SOME RESULTS ON THE ONE-LOOP EFFECTIVE ACTION IN CHIRAL BACKGROUND FIELDS*

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Some results on the effective action of a Dirac field in the background of a chiral mass term are presented. They concern the convergence of the gradient expansion and a new numerical method for evaluating the zero point energy in a nonperturbative way.

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1. Introduction

The effective action of a quantum field in the background of a classical field configuration plays an improtant rôle in the quantization of classical solutions, leading to an energy correction or in providing an effective potential that contributes to the dynamics on the same level as the classical potential. Among the recent applications the quantum fluctuations of fermions in a background field, given by a chiral mass term of the hedgehog type,

$$m(\vec{x}) = m_\infty \exp\left(i\gamma^5 \vec{\tau}\vec{\varphi}(\vec{x})\right) = m_\infty \left(\phi^0 + i\gamma_5 \tau^a \phi^a\right) \tag{1.1}$$

has attracted interest in the context of the chiral quark model [1] or Nambu–Jona-Lasinio type models [2]. We will discuss here this application mainly. We will present some analytic results [3,4], concerning the validity (convergence) of the gradient expansion and a new numerical, nonperturbative method for evaluating the full renormalized zero point energy [5,6] in a Lorentz covariant way. It has been applied previously to the case of a scalar field [5] and a spin $\frac{1}{2}$ field [6] with a scalar, r dependent mass term. We will present here results for the chiral mass term (1.1).

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In Section 2 we will define the action and comment on its perturbative expansions and especially on the convergence of the gradient expansion. In Section 3 we will describe the numerical evaluation. The results will be presented and discussed in Section 4. Concluding remarks are given in Section 5.

2. The effective action and its perturbative expansion

The (one loop) effective action for the fermion field is defined as

$$S_{\text{eff}} = -i \ln \det \left\{ (i\gamma\partial - m(\vec{x})) / (i\gamma\partial - m_\infty) \right\} , \tag{2.1}$$

where we have divided by the free effective action. $m(\vec{x})$ is the chiral mass term given in (1.1) and m_∞ its limit as $r \to \infty$. There are many ways to give this formal expression a concrete meaning; the most immediate one is to diagonalize the generalized Dirac operator and to do the eigenvalue sum (or rather integrals), a procedure which implies some regularization, and there are many of them (see *e.g.* [7]). We will not be very much concerned with this question since we will consider the renormalized action, which finally should not depend on the specific regularization chosen. This implies that we choose a way of evaluating the effective action that is close to the usual perturbation expansion. We will do so even when applying below a numerical method that gives the full nonperturbative result. In the following we assume that all the formal manipulations are performed on that part of the effective action that is obtained by omitting the divergent diagrams in the expansion with respect to $\Delta m(\vec{x}) = m(\vec{x}) - m_\infty$ ("Δm-expansion"):

$$S_{\text{eff}} = i \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr} \left\{ \left(S_{\text{E}}^{(0)} \cdot \Delta m \right)^n \right\} , \tag{2.2}$$

where $S_{\text{E}}^{(0)}$ denotes the free Euclidean Dirac propagator. For a static background field the effective action is related to the zero point energy by $S_{\text{eff}} = -\tau E_0$ where τ denotes the (Euclidean) time. One can then relate E_0 to the time Fourier transform of the full Dirac propagator in the external field $S_{\text{E}}(\vec{x}, \vec{x}', \nu)$ as

$$E_0 = -i \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \nu \int d^3x \operatorname{Tr} \left\{ S_{\text{E}}(\vec{x}, \vec{x}', \nu) - S_{\text{E}}^{(0)}(\vec{x}, \vec{x}', \nu) \right\} . \tag{2.3}$$

$S_{\text{E}}(\vec{x}, \vec{x}', \nu)$ satisfies

$$(i\nu - H)S_{\mathrm{E}}(\vec{x},\vec{x}',\nu) = -\delta^3(\vec{x}-\vec{x}')\,, \tag{2.4}$$

with

$$H = -i\vec{\alpha}\,\vec{\nabla} + \gamma^0 m(\vec{x})\,. \tag{2.5}$$

Introducing

$$H_0^2 = -\Delta + m_\infty^2 \tag{2.6}$$

and

$$V(\vec{x}) = i\vec{\gamma}\,\vec{\nabla} m(\vec{x})\,, \tag{2.7}$$

one can expand S_{E} as [8]

$$S_{\mathrm{E}} = \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu^2+H_0^2}V\right)^k \frac{1}{\nu^2+H_0^2}(i\nu+H)\,, \tag{2.8}$$

the so-called gradient expansion. The gradient expansion has been used in the present context to derive Skyrme type terms from the one loop effective action [9]. We think, however, basing on the results of Ref. [3], that it is not helpful for the discussion of stability. The gradient expansion allows power counting arguments [3,9] for large and small R, where R is the scale of the hedgehog characterized by writing the chiral angle as $\Theta(r) = \Theta_0(r/R)$. The contributions vanish for N odd. One finds that the term of order $N = 2n$ behaves for small R as R^{2n-1} for $n > 1$ and as $R\log R$ for $n = 1$. Logs appear also for $n > 1$, but multiplied with higher powers of R. For large R the leading behaviour of the $2n$-th order term is R^{-2n+3}; the resulting expansion with respect to powers of $1/R$ is sometimes also called gradient expansion. Which of these power counting results is relevant depends obviously on the convergence properties of the series (2.8) and one can indeed derive such properties [3], since the expansion is of the form of a resolvent expansion. Obviously (for a more subtle treatment see [3]) one has essentially to estimate the Hilbert Schmidt norm of the operator $(\nu^2+H_0^2)^{-1}V$. One finds, after taking the scale parameter out of the integral,

$$\|\,(\nu^2+H_0^2)^{-1}V\|^2 = \frac{4Rm_\infty^2}{k}\int_0^\infty dx\left[(x\Theta_0'(x))^2 + 2\sin^2\Theta_0(x)\right]\,. \tag{2.9}$$

This implies that the gradient expansion for a hedgehog is convergent for small R in $3+1$ dimensions. This contradicts the intuition that the expansion should be good for large R since then "the gradient becomes small". If

the chiral angle has to change from π at $r = 0$ to zero at $r = \infty$, a small gradient implies a large extension of the hedgehog and this leads to a factor R^3 in the norm, while the gradient lowers this by a factor R^{-2} only (the situation is of course different in $1 + 1$ dimensions). This convergence at small R does not imply **necessarily** that the large R expansion with respect to $1/R$ makes no sense. We just do not see how one could prove that it does. However, in $1 + 1$ dimensions where the rôle of large and small R is reversed the conclusions drawn from the small R expansion are certainly wrong. Also, in $3 + 1$ dimensions a bound state appears that persists for large R and the behaviour of the effective action for large R must necessarily depend on how it is taken into account (see the end of Section 4).

An immediate consequence of the convergence at small R is that the leading $R \log R$ behaviour will not be compensated by higher order terms and the vacuum instability [10] implied by it will persist. Using the same expansion, but another trace one can also [8] derive an expression for the fermion number induced by the background field. It turns out [4] that the induced baryon number vanishes at $R = 0$ at least as R^4. If it is constant, as one may derive generally, it must be zero throughout, unless one changes the physical state at the value of R where the bound state crosses $E = 0$. For large R the lowest (third) order result goes to the topological invariant

$$N^{\text{top}} = \frac{1}{12\pi^2} \int d^3 \, \epsilon_{abcd} \epsilon_{ijk} \phi^a \nabla_i \phi^b \nabla_j \phi^c \nabla_k \phi^d \,, \tag{2.10}$$

which is 1 for the standard hedgehog. However, the gradient expansion does not converge there, so the association of the topological number with the induced baryon number [11] cannot be based on the gradient expansion [4], although this is usually done. There is a relation between topology and the bound state, though: It is indeed easy to see that for $R \to \infty$ the Dirac equation for the $\mathrm{K}^{\mathrm{P}} = 0^+$ can be solved explicitly and that $E(0^+) = m_\infty \cos(\Theta(0))$ in this limit. However, it is hard to see how third order perturbation theory could know this.

While these results give some insight into the behaviour of the zero point energy near $R = 0$, the higher terms in the gradient expansion are hardly accessible to analytic evaluation and if one wants to learn about the behaviour of $E_0(R)$ at finite $R \approx O(1/m_\infty)$ one will have to have recourse to a numerical calculation. This will be discussed in the next Section.

3. Numerical evaluation of the zero point energy

Various approaches have been taken to find the numerical value of the zero point energy at finite values of R. The most immediate one is to find the eigenvalue spectrum of Dirac operator. This has been done in Ref. [12],

making the spectrum furthermore discrete by introducing a space boundary. This can be avoided by using the scattering phase shift, as it was done for the Sine Gordon model in the seventies [13] and recently also [14] for the chiral quark model. These calculations imply noncovariant intermediate cutoffs, and even if one renormalizes at the end this is a dangerous method [14]. Since the background field breaks Lorentz invariance, it is important to be on safe ground with Lorentz covariance. In the case of the self energy of quarks in a bag [15] — which is even finite for $m_{\text{quark}} = 0$ — only those results were correct and consistent with each other where the leading order graphs had been taken out before the numerical evaluation. Level summation techniques failed to give correct results. The situation may be less critical here and in cutoff theories one may not even care about Lorentz covariance but it is certainly useful to have a method [13,14] that respects it.

In order to avoid inessential complexity we will consider a very simple case, that of a scalar field with an r dependent mass term. The calculation in the chiral quark model can be reduced to this problem except for having a field with several components. In analogy with (2.3) we can express the zero point energy as

$$E_0 = -\int \frac{d\nu}{2\pi}\nu^2 \int \left(G(\vec{x},\vec{x}',\nu) - G^0(\vec{x},\vec{x}',\nu)\right) d^3x\,, \tag{3.1}$$

where $G(\vec{x},\vec{x}',\nu)$ satisfies

$$\left(-\Delta + \nu^2 + m_\infty^2 + V(\vec{x})\right) G(\vec{x},\vec{x}',\nu) = \delta^3(\vec{x}-\vec{x}')\,, \tag{3.2}$$

where the potential V is equal to $m^2(\vec{x}) - m_\infty^2$ and will be assumed to depend on $r = |\vec{x}|$ only. According to what we have discussed above we should take out the orders (0) to (2) and do them separately, using the covariant expressions derived from expanding the effective action, renormalize them and add the finite parts back. We then do not need cutoffs in the numerical calculation and the order in V is a Lorentz invariant concept. It is convenient to introduce the notation $\overline{(n)}$ for an expression that corresponds to the exact summation of the expansion in V from order n to ∞. In this sense we have now to evaluate $E_0^{\overline{(3)}}$ and therefore $G^{\overline{(3)}}$.

Using the spherical symmetry we expand

$$G(\vec{x},\vec{x}',\nu) = \sum (2l+1)\, g_l(r,r',\nu)\,, \tag{3.3}$$

where $g_l(r,r',\nu)$ is the Green function associated with the differential equation for the Euclidean partial waves

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - V(r)\right) f_l(r) = \kappa^2 f_l(r)\,. \tag{3.4}$$

The free ($m = m_\infty$) solutions of this equation are the modified spherical Bessel functions $i_l(z)$ and $k_l(z)$ where $z = \kappa r$ and $\kappa^2 = \nu^2 + m_\infty^2$. Let $f_l^+(r) = k_l(z)\,(1 + h_l^+(r))$ and $f_l^-(r) = i_l(z)\,(1 + h_l^-(r))$ be the solutions regular at $r = \infty$ and $r = 0$ respectively and let $h_l^\pm$ satisfy the boundary condition $h_l^\pm(r) \to 0$ as $r \to \infty$. Then

$$g_l(r, r', \nu) = g_l^{(0)}(r, r', \nu)\,(1 + h_l^+(r_>))\,(1 + h_l^-(r_<))\,, \tag{3.5}$$

where $g_l^{(0)}(r, r', \nu) = -\kappa\, i_l(z_<)\, k_l(z_>)$ is the free Green function. The $h_l^\pm$ can be evaluated by numerical integration of the differential equation obtained by substituting the $f_l^\pm$ in (3.3):

$$\left\{\frac{d^2}{dr^2} + 2\left(\frac{1}{r} + \kappa\frac{b_l^{\pm\prime}(r)}{b_l^\pm(r)}\right)\frac{d}{dr}\right\} h_l^\pm(r) = V(r)\,(1 + h_l^\pm(r))\,, \tag{3.6}$$

where we have introduced $b_l^+ = k_l$ and $b_l^- = i_l$. The solutions $h_l^\pm$ vary only in the region where the potential changes and become constant for small r (and vanish for large r). They are obviously of first and higher order in the potential, *i.e.* of order (1).

After we know how to evaluate the full Green function g_l we have to find a way to remove from it the leading orders in V. This implies that we have to find a way to remove the leading orders from the $h_l^\pm$. This can be done by applying the integral equation associated with the differential equation (3.6) and the boundary conditions. We have

$$h_l^\pm(r) = \int_0^\infty dr'\, r'^2 K_l^\pm(r, r', \nu)\, V(r')\,(1 + h_l^\pm(r'))\,. \tag{3.7}$$

Here $K_l^\pm(r, r', \nu)$ are kernels corresponding to the boundary conditions of $h_l^\pm$. For their explicit forms see [14]. The right hand side can be expanded with respect to V. We have in symbolic notation

$$h_l^\pm(r) = \sum_{n=1}^\infty h_l^{\pm(n)}(r)\,; \quad h_l^{\pm(n+1)} = \int K_l^\pm\, V\, h_l^{\pm(n)}\,; \quad h_l^{\pm(0)} \equiv 1\,. \tag{3.8}$$

Using these relations we can calculate $h_l^{\pm(n)}$ for $n = 1$ and 2 and subtract it so as to retain only the order $\overline{(3)}$ part of g_l. This means, however, that we have to remove almost all of the $h_l^\pm$ and of g_l by subtraction since the divergent parts dominate of course the integrands. This is numerically hazardeous. We note that we can use the integral equation in a better way,

using the fact that the numerical solutions of the differential equation $h_l^{\pm}$ are exact and of order $\overline{(l)}$. Therefore we may obtain $h_l^{\pm(n)}$ as (in symbolic notation)

$$h_l^{\pm\overline{(n+1)}} = \int K_l^{\pm} V h_l^{\pm(n)}; \qquad h_l^{\pm\overline{(1)}} = h_l^{\pm}, \tag{3.9}$$

instead of obtaining it by subtraction of the first $(n-1)$ orders from $h_l^{\pm}$. If calculated as a difference $h_l^{\pm\overline{(n)}}$ will have the same absolute error as $h_l^{\pm}$, while it will have only the same relative one if calculated via Eq. (3.9). The reduction of the amplitudes is of the order 10^{-5} at some values of r and l if the two leading orders are subtracted, so the gain in numerical accuracy is important and vital for a reliable calculation. Once we can calculate $h_l^{\pm(n)}$ and $h_l^{\pm\overline{(n)}}$ we can express $g_l^{\overline{(3)}}$ as

$$g_l^{\overline{(3)}} = g_l^{(0)} \left\{ h_l^{+\overline{(3)}} + h_l^{-\overline{(3)}} + h_l^{+(1)} h_l^{-\overline{(2)}} + h_l^{-(1)} h_l^{+\overline{(2)}} + h_l^{+\overline{(2)}} h_l^{-\overline{(2)}} \right\}, \tag{3.10}$$

which contains only terms that are genuinely of at least third order. It remains to perform the integration over r, the summation over l and the integration over ν. For $R = 0$ (m_∞^{-1}) summations and integrations converge well. If R is increased beyond $5\, m_\infty^{-1}$ one has to go to values of l of the order of 20; then the typical factors r^{2l+1} occuring in the integral kernels lead to strongly varying integrands and either the CPU time increases strongly or the accuracy becomes poorer.

4. Numerical results

We present in Fig. 1 the results of a calculation of the zero point energy of fermions in the chiral model. R and E_0 are given in units of m_∞^{-1} and m_∞ respectively. For $\Theta_0\,(r/R)$ we have chosen the function

$$\Theta_0(x) = \pi\,(1 - \tanh x)\,. \tag{4.1}$$

the solid line shows the result for the finite part $E_0^{\overline{(3)}}$; the dash-dotted line is the finite part of $E_0^{(2)}$ the renormalized second order contribution that can be evaluated analytically up to integrations. $E_0^{\overline{(3)}}$ behaves as R^3 at $R = 0$, as it should according to our power counting result (Section 2) because it is actually of order $\overline{(4)}$ since the third order does not contribute. It has a kink at $R \approx 2.7$ which is related to the fact that there a bound state level (dotted line) crosses $E = 0$ and therefore also the ν integration contour in the complex ν plane. If the negative of its energy is added, implying

that the state continues to have baryon number 0 and implying analytic continuation of $E_0(R)$ considered as a complex function, $E_0(R)$ continues as the dashed line. It is evident that this curve has no tendency to show the R^{-1} behaviour expected from the $1/R$ expansion mentioned in Section 2 for a contribution of fourth order. The originall result (full line) decreases possibly as $1/R$ but it is hard to understand how this could be derived from the gradient expansion. The finite part $E_0^{(3)}$ is especially for $R > 1$ of the same size as the perturbative second order contribution, so it has certainly to be taken into account in the dynamics of chiral hedgehogs. This agrees with similar results of [12,14].

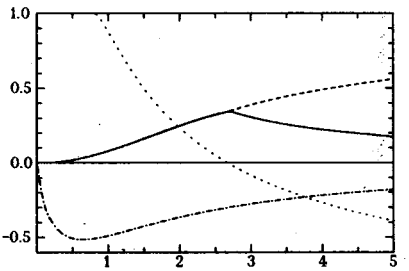

Fig. 1.

5. Conclusion

We have presented in this lecture some results on the effective action induced to the Dirac vacuum by the presence of a chiral mass. We have pointed out the limitations of the gradient expansion and we have described a numerical nonperturbative method which allows to calculate the renormalized zero point energy in a covariant and numerically stable way. Its application to recent models should be useful.

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