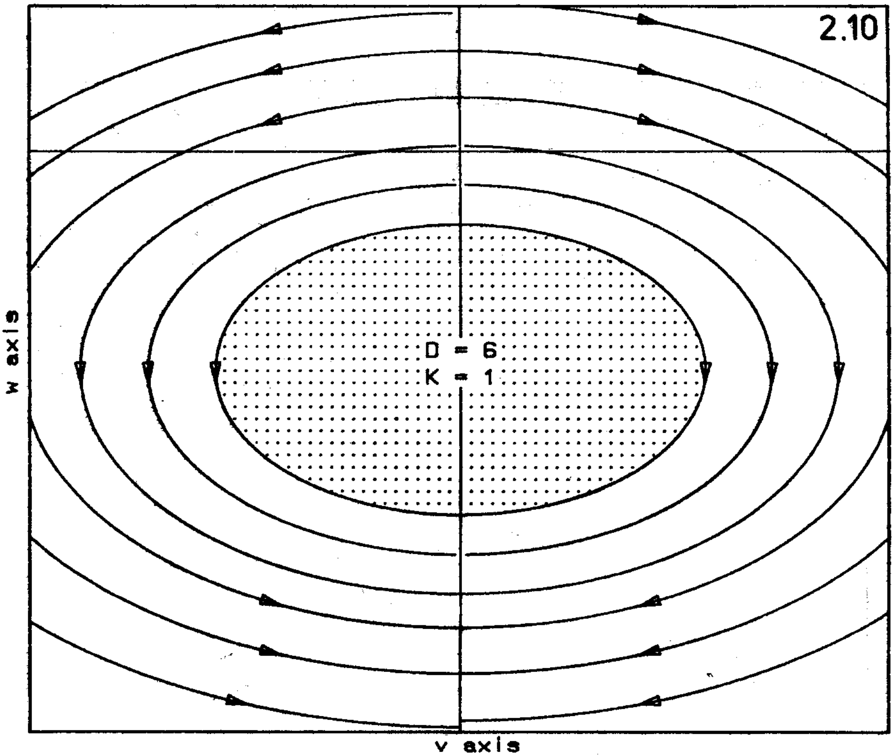
$$\frac{dv}{d\tau_1} = Dv+3vw+\frac{2\Lambda}{D+2}v^3,$$

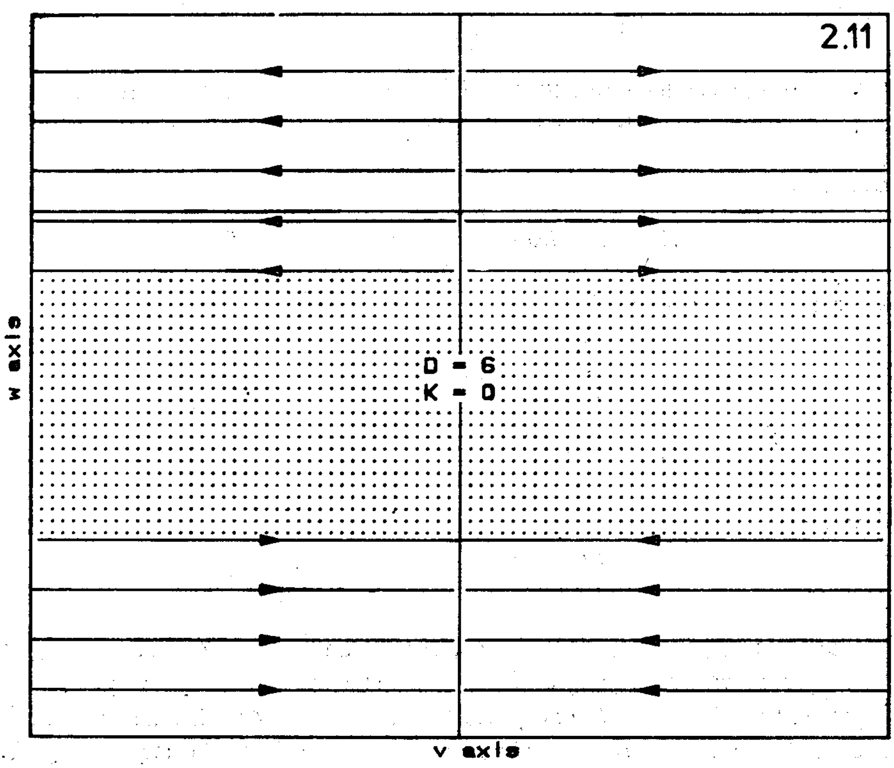
$$\frac{dw}{d\tau_1} = 2Dw+2w^2+\frac{D(D-1)}{3}-\frac{2(D-1)}{3(D+2)}\Lambda v^2-\frac{2\Lambda}{D+2}v^2 w, \tag{19}$$

in the domain:

$$K\left(-\frac{D(D-1)}{2}-3Dw-3w^2+\Lambda v^2\right) \geqslant 0. \tag{20}$$

As it can be seen, from the phase portraits obtained for the considered dynamical system in projective variables (z, u) and (v, w), the structure of the phase plane is in principle independent of dimension D, except for the case $D = 1$. Another dimension explicitly included in the portraits is $D = 6$, which is distinguished by the superstring theories.


2.10
D = 6
K = 1
w axis
v axis


2.11
D = 6
K = 0
w axis
v axis

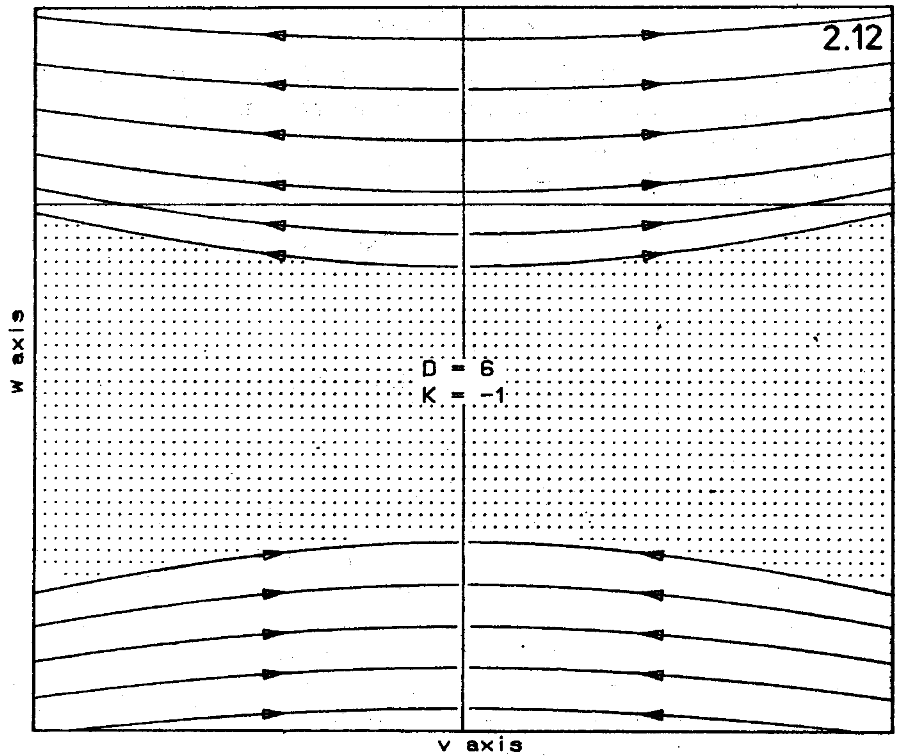


Fig. 2.10–2.12. Phase portraits for model FRW×T^D with massless scalar field in projective coordinates (v, w), for $D = 6$, and $K = 1$, 0 -1

Furthermore, one should notice that all typical critical points are situated at the boundaries of the corresponding physical consitions. (A critical is called — typical, for $t \to \infty$, if it is an attracting point. A solution, for $t \to 0$ ($\tau \to -\infty$), is typical if the corresponding critical point is repelling.) [1, 2].

Another property of typical critical points, which can be seen from the phase portraits, is that they lie at or below the straight line $u = 0$, $w = 0$.

Among five cases considered here, the phase portraits of the FRW×T^D model with a mssless scalar field ($p = p' = \varrho$) are peculiar in a sense. They are represented not by isolated critical points, but by infinitely many of them, constituting straight lines $z = 0$, $-\infty < u < \infty$ and $v = 0$, $-\infty < w < \infty$ ("singular straight lines"). Points situated on and above the non-physical region are unstable repelling points, whereas those situated below the non-physical region are stable attracting points. It should be also noticed that the straight line $z = 0$ corresponds to the singularity of the physical space ($H \to \infty$), and the striaght line $v = 0$ — to the singularity of the internal space ($h \to \infty$).

For a model with negative curvature ($K = -1$) in variables z, u, besides the singular straight line there are also two attracting knots $z_0 = \pm\sqrt{-\frac{1}{K}}$, $u_0 = 0$, which correspond to stable solutions with a static microspace ($y = 0$).

2. *Conclusions*

In this paper, the dynamics of the complete class of multidimensional cosmological models with the topology $FRW \times T^D$ and a source in the form of hydrodynamic energy--momentum tensor was investigated.

The phases portraits have been (as in the Part I), made with the help of the computer program "Dynamic".

As in the Part I of the present paper, we shall not discuss particular cases, but rather point to some general conclusions.

1) For models with the closed internal space and $D > 1$, the typical state of the metric, for large times is $R \to \infty$, $r \to 0$, and $t \to t_0$. In other words, after a finite period of cosmological time the model attains a singular state, in which the size of the physical space is much greater than that in the internal space. Rosenbaum, Ryan, Urrutia and Matzner in their paper of 1987 [3], showed that in the model with dust matter there are singularities of the kind which they called "crack-of-doom" singularities. The authors considered the case of one-dimensional internal space ($D = 1$) and stated that the dust models can reach a state in which, after a finite time t_0, the internal space attains the zero size, while the physical space exhibits a regular behaviour (a singularity appears in the total space, but it is "hidden" from an observer in the physical space). Our considerations show that such states can actually occur, but they imply the existence of one additional dimension and are independent of the form of matter (for the dust case $D = 1$, dynamical equations are integrable; this was probabaly a reason that such singularities had been noticed at all). The models for $D = 1$ are very peculiar ones, since $D = 1$ is a bifurcation parameter and the phase portraits for $D > 1$ have a quite different topological structure. To sum up, typical states of the metric are those in which $R \to \infty$, $r \to 0$ and $t \to t_0$ ($D > 1$); therefore, the crack-of-doom singularities may appear only after the physical space has grown in size to infinity. It should be noticed that in multidimensional cosmological theories the closed character of physical space does not imply that it is of a finite size ($R < R_{max}$, as it is in the case of classical $(1+3)$-dimensional closed models).

2) Let us consider models with a source in the form of hydrodynamic energy-momentum tensor. It can be shown [3] that quantum effects due to a massless scalar field in an external gravitational field at high temperatures, give rise to the hydrodynamic energy--momentum tensor of radiative matter $\left(p = \frac{\varrho}{D+3}\right)$. The effects of classical massless scalar fields correspond to the case $p = \varrho$ discussed in the present paper.

APPENDIX

General remarks useful in determining possible evolution patterns from phase portraits

1. Phase trajectories are scaled with different time parameters: Time parameter τ $\left(d\tau = \frac{dt}{R}, \; d\tau_1 = x\, d\tau\right)$ is a strictly monotonic function of time t $\left(\frac{d\tau}{dt} = R > 0\right)$. The

new time parameters τ_1, τ_2 are strictly monotonic functions of τ, for $x > 0$ and $y > 0$, respectively, and of $-\tau$, for $x < 0$ and $y < 0$, respectively. Accordingly, the attracting critical points occuring in the region $x < 0$ and $y < 0$ are repelling critical points in time τ or t.

2. In asymptotic states, typical states of the metric are always situated at boundaries of the relevant constraint conditions and are represented by Kasner asymptotics:

$$3p_1+Dp_2 = 3p_1^2+Dp_2^2 = 1, \quad p_{1\pm} = \frac{3\pm\sqrt{3D(D+2)}}{3(D+3)},$$

$$p_{2\pm} = \frac{D\pm\sqrt{3D(D+2)}}{3(D+3)}.$$

The asymptotics of corresponding solutions, in time t, have the form:
near the initial singularity

$$R \propto t^{p_{1+}}, \quad r \propto t^{p_{2-}}, \quad R \propto t^{p_{1-}}, \quad r \propto r^{p_{2+}}, \quad t \to 0. \tag{A1}$$

$$R \propto t^{p_{1-}}, \quad r \propto t^{p_{2+}}, \quad t \to 0. \tag{A2}$$

and near the final singularity

$$R \propto (t_0-t)^{p_{1+}}, \quad r \propto (t_0-t)^{p_{2-}}, \tag{B1}$$

$$R \propto (t_0-t)^{p_{1-}}, \quad r \propto (t_0-t)^{p_{2+}}, \quad t \to t_0. \tag{B2}$$

3. Asymptotic states, in which the macrospace and the microspace are comparable in size, are represented by non-stable saddle points. For instance, asymptotic solution for radiations are of the form:

$$R \propto r \propto (t)^{\frac{2}{D+4}}, \quad \text{near the initial singularity,}$$

$$R \propto r \propto (t_0-t)^{\frac{2}{D+4}}, \quad \text{near the final singularity.}$$

4. Singularities of the crack-of-doom type are always connected with the presence of the critical point $v_0 = 0$, $w_0 = 0$, i.e. a singularity occurs in the internal space, and thus in the total space, for arbitrary values x. The corresponding asymptotic solutions near singularities of this type are of the form (B1) and (B2). If $t \to t_0$, $R \to$ const for $D = 1$, and $R \to \infty$ for $D > 1$, while $r \propto (t_0-t)$ for $D = 1$, and $r \propto (t_0-t)^{p_{2+}}$ for $D > 1$ ($p_{2+} < 0$). The asymptotic B2 is valid for a space-time with an expanding physical space and a contracting internal space, and the asymptotic B1 for symmetrical solutions with a contracting physical space and an expanding internal space. Let us notice that the initial singularity in the FRW$(K = -1)\times T^1$ model has the structure of a crack-of-doom singularity. Because of a dynamics of additional dimensions, although the physical space has no initial singularity, the singularity in the total space results from that of the internal space (which expands from a singularity to a constant size). From the phase portraits

it can be seen that the final singularities of the crack-of-doom type are typical features of models with closed physical space; this models exhibit the following behaviour $R \to \text{const}$ for $D = 1$, and $R \to \infty$ for $D > 1$, when $t \to t_0$. For models with flat physical space, this state is attained for $t \to \infty$.

5. Let us notice that the phase trajectories in the system (z, u) and (v, w) are symmetrical with respect to the axes $z = 0$ and $v = 0$, respectively. In the region $z < 0$ and $v < 0$, times τ_1 and τ_2 are monotonic functions of $-t$. From the phase portraits it can be seen that the crack-of-doom singularities are stable final singularities (in time t) within the class of solutions with contracting internal spaces, and stable initial singularities within the class of solutions with expanding internal spaces.

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